INTERNATIONAL CENTER FOR MATHEMATICS

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Editorial

In the second issue of bulletin of 2016, we present two articles with the recent trends in *Dynamical Systems* and in *Numerical Analysis*, an article about the *Willmore Conjecture* and the work of ANDRÉ NEVES, as well as several summaries and reports of some of the activities partially supported by CIM. Inserted in the cycle of historical articles, we include a piece about the work developed by PEDRO NUNES to celebrate the revival of the *Pedro Nunes' Lectures*.

The Pedro Nunes' Lectures is an initiative organised by CIM and SPM, with the support of the Fundação Calouste Gulbenkian, for promoting short visits to Portugal of outstanding mathematicians. Each visitor is invited to give two or three lectures in Portuguese universities about recent developments in Mathematics, its applications and cultural impact. This year, the invited lecturer was JEAN-PIERRE BOURGUIGNON who gave us a very interesting interview that is one of the highlights of the present issue.

We recall that the bulletin welcomes the submission of review, feature, outreach and research articles in Mathematics and its applications. The CIM Bulletin has recently been assigned an ISSN number for its print and electronic versions.

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https://www.fc.up.pt/pessoas/jmfreita/

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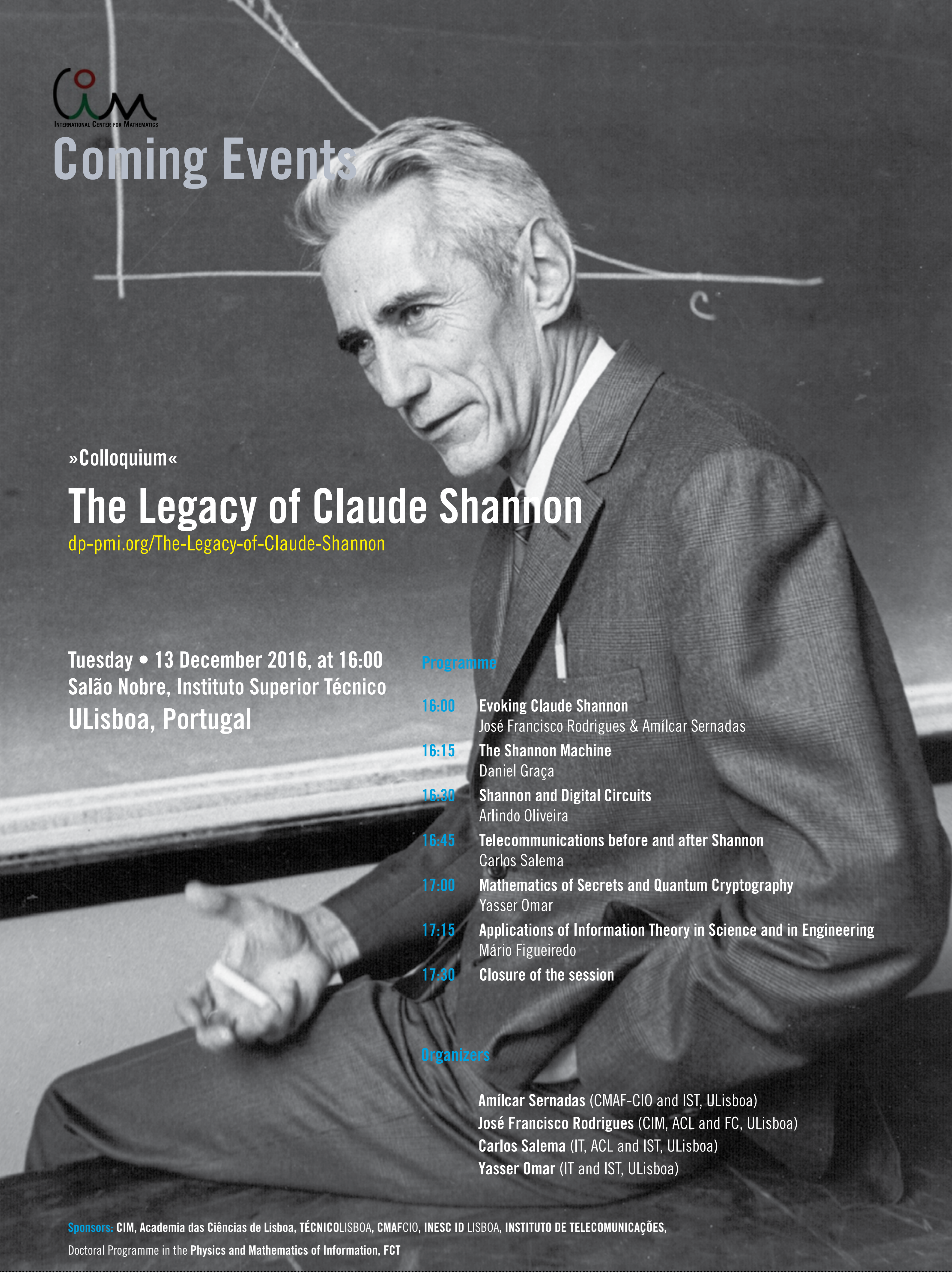

Coming Events

»Colloquium«

The Legacy of Claude Shannon

dp-pmi.org/The-Legacy-of-Claude-Shannon

Tuesday • 13 December 2016, at 16:00
Salão Nobre, Instituto Superior Técnico
ULisboa, Portugal

Programme

16:00	**Evoking Claude Shannon** José Francisco Rodrigues & Amílcar Sernadas
16:15	**The Shannon Machine** Daniel Graça
16:30	**Shannon and Digital Circuits** Arlindo Oliveira
16:45	**Telecommunications before and after Shannon** Carlos Salema
17:00	**Mathematics of Secrets and Quantum Cryptography** Yasser Omar
17:15	**Applications of Information Theory in Science and in Engineering** Mário Figueiredo
17:30	**Closure of the session**

Organizers

Amílcar Sernadas (CMAF-CIO and IST, ULisboa)
José Francisco Rodrigues (CIM, ACL and FC, ULisboa)
Carlos Salema (IT, ACL and IST, ULisboa)
Yasser Omar (IT and IST, ULisboa)

Sponsors: **CIM, Academia das Ciências de Lisboa, TÉCNICO**LISBOA, **CMAF**CIO, **INESC ID** LISBOA, **INSTITUTO DE TELECOMUNICAÇÕES,**
Doctoral Programme in the **Physics and Mathematics of Information, FCT**

Mathematics and Literature
An International Workshop held in Óbidos, Portugal, the 1st October 2016

by José Francisco Rodrigues*

The *Centro Internacional de Matemática*, in partnership with the *Portuguese Mathematical Society* https://www.spm.pt/ and the *Science Museum of the University of Coimbra* http://www.museudaciencia.org/, organized a one day international workshop on *Literature and Mathematics*, with the collaboration and support of the *Centro de Matemática, Aplicações Fundamentais e Investigação Operacional of the University of Lisboa*.

This initiative took place at the beautiful Bookstore SANTIAGO, on the occasion of the FOLIO 2016 — *Óbidos International Literature Festival* http://foliofestival.com/ — organised and hosted by the medieval town of Óbidos http://www.obidos.pt/.

* Centro de Matemática, Aplicações Fundamentais e Investigação Operacional da Faculdade de Ciências da Universidade de Lisboa

Figure 1 — A glimpse on the participants at the SANTIAGO bookstore

The workshop, with an average above twenty participants, consisted of ten presentations, covering not only significant aspects of mathematical influence in the literature, interactions between mathematics and poetry, but also on the raise of graphic novel as a mean of communication of Mathematics stories.

Sydney Padua, a cartoonist based in London and author of the recent book *The Thrilling Adventures of Lovelace and Babbage*, spoke on how Ada Lovelace saw in Babage's machine a way to create what she called a "Poetical Science": combining metaphor and mathematics to anticipate the digital age. In her talk she also told the story of these two fasci-

Figure 2 — Sydney Padua explaining her recent book The Thrilling Adventures of Lovelace and Babbage.

Figure 3 — Bernard Hodgson and his mathematico-literary gleanings.

nating and brilliant eccentrics, and discussed her process of primary-source research and creative transformation. Jorge Buescu, from the U. Lisboa, made a presentation on this last decade trend on the comic book-style works as a new literary genre and unexpected interaction between Mathematics and Literature, which are nothing short of spectacular.

Bernard Hodgson, from the U. Laval, Canada, talked on his mathematico-literary gleanings, a series of short papers he is currently writing for *Accromath*, a bi-annual magazine published in Québec and whose mission is the popularisation of mathematics amongst secondary school teachers (and their pupils). With a few examples, mostly taken from the

Figure 4 — António Machiavelo speaking about Proust and Luisa Malato reciting poetry of José Anastácio da Cunha.

Figure 5 — Alexander Nazarov reciting Akhmatova's poem *In Vyborg*

French literature, he showed how ingredients with an explicit mathematical flavour can be found in literary works, and how he sees their potential for the context of secondary education—for example, the extent to which these excerpts could be used as a starting point for a mathematical activity in the classroom. He presented examples from authors such as Marcel Pagnol, Boris Vian, Raymond Queneau (and other OULIPO members), as well as Franquin, the *father* of the famous and hilarious Gaston Lagaffe.

António Machiavelo, from the U.Porto, impressed the participants with the mathematical metaphors that he found in the monumental work of Marcel Proust *In Search of Lost Time*. Going over these deep metaphors, he showed how Proust had very accurate ideas on some non-trivial mathematics and he pointed out some of their philosophical relevance. He explained how he found a sort of loop in the entire book with some delicious self-references with a strong mathematical flavour. Reporting on more or less explicit references to mathematical ideas that abound in Jorge Luis Borges's short stories, like infinity, recursivity, equality, logical paradoxes, Jerôme Germoni, from the U.Lyon 1, aimed to unravel a few reasons why mathematicians often like Borges so much. In his talk, he pointed out not only a few of those structures, but also more hidden resonances where the short story turns out to be unexpectedly similar to mathematical thinking.

The amazing interactions between Mathematics and Poetry were illustrated in three presentations. Darya Apushkinskaya, U. Saarbrucken, and Alexander Nazarov, U. St.Petersburg, told about the friendship between the great Russian poet of the twentieth century Anna Akhmatova and the prominent mathematician Olga Ladyzhenskaya, while Carlota Simões, U. Coimbra, and Carlos Santos, A. Ludus, Lisboa, spoke about Camões and Mathematics. Luís de Camões (1524–1579), the great the Portuguese poet of Renaissance, had a clear and accurate knowledge of XVI century's astronomy. For his epic poem *Os Lusíadas*, it is known

Figure 6 — The participants from left to right: J.F.Rodrigues, P.P.Pálfy, A.Machiavelo, A. Nazarov, C.Toffalori, J.Buescu, B.Hodgson, C.Simões, J.Germoni, S.Padua, C.Santos, D.Apushkinskaya, M.L.Malato

today that the the main source for astronomic references was the mathematician and Royal Cosmographer Pedro Nunes (1502–1578). They also presented several fascinating aspects of Camões' sonnets. José Francisco Rodrigues, U. Lisboa, and Maria Luísa Malato, U. Porto, briefly overviewed the life and work of José Anastácio da Cunha (1744–1787), a Portuguese progressive thinker, modern mathematician and talented poet. As a mathematician, he is known by his deep anticipation on the foundations of infinitesimal analysis, appreciated by Gauss and highlighted by Yushkevich, and as a proto-romantic poet he is considered by Fernando Pessoa to "represent the first white glimmer of dawn on the horizon of Portuguese literature, for he represents the first attempt to dissolve the hardened shape of traditionalist stupidity by the usual method of multiplied culture contacts".

The contribution by Carlo Toffalori, from the U. Camerino, Italy, illustrated the image of Mathematics in Dostoyevsky's novels as clearly negative, by explicitly accusing mathematical determinism of being arrogant and oppressive and by comparing truth and freedom in Mathematics and in the vision of the Russian writer.

Finally, Péter Pál Pálfy, from the Hungarian Academy of Sciences, introduced Péter Esterházy (1950–2016), an outstanding postmodern author, passed away in July this year. Coming from one of the most famous Hungarian aristocratic families he was allowed to study only a subject furthest away from ideology: mathematics. Although he had worked only four years as a mathematician, his creative power and the surprising connections in his writings show that mathematics had a deep influence on his literary works.

Acknowledgement: José Francisco Rodrigues, author of this notice, thanks Graça Brites for the photographies of the Workshop numbered 2, 4, 5 and 6, and José Pinho, coordinator of FOLIO MAIS and his collaborators in Óbidos for their hospitality.

Topology of Manifolds, Lisbon

By Gustavo Granja*

The conference Topology of Manifolds, Lisbon was held at the National Museum of Science and Natural History of the University of Lisbon, from June 27th to July 1st 2016. The event was partially supported by the following institutions: *National Science Foundation* (U.S.A.), *Centro de Análise Matemática, Geometria e Sistemas Dinâmicos* (IST, Universidade de Lisboa), Fundação Calouste Gulbenkian, *Fundação para a Ciência e Tecnologia* and *Centro Internacional de Matemática*.

Michael S. Weiss. Picture by A. Ranicki

The conference brought together 120 experts on Algebraic and Geometric Topology, mostly from the United States and Europe. The program consisted of 11 invited talks and 14 contributed talks. The scientific committee of the conference consisted of G. Arone (University of Virginia), A. Ranicki (University of Edinburgh) and U. Tillmann (Oxford University) and the organizing committee consisted of P. Boavida (IST, University of Lisbon), S. Galatius (Stanford University), G. Granja (IST, University of Lisbon) and P. Lambrechts (Université Catholique de Louvain).

The conference was an opportunity to celebrate the 60th anniversary of Michael S. Weiss, one of the foremost contributors to the subject of the conference in the last 35 years. Michael Weiss holds the Alexander von Humboldt Professur at the Westfälische Wilhelms-Universität Münster since 2012, having joined Münster from the University of Aberdeen. His contributions to homotopy theory and geometric topology include outstanding work on Automorphisms of Manifolds and its relation to Algebraic K-theory (with Bruce Williams) [1], the development of Homotopy Functor Calculus (specifically Orthogonal Calculus [2] and Calculus of Embeddings [3]) now called Goodwillie-Weiss Calculus. He is perhaps most famous for the solution (with Ib Madsen, in 2002) of the Mumford Conjecture on the stable homology of the mapping class group [4]. Weiss received the *Fröhlich Prize* of the London Mathematical Society in 2006.

The circle of ideas involved in the Madsen-Weiss solution of the Mumford conjecture has exploded into a new and very active field of Algebraic and Geometric Topology — the study of moduli spaces of manifolds and cobordism categories [5] — which was heavily represented in the program of the conference. There were also many talks on the related and extremely active field of homological stability, as well as talks on the more classical subjects of surgery and Algebraic K and L-theory.

Highlights of the conference included Michael Weiss' own talk proving the existence of exotic Pontryagin classes for topological bundles of Euclidean spaces and Alexander Kupers' talk proving the finite generation of the homotopy groups of the diffeomorphism groups of disks relative to their boundary (in dimensions not equal to 4,5 and 7) drawing on Galatius and Randal-Williams's work on parametrized surgery [6] as well as Goodwillie-Weiss calculus.

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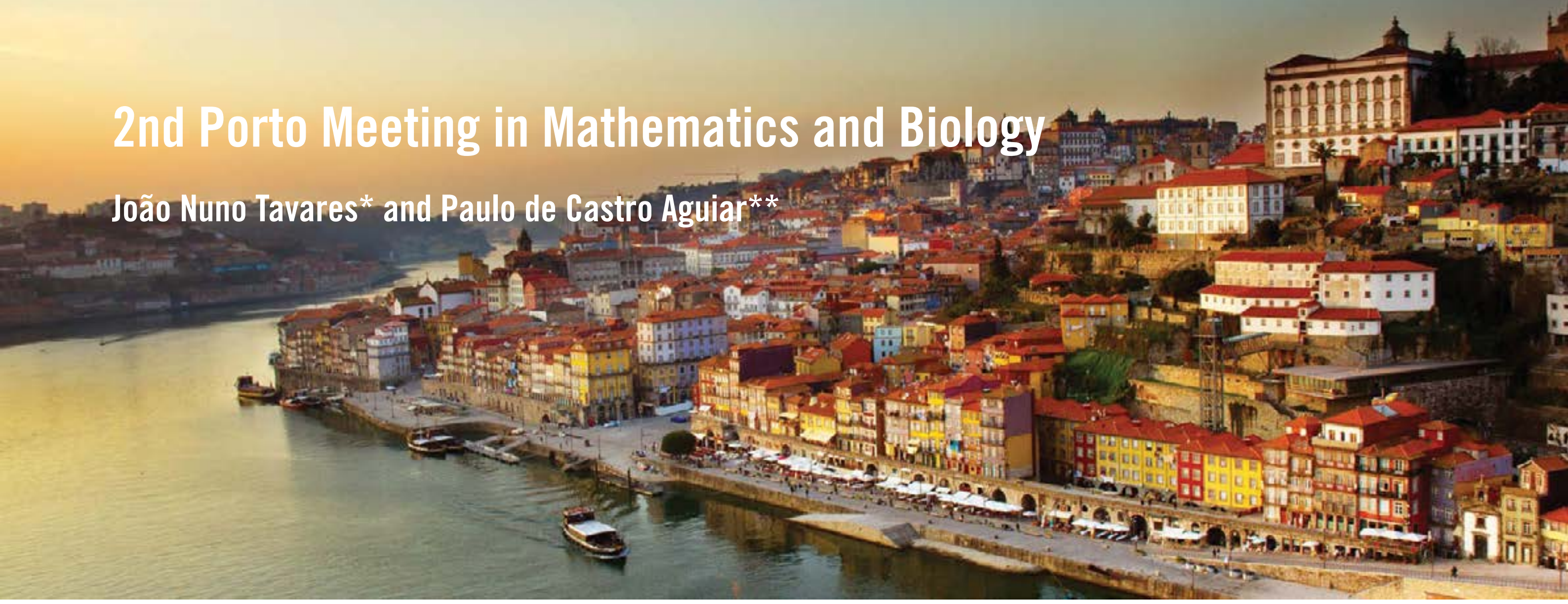

2nd Porto Meeting in Mathematics and Biology

João Nuno Tavares* and Paulo de Castro Aguiar**

In the past days 15 to 17 June 2016, held the second edition of the so entitled *Porto Meetings in Mathematics and Biology*, with the promoting of the *Faculty of Sciences of University of Porto*, the *Institute for Research and Innovation in Health* (i3s), the *Mathematics Center of the University of Porto* (CMUP), and also counted with the sponsorship of the CIM.

The purpose of this second meeting (the first was held on 23 and 24 June 2015) was to promote interaction between mathematicians, physicists, engineers, statisticians, biologists and clinicians to discuss the application of quantitative analysis methods to biological problems. More specifically this year's edition was dedicated to the theme *Systems Biology*.

Systems Biology is a new methodological paradigm that transformed 21st century research in Biology. Biology has become increasingly cross-disciplinary as biologists, computer scientists, engineers, mathematicians, physicists and physicians, work together to develop the high throughput technologies and computational/mathematical tools required for this new biology — all driven by the contemporary needs of biology and medicine. The systemic approach to biology is not new, but recently gained new impact, mainly due to the remarkable progress of experimental and computational (Bioinformatics) methods, each time most ingenious and powerful. We have now a golden opportunity to uncover the essential principles of biological systems that enable us to understand them in their entirety by investigating: (1). the structure of the systems, such as genes, metabolism, and signal transduction networks and physical structures, (2). the dynamics of such systems, (3). methods to control them, and (4.) methods to design and modify them for desired properties.

This conference was part of a set of initiatives that are designed to promote scientific interactions between mathematicians, biologists and clinicians (epidemiologists, imunologists, etc.) to facilitate the multidisciplinary research on topics of common interest.

The conference has consisted of the following 4 courses: *Introduction to Dynamic Mathematical Modelling in Systems Biology*, by Brian Ingalls (Department of Applied Mathematics, University of Waterloo; *Metabolic Network Modelling: Genome-scale Reconstruction, Flux Balance Analysis, and Applications to Caenorhabditis elegans Metabolism* by Lutfu Safak Yilmaz (Walhout Lab, Program in Systems Biology, Department of Biochemistry and Molecular Pharmacology, University of Massachusetts Medical School, USA); *Introduction to modelling noise and cell-to-cell variability in signalling networks* by Maciej Dobrzynski (Systems Biology Ireland, Conway Institute Belfield, Dublin, Ireland); *Optimization and parameter estimation (with COPASI)* by Pedro Mendes (Director of Mendes Research Group, School of Computer Science, Manchester Institute of Biotechnology, UK), and of the following Plenary lectures *Aging, Cancer and Neurodegenerative diseases* by Lloyd Demetrius (Max Planck Institute for Molecular Genetics at Berlin, Germany, and the Department of Organismic and Evolutionary biology, Harvard University; *Patterns of gene expression across multiple tissues and individuals* by Pedro G Ferreira (i3S, Porto); *In silico metabolic engineering* by Miguel Rocha (Departamento de Informática da Escola de Engenharia da Universidade do Minho, and finally *Using time series data to reconstruct gene signaling networks* by Joel Arrais (DEI/FCTUC, University of Coimbra).

All the information's are available in the conference website http://cmup.fc.up.pt/cmup/biomath/

* CMUP e GEMAC

** Institute for Research and Innovation in Health (i3s)

An Interview

with Jean-Pierre Bourguignon

by **Carlos Florentino** e **Jorge Milhazes Freitas**

Jean-Pierre Bourguignon holds an engineering degree from École Polytechnique and a PhD in mathematical sciences from the University Paris VII. A differential geometer by training, he has since pursued his interest in the mathematical aspects of theoretical physics. He is the President of the European Research Council as of 1 January 2014. He was the Director of the Institut des Hautes Études Scientifiques (IHÉS) from 1994 till 2013 and president of the European Mathematical Society from 1995 to 1998.

Jean-Pierre Bourguignon visited Portugal in July 2016, when he delivered an opening speech for the Encontro Ciência '16. This was the perfect occasion to make an interview, which flowed as an enriching and pleasant conversation, full of personal insights and experiences, with a mathematician who also occupied several high profile positions.

Jean-Pierre Bourguignon was invited to give three Pedro Nunes Lectures, which were delivered at the Universities of Aveiro, Porto and Lisbon, in October 2016. The Pedro Nunes Lectures is an initiative of CIM, which aims at bringing outstanding mathematicians to Portugal in order to encourage the interest in Mathematics and, in particular, in research in Mathematics.

How did you become a Mathematician? And how did you get interested in Differential Geometry?

Well, it's kind of an unusual story. When I was in secondary school, I was very much interested in Literature and Philosophy. So, this is really what I thought I could be involved in although I had good grades in Mathematics. I was in a small *Lycée*, and had the same Math teacher for five years, which is unusual. He was teaching very efficiently, and using the best students to help the others. So, very early on, I was asked to explain mathematics to others, and now I am sure this played a critical role in my being comfortable with Mathematics.

Then, in the last year of secondary school in France, I moved to a much bigger *Lycée*. Here, I had a very challenging teacher, known to be a remarkable mathematician. All of a sudden, I was confronted with somebody who was saying something which I perceived as interesting and important but that I could not understand. Just to show you to what extent I suffered, being used to having good grades in Math, my first grade with this teacher was 0.5 out of 20. It wasn't the worst grade, as some people had 0.25 and others zero. Actually, he was teaching some form of Linear Algebra, without it being supposed to be taught! So, I started studying Mathematics by myself. Fortunately, at the same time, I had a Physics teacher who convinced me that I was not that bad, and I was successful in Physics. Slowly, I also recovered in Mathematics.

Then, I decided to go on studying Science and went to the *Classes Préparatoires* where I realized that, because I had been thinking by myself, I could be among the bests in a class where some students already had received some prizes. From that on, I thought that maybe I could be able to do Mathematics as a profession. A year after, when preparing for the competition to enter the *Grandes Écoles*, I had another good mathematician as teacher, but his grading method was very peculiar. He would grade according to what he thought you could do. So, if he was expecting something great from you, then you could have terrible grades, and next to you there could be someone from whom he had no such expectation, who would end up having better grades than you. So, even though I was understanding better than others, I was lost, and it wasn't clear I would be able to do Mathematics.

I successfully passed the entrance competition to *École Polytechnique* where the Mathematics courses were very solid — Gustave Choquet was one of my teachers — but I realised others, e.g. at École Normale Supérieure, were learning much more Mathematics than me. At this time, the teachers I had in Mechanics were very disappointing and confusing and, with a small group of students, we organized some kind of *pirate courses* to replace teachers. So, I learned a lot, and read all possible books I could find on Mechanics: Arnold, Truesdell, Sedov, etc.

So, at the end of my two years as student at *École Polytechnique* and one year of *Diplôme d'Études Approfondies* in Mathematics studying *Sheaf Theory*, I decided to try and study Mechanics for a PhD.

I already had a clear idea of what I wanted to do: I wanted to solve the Euler equations of the motion of fluids using a technique introduced by Vladimir Arnold based on the search for geodesics of the group of diffeomorphisms. However, at the time, most of the teachers in Mechanics in Paris were quite senior people, and when I approached them, they all basically told me the same thing "no, you are not going to do what you want, you are going to do what we tell you to do". And actually, shortly afterwards, Claude Godbillon, with whom I had spoken about my project, just forwarded to me the just published work by David Ebin and Jerrold Marsden in which they solved the Euler equations using precisely the method I had in mind. So, I was again lost, and went back to the subject closest to Mechanics: Differential Geometry.

With this in mind, I approached Marcel Berger, and started to work on a PhD with him. Then, I could convince him to invite David Ebin to France to give a course, so that I could learn more systematically Global Analysis at a moment when it was not considered so important. Earlier, I had received from Choquet a good training in Analysis. Therefore, I could combine Analysis and Geometry, and started working on non-linear partial differential equations (PDEs) with a reasonably solid background. Accept my apologies for such a long story. As you have now read, my idea of doing research in Mathematics did not come so straightforwardly.

Do you have a favorite mathematician that has particularly inspired you?}

Besides Berger of course, who was very generous with his time to share his broad knowledge about Geometry with me, a person who had quite some influence on me was Shiing Shen Chern.

Very interestingly, in 1972, I was invited to Stony Brook by Jim Simons He had attended one of the lectures I gave in Berger's seminar in June 1972, in Paris. The next day, I had on my desk a fax from Stony Brook, offering me an Assistant Professor position there. At the time, I did not hold a PhD; I did not even bother to get a *Thèse de Troisième Cycle*, as I had already a position at the CNRS on the basis of a small article written while at the École polytechnique. After an intense discussion with my family we decided to take the chance and go.

I spent the year 1972–1973 in Stony Brook, which was then really the *Mecca* of Differential Geometry, with 14 mathematicians in this field in the Mathematics Department. Can you imagine? There was of course Simons himself, John Millson was a student there, James Ax, John Thorpe, Leonard Charlap, Jeff Cheeger, Detlef Gromoll, Wolfgang Meyer, Shing Tung Yau and several others. Actually this was a fantastic opportunity to become very close to Yau, who was enjoying his first position (at age 23) after having been a student of Chern. Being in Stony Brook was an unbelievable chance.

During the summer of 1973, I was invited by Robert Osserman to Stanford, and spent the whole summer there. While I was at Stanford, I got a phone call from Chern saying that he would like to have lunch with me, in Berkeley. At the time, I was 26, did not have a PhD, and here is Chern calling me to have lunch. I was just amazed! Actually, I was told later that he was doing that with a number of young people. Nevertheless, being called by Chern was something special. We had a very interesting discussion, he wanted to know what I was doing. By then, in France doing Differential Geometry was more or less proving that you were not a real mathematician. If you were one, you would be doing Algebraic Geometry or Number Theory, Differential Geometry was viewed by a number of people as a secondary subject considered technical. When I came back from the US, I thought that maybe what I was doing was not so silly. After all, Chern and a lot of other people were interested in it. In this way, and a few other ways later on, S.S. Chern had a lot of influence on my career.

Moreover, in September 1973 there was a Summer Institute of the American Mathematical Society (AMS) on Global Analysis, which actually was a turning point of the whole theory. At the time, working with Yau, we were trying to disprove the so-called Calabi conjecture, a major conjecture in Kähler Geometry, and we published a paper on it, showing that at least quotients of K3 surfaces dit not admit a metric with SU_2-holonomy. During that summer, Yau thought that he had disproved the conjecture.I attended the lecture he gave there to Calabi and Chern, but actually there was a gap. Finally, two years later he proved that the conjecture was *true*.

So, this visit to the US changed my perspective on my own work a lot, I was exposed to fantastic mathematicians, and I improved substantially my knowledge and practice of English.

Another important point is that I was blocked for defending my *Thèse d'État* because somebody had announced the result I was trying to prove, namely a stratification of the space of Riemannian metrics, in a Physics journal. And he never replied to any of my letters, neither to those of Berger, asking if he had proved it or not. Finally, in September 1973, he was also attending that conference and I could ask him directly: "Do you have a full proof?" and he said: "I am writing for physicists, so why should I have the full proof?" But he agreed to have Berger speak to him, and, some half a year later back in France, I could defend my *Thèse d'État*.

Among your many results and achievements, is there one that you are particularly proud of?

Well, there is one which I am proud of and with which goes a somewhat crazy story. In 1979 Blaine Lawson was in Paris, with Marie-Louise Michelsohn, his wife, visiting IHÉS, and I was meeting them regularly. At the time, the topic of Gauge Theory had become popular and much in demand from physicists. As I had studied physics quite seriously at École polytechnique, in particular quantum physics, I had an advantage over other mathematicians. I think I understood quite solidly what was behind Gauge theory and quantum effects. So, at some point, I was asked by physicists to give a course on the differential geometric background needed to develop Gauge theory. One day, before starting the course, I came to Lawson and showed him the outline of the course, and in passing, I mentioned to him what I knew about one of the conjectures that physicists were very much looking into, concerning the critical points of the Yang-Mills functional on the four sphere S^4. He looked at me and asked "What can you exactly do?" After I explained what I could prove, he said "I think I know how to do the missing half!". So, just by talking, we had the proof of a nice theorem. The heart of the matter is that I had understood how to use ideas of Jim Simons to go from 5 dimensions to 4 dimensions, but I was stuck at one point. In 5 dimensions the Yang-Mills functional is non-degenerate, but in 4 dimensions it is degenerate. I did not know how to get rid fully of the degeneracies, but Blaine did. We could very quickly publish an announcement in the Proceedings of the National Academy of Sciences. We decided to do it jointly with Jim Simons because we knew that he had just decided to quit mathematics. At this time he was not famous nor a rich person. The full article with Lawson was published in Communications in Mathematical Physics and it's one of my best papers.

There is another one which I like very much, but remains partly a mystery to me. It's about proving that a metric on a 4-dimensional manifold whose curvature is harmonic, is actually an Einstein metric, i.e. one for which the Ricci curvature is a constant multiple of the metric. The way to prove the result is, I think, very peculiar, because it uses the fact that, if you apply a certain identity called a Weitzenböck formula to the curvature tensor that you need to view as a harmonic vector-valued 2-form, it satisfies a generalized Laplace equation from which you can derive a peculiar pointwise algebraic commutativity property. From this property you can get information on the integrand of the signature of your 4-manifold Hence, under the topological condition that the signature is non-zero, harmonicity of the curvature — a third order condition on the metric — implies that the metric is Einstein, which is a second order condition.

I like this theorem very much because it brings together non-trivial facts about PDEs and Topology, but still the reason why it works remains mysterious to me.

Scientifically speaking, do you have any particular unfulfilled goal that you still would like to accomplish?

Oh, many. Well, the first one is the first problem suggested to me by Berger, I worked on it several times in my career: namely to decide whether $S^2 \times S^2$ admits metrics of strictly positive sectional curvature. This is still an unsolved problem. One can ask the same question for products of spheres in all dimensions. My guess is that the situation for $S^2 \times S^2$ may not be the same as the one for $S^3 \times S^3$. I tried many things and many people tried also, since it is a question which can be formulated in easy terms. My first publication actually was to show that, in fact, there is no such metric in the vicinity of the standard product metric of $S^2 \times S^2$. It's far from the final solution of the problem, but at least it shows that the problem is non-trivial.

It is widely acknowledged that Physics has had a long tradition of providing important challenges for mathematics research in particular for geometry, such as General relativity and Quantum Mechanics.

What physical theory do you think will have an analogous impact and provide the next big challenge for geometric research in this century? String Theory? Supersymmetry?

It's a complicated question. This influence has already happened to an extent people would have never believed. String Theory (ST) had an impact in particular towards Algebraic Geometry. For example, Kontsevich has a totally new way of thinking about Geometry using categories of higher order, which is certainly inspired by the challenges posed by ST. One theory which I personally spent quite some time on is supergravity (SG). Of course, it is not clear whether physicists are so interested in this theory anymore, but what I find really interesting is the way it combines classical DG with the study of connections with torsion. In SG there is, besides the usual structure, a 3-form. The geometry of such objects has been investigated recently by Nigel Hitchin and some people around him. And I think there is more to be said, in particular in connection with geometries with special holonomy (G_2 in 7 dimensions and $Spin_7$ in 8 dimensions).

Another area in which I was involved is the fantastic progress in the theory of systems of non-linear PDEs which came from the study of the Einstein equations. Actually, I taught General Relativity for 15 years at École polytechnique. Since the work of Demetrios Christodoulou and Sergiu Klainerman, as well as others who followed, we now have an understanding of the kind of regularity which is needed to guarantee the existence of solutions to the Einstein equations. I think this is a domain in which fantastic progress has happened thanks to both geometric ideas and sophisticated physics and mathematics. For me it is one of the most amazing achievements of the last 20 years.

This is a speculation now: do you think sometime soon Quantum Field Theory (QFT) will be placed in a rigorous mathematical basis?

There are some versions of QFT which are rigorous, but these are not the ones that physicists find the

most relevant. We always face the dilemma: on the one hand one can make the theory rigorous; on the other hand, one is not touching what the physicists consider to be the heart of the matter. Probably, we are missing some new mathematical concepts and background, and I wouldn't be surprised if one has to look at it from a very different perspective. In some recent approaches by people like Kontsevich using a new geometry involving higher categorical structures, the level of abstraction and the sophistication of the algebraic machinery seems to completely kill the geometry behind it. Not for him of course.

Another question that is talked a lot about is Alain Connes' programme of non-commutative geometry (NCG). It's a point of view providing very interesting approaches to theoretical physics. His belief, and there is evidence to support it, is that the Standard Model (SM) of particle physics has an internal structure which is much more meaningful than usually assumed. For many physicists, the SM is something where various pieces fit in a quite *ad hoc* way as the values of some coefficients in the SM were obtained through measurements. But for Connes, using NCG, these constants are really built into it for geometric reasons. So far, physicists are looking at this with some kind of a smile, as experiments should tell you which values are correct. As you know, the mass of the Higgs boson is not the one supersymmetry was predicting, and at some point, Connes thought that the LHC [Large Hadron Collider, CERN, Geneva] had proved one of his predictions to be wrong. But now, his latest conclusion is that he had made a mistake in one of his estimates and now, after correcting it, he gets a value for the mass of the Higgs particle compatible with experiments.

I think Connes'geometric approach is extremely interesting, in particular because it allows to put on an equal footing discrete and continuous spaces. This plays an important role in physical theories, which may have to deal with discrete or continuous objects, but also in Number Theory.

Throughout your career you have assumed several high profile administration positions, such as president of the Société Mathématique de France, president of the European Mathematical Society, director of the Institut des Hautes Études Scientifiques and president of the European Research Council. Portugal has been going through a severe financial and social crisis, which meant that only very few positions for mathematicians have been opened in the past years. Nonetheless, the PhD programmes in Mathematics have grown and have become quite successful. Given your experience, do you have any advice for these young researchers who have just finished (or are about to finish) their PhD, in terms of career opportunities?

Well, this is a big question. Maybe I should remind you that I was among the people who reviewed Portuguese Mathematics during the nineties. This was an extremely interesting exercise. At that time, and it has nothing to do with the quality of people, a number of researchers there were really looking at narrow and sometimes bizarre problems. And so, since the landscape was dominated by senior people doing at times somewhat routine research, these evaluations brought up a broader perspective that some younger people were able to take up when there were not even proposing them spontaneouly themselves.

So, for young people in order to be ambitious and develop research at the highest level, to have a clear idea of what their career path can be is critical, I even mentioned it in my speech this morning [Ciência 2016 — Encontro com a Ciência e Tecnologia em Portugal, 4–6 Julho, Lisboa]. It is fundamental that policy makers understand that, to have leading researchers, at some point one has to offer them a decent career perspective. This is exactly what happened in France in the early 1990s and this led to the generation of Jean-Christophe Yoccoz and Pierre-Louis Lions.

But still, there is one point that I would like to make here, namely that the possibilities for mathematicians to be employed are much broader now than they used to be, for several reasons. First of all, the interfaces of Mathematics with a number of other disciplines developed fantastically in the last thirty years: There are new interfaces with Biology and Medicine, for example, touching many areas of Mathematics, not just Statistics; but also with Social Sciences or Humanities there are many possibilities of involving mathematicians. I think it is quite important for the next generation of mathematicians to be exposed to several fields. Of course, in the end, people do what they feel is interesting. But, at some point, teachers must understand that you can become a mathematician in many more ways than one used to. You have to let students choose what is most appealing for them, but it would be a mistake to say that to do Mathematics you have to do Algebra, Geometry, Analysis, and so on. It's very important to expose students to various possibilities.

I'm not sure you know the figures, but for France, today one Mathematics Ph.D. out of two takes a job outside academia. In the early 1990s around 90% would stay in academia. So, this has broadened the perspective for students in Mathematics. There are people working in many different environments. Also many companies now want to have mathematicians as members of their teams. I often give the example of Veolia, a company doing transportation, garbage collection and many kinds of things, which employs many engineers. Talking to the head of research 4 or 5 years ago, he told me that, at that moment, 8% of the engineers had a strong mathematics background, and the objective was, by 2025, that 20\% of all engineers should have a broad mathematics background. This means that the number of people with very sophisticated mathematics knowledge employed in companies will grow considerably in the years to come.

Another point that I would like to make, is that three European countries have now studied what is the impact of advanced Mathematics in their economies: The UK, Netherlands and France. The conclusion was that the impact of advanced Mathematics was much bigger than people ever thought. In the case of France, the figure is 15% of all GDP is directly related to advanced Mathematics. And the number of jobs induced by this use is above 2 millions. The report can be found on the website of the Société Mathématique de France. It shows that mathematicians have been, in some sense, collectively underestimating their impact on Society, and that there are now many more ways of being a professional mathematician than before. But of course, it depends a little on how each country is dealing with this issue.

In Portugal, there are few purely research permanent positions, in contrast to the French CNRS. What do you think about this? Would you have any advice for the Portuguese government with respect to this issue?

I was an employee of CNRS for 44 years, and it is clear to me that I owe my career to this organisation. But when one considers the overall organization of the academic personnel involved in Mathematics in France, one finds that 85% are holding positions at higher education institutions and that only 15% are employed by the CNRS. Of course, given the size of France, the number of mathematicians employed by the CNRS exceeds 400. In a number of cases, the researchers from CNRS still teach somewhat, but of course less than if they were holding a regular teaching position.

So, the right thing to do is to make sure that, in a given country, there are enough positions to give a relief from teaching to a significant number of people. It should be possible, for example, that for 5 years, someone takes a relief from teaching in order to pursue research more intensely.

Actually, in France, one thing that was organized with this in mind was the *Institut Universitaire de France* (IUF). A national selection done both at the junior and at the senior levels, allows people to be relieved of one third of their teaching duties and get some extra support to do research in their own home institutions. Being a member of the IUF is seen as a very distinguished position with a positive impact on both the recipient researcher and his or her university.

This structure works quite well. Hence, I think this is another way of funding research personnel which is less expensive than having permanent research positions. It also helped to recognize that, for some people, the teaching load was too heavy to achieve excellence in research.

You are definitely a person who travelled the world. How do you see Portugal in terms of its scientific development?

Since I have been president of ERC, I lost a little bit contact with what different countries have been doing from a strict mathematical point of view. But since the middle of the nineties, I would say the transformation has been quite positive. Nowadays, many more people are exposed to international competition and all in all Portuguese mathematicians have been very successful, in particular young ones.

I understand that the recent years have been tough, as I heard from several Portuguese colleagues. But I think that taking a longer perspective, Portugal has really gone through a long transition. Actually, I think n the first years the efforts on the side of the Portuguese government were really important, with a significant increase in the number of funded research projects. I would like to stress that that these projects were evaluated by international panels. This was a smart move particularly in a small country like Portugal, where most people know each other very well, maybe too well. So, globally, I would say that the evolution has been very positive. I am not saying this to be nice. You may have noticed that I tend to be blunt.

Here, I must mention the very positive, in my opinion, influence José Mariano Gago had in this respect. We became friends and we exchanged on a regular basis on European issues. With Philippe Busquin, he played a critical role in the establishment of the ERC. He left us much too early.

As mentioned before, you have been the president of the European Mathematical Society (EMS). How do you see the importance for Europe to have a Mathematical Society?

It took a long time for the EMS to develop. Actually, you may not be aware since you are too young to have witnessed how slow the process was. Part of the problem was a remake of the traditional disagreement between the British and the French about the level of integration of the European process. Fortunately, there were the Germans to bring us together. I am serious about that. Two models were indeed competing: a British one where the EMS should be a society of societies with no individual members, and another one, supported by the French, according to which the EMS should be a much more integrated structure with individual members. The compromise was to have both, which is actually the current situation in the EMS, showing the compromise found was a good one.

I remember, in particular, the controversial foundational meeting in 1990 in Madralin. It was a not so gentle fight. Fortunately, the person who had been chosen to become the first president of the EMS, Friedrich Hirzebruch, imposed a mixed view which was accepted by Sir Michael Atiyah, who had been chairing the European Mathematical Council, which in some sense has been the matrix for the EMS later on. The key decisive step taken by Friedrich Hirzebruch was to ask Sir Michael Atiyah whether he would agree to become the member number one of the EMS, which of course would mean that he was accepting the compromise. He agreed.

But why should there be a European Mathematical Society? There are actually several obvious reasons. At the time, the European Commission was developing its framework programmes and mathematicians were unable to be present enough in this process. The only way was by having a lobbying power with a European flag in Brussels. So, the EMS played a role there and was able to force the presence of some meaningful programmes for mathematicians in the agenda.

Another important reason for me has been the need to enhance the development of the bibliographic databasis Zentralblatt Math (ZM) to avoid the monopoly of MathSciNet, property and one of the main providers of resources of the AMS. Attempts to get the two databases to cooperate had failed. It was of paramount importance that the European mathematical community could get organized to stand behind ZM and press for its modernization and presence worldwide. The EMS soon was a dynamic partner of the FachInformationZentrum Karlsruhe, the Heidelberg Akademie der Wissenschaft and Springer in ZM.

After this complicated start, I was completely surprised when Friedrich Hirzebruch invited me to be his successor. I could hardly believe that. But my relation with Hirzebruch was one of great respect. I greatly admired his efficiency and appreciated very much his efforts, for example, to develop the Max Planck Institute for Mathematics. I truly believed in the importance of the EMS and accepted the challenge.

It turned out to be a fantastic experience. I was very lucky with the excellent team who worked directly with me. We continued the work initiated during the previous presidence. I remember vividly for example the creation of an active website and the first years of the Journal of the European Mathematical Society. Other important achievements were attained in direction of applied mathematicians, as we managed to create contacts and start some studies between Mathematics and Industry.

In the past few years, the Mathematical and the general scientific community have been overwhelmed with the use of bibliometric data to assess and evaluate individuals and institutions. This has been happening in job and grant applications, individual evaluations at reputable universities, research institutions' evaluations by funding agencies, and so on. On the other hand, we have the San Francisco Declaration on Research Assessment signed by many important scientists and scientific organizations. Do you have a personal opinion on this matter that you would like to share with us?

This has become an important issue. I am fighting the use of bibliometrics to evaluate people in a very explicit way. With ERC panel members, I have been insisting that they do not to use that. Of course the temptation to use this information varies much from one discipline to the other. Disciplines where this is more or less routine are Biology and Biomedical Sciences but for Physics and Mathematics, for example, I have not seen any of the panels making real use of this.

Of course figures at certain levels can be useful to obtain a global picture. For example, at the ERC we sometimes use the 10% or 1% more cited papers figures globally for Europe or at the level of nations. But the idea of evaluating and funding individuals or teams based on bibliometrics is inappropriate, and there are several arguments against it. The first argument is that people have different publication and citation habits across different disciplines and subjects. Even within Mathematics, for example, the geometers do not quote and cite in the same way as analysts do. People doing applications have even more different patterns. A second argument is the fact that most of these data are using citations from the last three years, when the average age of citation of a mathematical paper is between eight and nine years, so using this type of citation information does not actually make any sense. Of course this varies a lot with disciplines because in some other fields a paper with more than three years of age has basically no value for quotation. This is of course not the case for Mathematics.

The other argument why I insist not to use this at the ERC is the following: the objective of the ERC is to fund ambitious projects with bright new ideas, and looking at passed data does not give much of a clue about the value of the project. Hence, I have been very explicit about this and, although I experience some resistance from biologists, the position of the ERC Scientific Council on this is very clear. We highlight the necessity of evaluating the potential of a good idea as the most important thing.

This does not mean that bibliometric data have no value. It just means that they have no place in the evaluation of individuals and can be used for the evaluation of research teams when properly aggregated at a large enough scale.

Research in Mathematics has a dual mode: fundamental research and applied research. Often they are closely connected and one stimulates the other. However, in certain fields or subjects, applications occur (if they occur) only after a very long time gap. In a society eager for technological advances, the pressure for financing almost exclusively applied research is overwhelming. Do you have any advice for people working in fundamental research on how they should proceed to have access to funding?

There are several sides to your question. First of all, at the level of the ERC we insist that we are dealing with frontier research. We do not want to discriminate between fundamental or basic or pure and applied or technological research. The truth is that, if you look at the ERC portfolio (and this was not decided a priori), 85% is pure or basic research and 15% is applied or technological research. But this can change over time.

The second comment is that people who decide policies are very often under pressure by politicians. For politicians the key issue is to have short term results. The reason is that the next election is tomorrow. In some countries like China, they do not care so much about short term results because the government has longer periods, like 20, 30 years, in mind. Therefore they initiate programmes like the new 5 year plan with a considerable focus on fundamental research because they want to build a community able and eager to develop new technologies in the future. We, as scientists and this is especially true for mathematicians, have to teach politicians how research really works. Research does not work as well when you tell people what to do. Actually, this should not be called research, this is development. When you do research, it is very difficult to anticipate what is going to come out

since you are dealing with the unknown. This does not mean you should not make specific efforts in some particular areas. The best response we can give to politicians is that they should adopt a balanced strategy. Clearly there can be top down priorities on topics like energy, climate change, etc. But at the same time, there should be a very significant percentage of research left at the initiatives of researchers using a bottom up approach. Then you have to make the case for numerous initiatives of researchers which turn out to be relevant for politicians. One such example is the recent use of perovskite minerals to build batteries which are much cheaper and have a very promising efficiency output when compared with other batteries. It came from a totally bottom up approach. I made this point to the Vice-President of the European Commission in charge of the energy portfolio, Maroš Šefšovič. The people who came up with this technology were not told to do that. This discovery just came from their own team dynamics.

Moreover, there are more short circuits coming from research projects not led by any a priori request but which can suddenly become a big story. The example I like to quote is the case of *CRISPR Cas9*, a new gene editing technique, which is actually used by bacteria for millions of years and was studied, in the 1980's, by Japanese researchers, who could not really understand the process at the time. Then, recently, the work was picked up by Jennifer Doudna and Emmanuelle Charpentier who paved the way for the discovery of this very promising new gene editing technique. In fact, to give an example of the impact of this breakthrough, at the ERC, last year, we had less than fifty projects using this technique and, this year, we have several hundreds. So, this is something that was spotted at a given time but not understood. Then, much later, someone managed to understand it so well that it became a new wide spread technique with a very promising and challenging future. This is a fantastic example of something which is most likely to have an enormous impact in several areas both from the economical and from the health point of view. And the key point is that all this happened just because people wanted to understand better something that looked mysterious. This is the perfect example to show that one cannot only rely on top down strategies but that bottom up is badly needed. So the key is to look for the right balance between the two.

Given the fact that research in Mathematics is most of the time less expensive when compared to other types of research, demanding intensive lab work, what do you think about the idea of reducing the huge amount of money for a single ERC grant and make them available to a larger number of people?

First of all, if you allow me, I am always surprised with this question because ERC allows people to ask for the amount of money they find appropriate to achieve their project. Recently a 200 000 Euro grant has been given for five years, which I think is not such a big grant. At no moment does the ERC press people to ask for large amounts of money. Most of the time it is the institution that presses the researcher to ask for more money, probably because of the 25% overhead it receives for each grant. So it really depends on the the people applying for the grants. It is true that the researcher can use this money to pay typically half of his or her salary, in line with the time dedicated to the project. If the institution is fair, then it should use that money to improve the support around the grantee, so that more people benefit from it. That is for example what the CNRS in France is doing: if the researcher decides to take half of his or her salary from the grant, then a large part of that money is distributed around him or her. I personally think it is a good way of lifting the spirit of all the people around the grantee.

The difficulty with giving very small grants is the fact that the administrative burden needed for putting in place a two million Euro grant is almost the same as the one for a 200.000 Euro. So, of course, by multiplying the number of grants by ten, for the agency, it would mean a huge increase in administrative costs. At the moment, the Executive Agency in charge of the ERC is managing about 5.000 grants, and there are only 90 people to do that. Another issue here is also to determine the European added value of distributing small grants. The national or even local levels are almost surely the right one to do that. This points to the fact that the support to research has to be thought in systemic terms: different means for distributing support should be in place and enough money be given in a recurrent way with decisions taken as close as possible to the researchers. I do not see any reason why the support to research has to be given only through projects. In order to develop completely new ideas, researchers need to be able to do it in a spontaneous and totally non bureaucratic way. Unfortunately, in a number of countries the balance has gone too much in the direction of supporting competitive projects. Not enough money has been left for recurrent support. This is, for me, a major mistake with, potentially, a very negative long term impact.

Concerning the ERC, another thing you need to keep in mind is that, in the end, the people who determine the amount of money granted are the panel members.

In fact, at the ERC, there is something sometimes referred to as the “Bourguignon policy”, which goes back to the time I was chairing the first panel distributing starting grants in Mathematics, because I insisted that the budget should be checked thoroughly. Already then, people tended to ask for an amount of money which was not related with the real needs of the project. So the mathematics panel I was in charge of did cut the budgets of some projects, because we felt the request made was not based on actual scientific needs. To conclude people should ask the amount of money they really need for thesatisfactory development of their project.

Pedro Nunes and Mercator: a Map From a Table of Rhumbs

by Pedro Freitas*

Pedro Nunes and the loxodrome

The 16th century was a period of great scientific and technological development in Europe. Portugal was no stranger to this general atmosphere, having developed new sailing techniques, which were necessary to navigate outside the Mediterranean, and below the Equator, where the North Star is not visible.

Along with more practical developments, there was also an interest in abstract physical and mathematical problems, inspired by concrete needs and questions. Among the people that were interested in these problems, stands the figure of Pedro Nunes (1502–1578), a remarkable mathematician and astronomer. Figure 1 is sculpture of Pedro Nunes in a monumento to the discoveries.

Pedro Nunes started his scientific studies in Salamanca, around 1517, where he got a degree in Medicine, in 1525 (this was the usual course of studies at the time for someone interested in a higher scientific education). He then returned to Portugal, and taught Moral Philosophy, Logic, and Metaphysics at the University of Lisbon (starting between 1529 and 1531). He was appointed Royal Cosmographer in 1529 and Chief Royal Cosmographer in 1547, a post that he held until his death.

Among his publications, we refer the *Tratado da sphera* (1537), which includes the *Treatise on certain doubts of navigation* and the *Treatise in defence of the nautical chart*, to which we will return, and *De crepusculis* (1542). In this second book, he studies, and solves, an important problem of his time: the determination of the duration of the twilight, depending on the latitude and the day of the year, and provides many other new and relevant observations, including the description of

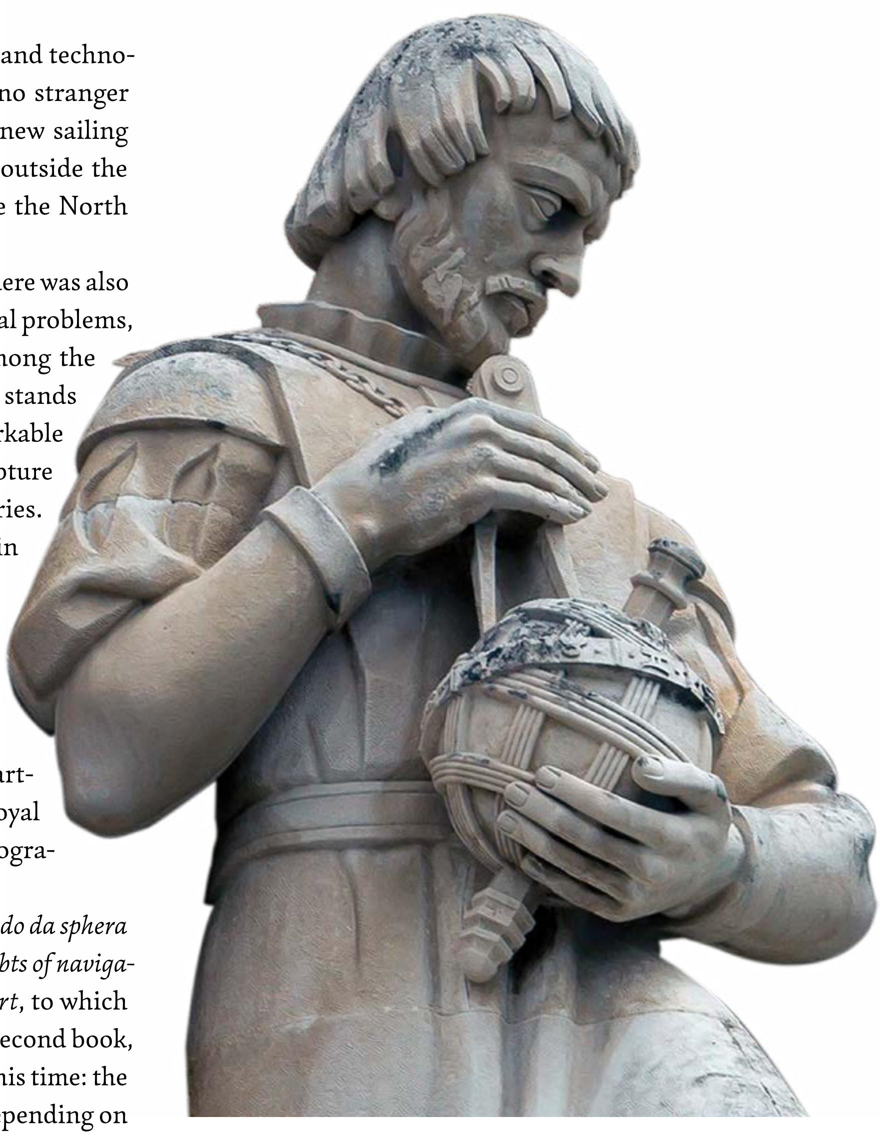

Figure 1.— A sculpture (by Leopoldo de Almeida) representing Pedro Nunes

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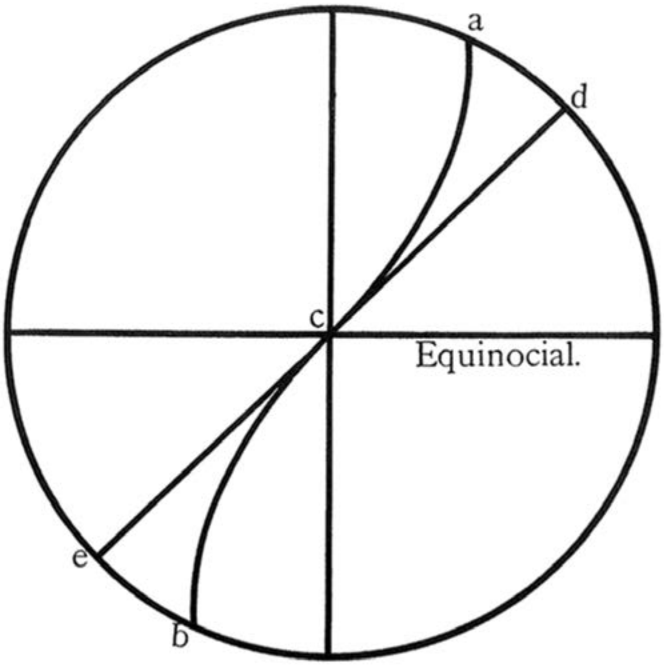

Figure 2.— A great circle (dce) and a rhumb line (acb)

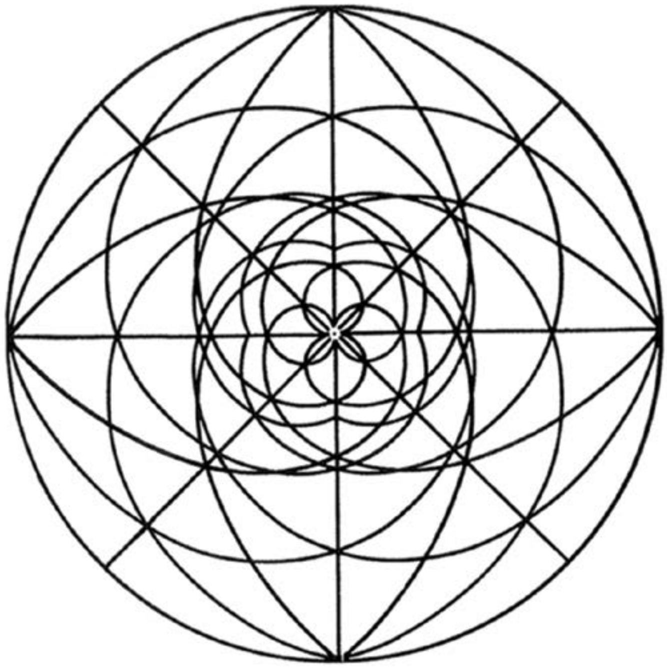

Figure 3.— Loxodromes seen from the pole in a figure by Pedro Nunes

a new instrument, the *nónio* (see [6] for a study of this book). In both works, Nunes gives a strictly mathematical treatment to practical problems. These books, especially *De crepusculis*, with its rigorous and extensive solution to an old problem, afforded Nunes a high intellectual standing, at an international level.

These and other works were later collected and expanded by Nunes in *Petri Nonii salaciensis opera*, his collected works, published in Basel, in Latin, in 1566 — a commented edition was recently organized, between 2002 and 2010, and published by Fundação Calouste Gulbenkian.

In the course of his career of Royal Cosmographer, Nunes came across problems relating to navigation, which were in great extent inspired by practical considerations. In [2], Pedro Nunes mentions some specific questions that the navigator Martim Afonso de Sousa asked him. One of them was about the correct way to navigate along a great circle, which is the path of least distance on a sphere (this is Nunes' reformulation of the original question). The knowledge of the north, given by the compass or the North Star, would allow keeping the ship at a course maintaining a given angle with meridians, and apparently there was a belief that this would ensure the ship would travel along a great circle. Pedro Nunes gave this problem quite some thought. Figure 2, taken from [2], and the accompanying text, show that he realized that this belief was misguided: the great circle is not the course the ship would take if this angle (called bearing) were kept constant.

Nunes distinguishes very clearly these two forms of navigation, noting that if one wants to navigate along a great circle, the bearing has to be constantly adjusted. The curve that the ship follows if the angle with meridians is kept constant came to be known as a rhumb line (this was the name used by Nunes), a loxodromic curve, or simply a loxodrome. The word rhumb refers to this constant angle with meridians, to be kept constant in order to navigate along the curve. The method Nunes suggests to correct angles in order to travel along a great circle, based on spherical trigonometry, turned out to be too difficult to implement on board, and apparently was never adopted by Portuguese sailors. In fact, Nunes suffered much criticism regarding the abstraction and complexity of his methods, in his time, even though it remains to be ascertained if such criticism was deserved or not.

At any rate, the treatment of this new curve,[1] the loxodrome, was extended to a theoretical level not seen before for non-conical curves. Figure 3, also taken from [2], and reproduced in the cover of the Proceedings of an International Conference held in Portugal in 2002, shows a few loxodromes as seen from the pole.

One of the results Pedro Nunes proved about this curve, only clarified and published a few years later in [3], is that it does not enter the pole (unlike what the figure above sug-

[1] Pedro Nunes makes a reference to Ptolomy's *Geography* when describing the loxodrome, but this is probably just a way to relate it to a famous name, as no reference of this curve has been found in Ptolomy.

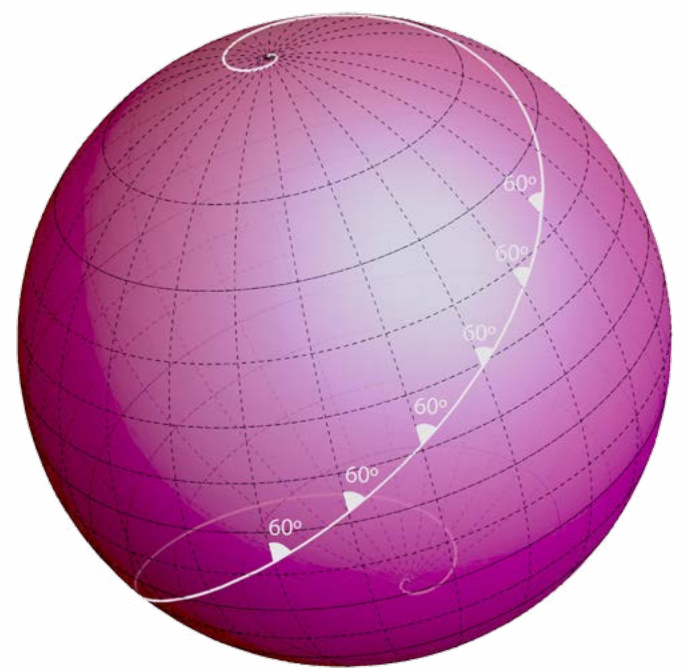

Figure 4.— A loxodrome making an angle of 60° with meridians

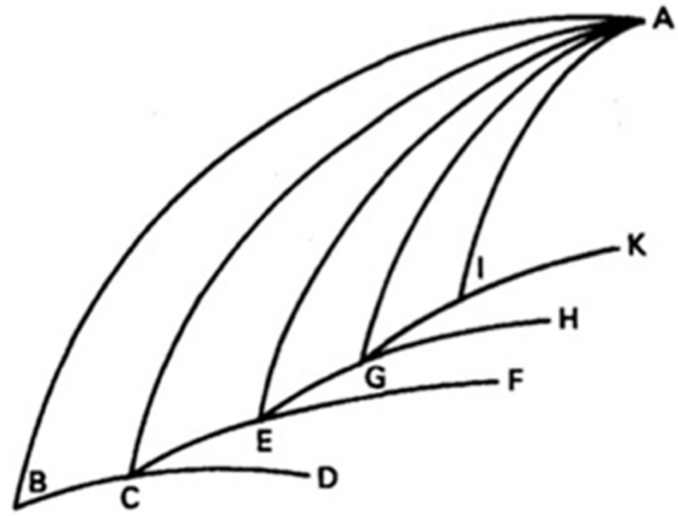

Figure 5.— Approximation of a rhumb by arcs of great circles

gests), yielding an *infinite line* on a sphere. Figure 4 shows a loxodrome on a sphere.

The equation for a loxodrome making an angle α with meridians is given by

$$\varphi = -\frac{\pi}{2} + 2\arctan e^{\lambda \operatorname{cotan} \alpha}$$

where φ is the latitude and λ is the longitude, and can be deduced from the definition of the curve using integral calculus (see [4] for a brief deduction of this formula and [5] for a more extensive study of this curve). The description of the curve in Nunes' time, however, was not done with a formula but with a table of pairs of longitudes and latitudes, along each rhumb. Usually, seven angles were chosen, making angles of 11,25° between them, evenly dividing the right angle between the equator and a meridian.

Nunes' method for constructing such a table is described in [3]. Once an angle was chosen, the rhumb is approximated by arcs of great circles. Figure 5 illustrates this.

In the figure, *A* represents the pole and the lines *BCD*, *CEF*, etc, are arcs of great circles. The lines originating at *A* are arcs of meridians. Nunes uses a theorem by Gebre about sines in spherical triangles to iteratively calculate the angles and lengths involved. After this description the method, Nunes includes an empty table, inviting the "laborious lads" to supply the calculations.

The Mercator chart

The expression *rhumb line* can also refer to a straight line, drawn on a navigation chart. When Pedro Nunes started working on these problems, there were no charts with the property that loxodromes would be represented as straight lines. In [2], Nunes actually refers to the need to create such a chart, which would make navigation problems much easier. The navigation could be charted along a certain rhumb, which would just be a straight line on a map, and correspond to a certain bearing, something that could be achieved with a compass.

The first chart with this property was published by Gerardus Mercator in 1569, and this cartographic projection came to be known as *Mercator projection*. Figure 6 (see next page) shows a modern map using this projection. The circles, called Tissot indicatrices, all have the same area on the globe.

The map shows that horizontal and vertical lines represent parallels and meridians, respectively, but parallels must be more spaced as the latitude increases (the scale factor at a given latitude is the secant of this latitude). This leads also to area deformation—for instance, Greenland looks as big as Africa, which is fourteen times larger—but it does have the property that loxodromes translate to straight lines.

The main problem of producing such a map is determining the rule for this increase in spacing of parallels. Mercator left nothing written about the method he used to calculate this spacing increment. He does refer that the rule for creating the map is that the proportion between lengths of meridians and of parallels should be the same in the map as in the globe. This implies angles are not distorted (in other

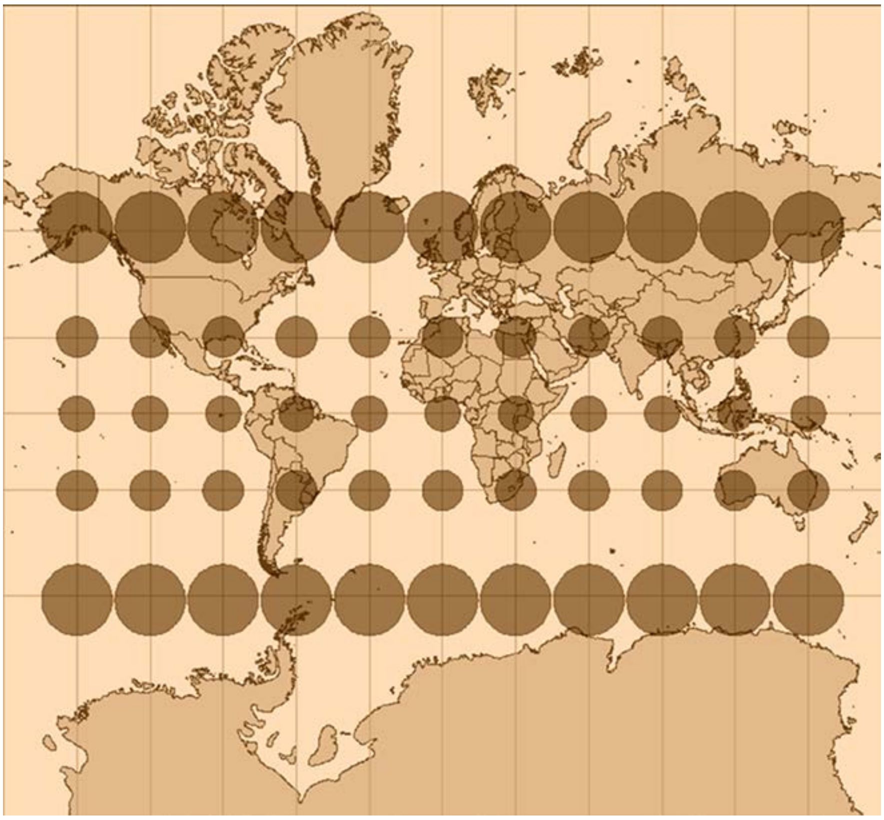

Figure 6.— A modern map using the Mercator projection (Stefan Kühn, Wikipedia).

words, it is a conformal projection). However, it does not provide a practical method for drawing the actual map.

This has been a problem of cartography for many years, several theories having been presented. There was a recent breakthrough, though, that originated in a detailed analysis of the errors in the original chart. We follow article [1] in describing this new approach.

First of all, one has to separate the calculation errors (inherent to the method) from errors due to the physical distortion of the sheet on which the map was printed (article [1] was the first one to distinguish these two types of errors). The key to detecting the physical distortion was a figure on the bottom right of the map, called *Organum directorium* (see Figure 7).

A graduated quarter circle with a mesh of meridians and parallels appears in this figure, along with the angles marked on the quarter circle. This means that we know what were the angles considered when making computations for the drawing of the parallels, and by comparing them to the actual angles and y-coordinates of the parallels on the map, we can ascertain what was the physical distortion the map suffered after printing, thus separating it from the errors due to calculations. As an illustration, we can see in the figure an angle α, along with a y-coordinate Y_α, which is used to draw a parallel.

Finally, a last question remains: after isolating the physical errors, what was the method used to draw the map? An answer that is simultaneously natural and ingenious consists of taking the property that the map should have and *turn it into the method* for drawing it. In other words: if the aim is that loxodromes should be straight lines, then consider one of these loxodromes, and make it a straight line! The (simplified) process is as follows: after drawing a graduated equator on a piece of paper, draw a straight line forming the given rhumb angle with the equator, and use a table of rhumbs to mark, on each longitude, the corresponding point on the rhumb line. This gives you y-coordinates for the parallels, which can then be drawn.

For this we need a table of rhumbs, a sequence of pairs of longitudes and latitudes, something that Pedro Nunes proposed. The authors of [1] tested a few tables available in 1569, and found that there was a remarkable match between the map and a table for the second rhumb (22,5°). This table used constant intervals of one degree of longitude, yielding differences in errors smaller than one fifth of a degree, when compared with the errors in Mercator's map.

Conclusion

Our story started about five hundred years ago, with Pedro Nunes' idea of a new curve on the globe, which would facilitate navigation, and his call for a map in which these curves would be straight lines. The map was created by Gerardus Mercator a few decades later, but the method of its making remained a mystery until our present days, only to be solved recently by Joaquim Gaspar and Henrique Leitão. And in this solution, Pedro Nunes happened to also have a side role, by launching the idea (along with a method) of constructing tables of rhumbs.

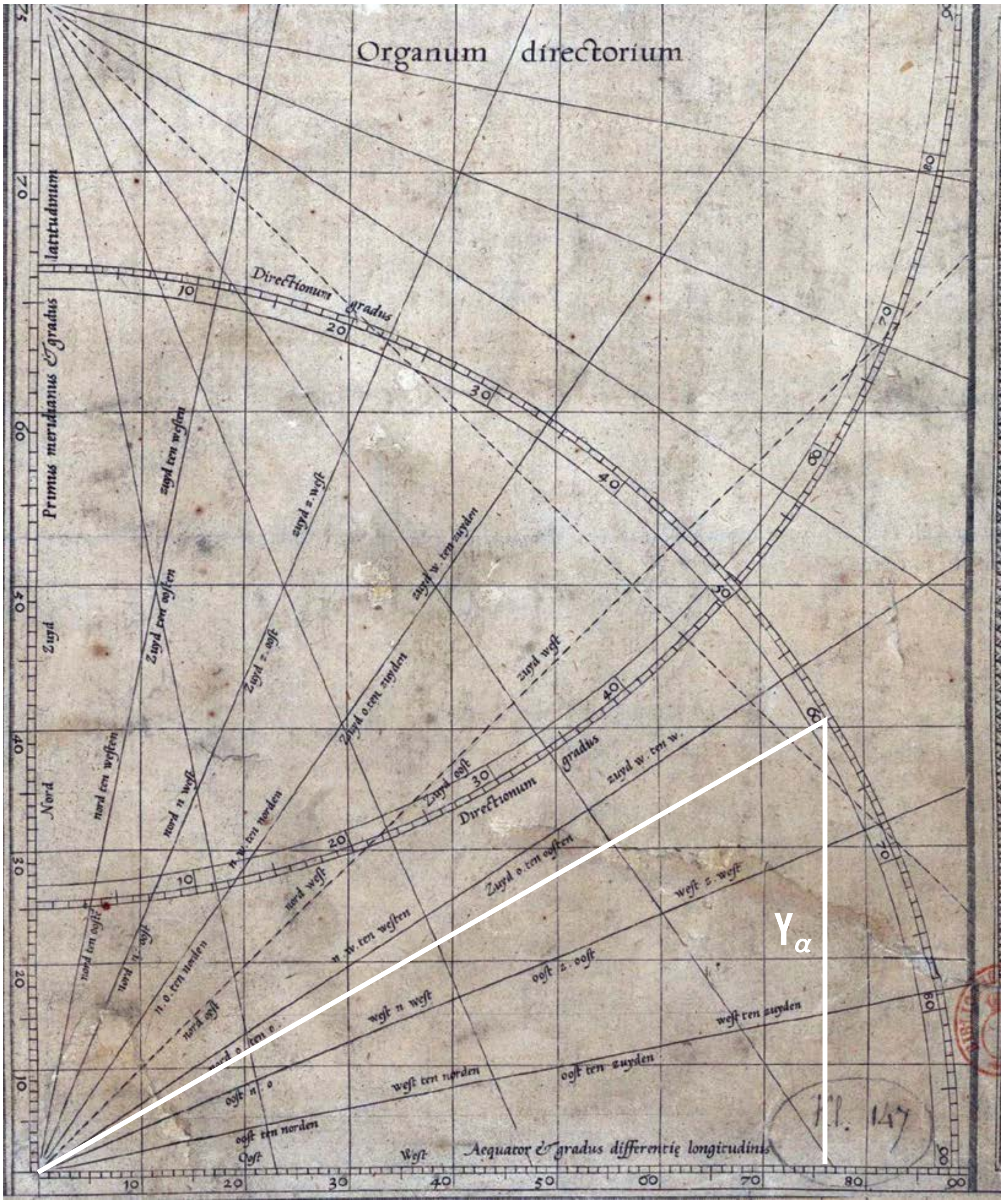

Figure 7.— Organum directorium from Mercator's 1569 map, taken from [1].

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For more information on Pedro Nunes' life and works, the reader can visit the site http://pedronunes.fc.ul.pt, *by Bruno Almeida and Henrique Leitão.*

High Order Finite Elements

by Sílvia Barbeiro*

1 Introduction

The great advances in computational mathematics over the last half century, driven by profound developments in numerical methods along with remarkable progresses in the field of high performance computing, are playing a major role in the scientific and engineering innovation.

Partial differential equations arise in the mathematical modelling of many physical, chemical and biological phenomena in a wide and diverse range of subject areas such as fluid dynamics, electromagnetism, material science, medical imaging. Very frequently is either impossible or impracticable to find closed form solutions to the equations under consideration and it is crucial to obtain numerical approximations to the unknown analytical solution.

When assigned with the task of solving numerically a partial differential equation, the first question one faces is tho choose an adequate method.

The demand for finding accurate numerical models for physical phenomena around complex geometries are making high order methods very attractive for practical applications. Among the possible choices, the discontinuous Galerkin (DG) finite element method, which ensures geometric flexibility and supports high order locally adapted resolution, appears to offer most of the desired properties.

The DG finite element method appeared in the literature back to 1973 in [16], as a proposal to solve the steady-state neutron transport equation. The first convergence analysis results were presented in 1974, in [13] and improved later for example in [12], [14] and [15]. The extension to nonlinear scalar conservation laws was achieved in late 1980's ([4]). Important progresses, namely the development of adaptive solution techniques and the extension to multidimensional cases and to unstructured grids, took place in the next two decades (see *e.g.* [5], [11]). Since the years 2000 there has been an explosion in activities and DG methods become widely used for solving a large range of problems, for example, electromagnetic wave's propagation ([8]), or fluid flow in porous media ([17]).

Being capable of producing highly accurate numerical solutions, DG methods gather many desirable features over the finite differences, finite volume and finite element methods, when used to derive spacial discretizations. The widely used finite differences, on top of being simple, lead to very efficient schemes in many problems. However they are not suitable to handle complex geometries. The finite volume method uses an element based approach and ensures geometric flexibility. Moreover it is locally conservative. The main drawback of the finite volume method is its limitation for achieve high-order accuracy on general unstructured grids. The need to solve geometrically complex large scale problems with higher-order convergence, justifies the huge interest in the flexibility offered by the finite element schemes, which is the natural choice in many problems. However, the basis functions are globally defined and consequently it is not straightforward to deal for instance with hanging nodes. While the mass matrix is sparse and typically well conditioned, finding, for instance, a steady state solution implies to solve a system that involves the global mass matrix. In addition, the finite element method is less natural when compared with finite volume method to deal with conservation laws, where there is a flow in specific directions. Discontinuous Galerkin methods fulfill the need of geometrical flexibility and locally adapted resolution. Some other features include local mass conservation, possible definition on unstructured meshes, *hp*-adaptivity with locally varying polynomial degrees.

There is likewise a wide variety of methods for the integration in time. For example, we can mention the fully explicit leap-frog method ([1]), or the classes of implicit and explicit Runge-Kutta type methods (e.g. [3],[10]), which reflect a method-of-lines approach with the time and space separately discretized. Explicit time-stepping schemes are computationally very effective. Nevertheless, those methods are only conditionally stable. If an explicit time integrator is considered, the maximum time step size allowed is related with the smallest elements of the spatial mesh. Locally refined meshes often obstruct the efficiency for the simula-

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tion of time-dependent phenomena, because of the stringent stability constraint caused by the existence of some small elements in the spatial mesh. This could be the case when the problem involves modelling small structures with complex shapes and consequently a very fine mesh is needed at some spatial locations. As an example we mention the use of Maxwell's equation to model the electromegnetic wave's propagation in the human retina described in [2] and [18]. Simulating the full complexity of the retina, in particular taking into account the variation of the size and shape of each structure, demands the use of a spatial mesh which reflects that level of detail. This is remarkably limitative for the choice of the time step in the case of explicit time-stepping schemes. By taking smaller time-steps precisely where the smallest elements are located, local time-stepping methods ([9]) become a possible approach. Another interesting choice, is to consider locally implicit time-schemes ([6]). Here we highlight another alternative, which is to consider the DG method in time. In contrast to explicit Runge-Kutta methods, the DG in time is unconditionally stable ([7]). This idea suggests the use of DG methods in a space-time approach, giving a framework for high-order accurate methods. In this technique, time is considered as an extra dimension and it is treated in a similar fashion as the spatial coordinates.

The advantages of DG methods for space, time or space-time integration, include their flexibility on the choice of meshes and thus their capacity to handle complicated geometries, their potential for error control and mesh adaptation, their possible definition on unstructured meshes. The possibility of parallel implementation attenuates the major drawbacks which are high memory requirements and computational cost.

In spite of the theoretical developments, which encourage the use of high order finite element methods, the range of polynomial degrees used in finite element computations for practical applications and in commercial codes is usually rather small. In many cases, this fact is due to computational efficiency rather than any theoretical issue. The search of efficient solvers for the linear systems originated from the DG finite element approach is nowadays a trend of utmost importance.

In what follows we will briefly discuss the formulation of the DG finite element method for linear wave problems. We will also summarise the theoretical convergence properties to give an appreciation of what can be expected in terms of accuracy of the schemes.

2 The continuous setting

Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^d$, $d \geq 1$, and let $T > 0$ be a finite time. We consider the following linear evolution problem: find $u : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} + Au &= f \quad \text{in } \Omega \times (0, T], \\ u(., 0) &= u_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_- \times (0, T], \end{aligned} \tag{1}$$

where A is a first-order linear differential operator

$$Au = \beta \cdot \nabla u + \sigma u,$$

$\beta : \Omega \to \mathbb{R}^d$ is a given Lispschitz convection field, $\sigma : \Omega \to \mathbb{R}$ is a bounded reaction term, $f : \Omega \times [0, T] \to \mathbb{R}$ is the source term, $u_0 : \Omega \to \mathbb{R}$ is the initial datum, and Γ_- is the inflow part of the boundary defined as

$$\Gamma_- = \{x \in \Gamma : -\beta(x) \cdot n > 0\},$$

with n denoting the outer normal unit vector to Γ. The outflow boundary, Γ_+, is defined by $\Gamma_+ = \Gamma \backslash \Gamma_-$. We make the following hypothesis on the data

$$\sigma(x) - \frac{1}{2}\text{div}\beta(x) \geq \mu_0 > 0 \quad \forall x \in \Omega.$$

Let us consider the space

$$V = \{v \in L^2(\Omega) : \beta \cdot \nabla v \in L^2(\Omega), v|_{\Gamma_-} = 0\},$$

endowed with the norm

$$\|v\|_V^2 = \mu_0 \|v\|^2_{L^2(\Omega)} + \|\beta \cdot \nabla v\|^2_{L^2(\Omega)}.$$

Assuming that $f \in C^0([0, T], L^2(\Omega))$ and $u_0 \in V$, taking the L^2-inner product, from (1) we obtain the following variational problem: find $u \in C^0([0, T], V) \cap C^1([0, T], L^2(\Omega))$ such that, $\forall v \in L^2(\Omega)$, $\forall t \in (0, T]$,

$$\begin{aligned} &(\frac{\partial u}{\partial t}(t), v)_{L^2(\Omega)} + (Au(t), v)_{L^2(\Omega)} = (f(t), v)_{L^2(\Omega)}, \\ &u(0) = u_0. \end{aligned} \tag{2}$$

Using the relation

$$\begin{aligned} &(\beta \cdot \nabla u + \sigma u, u)_{L^2(\Omega)} = (\sigma - \tfrac{1}{2}\text{div}\beta, u^2)_{L^2(\Omega)} \\ &+ \tfrac{1}{2}((\beta \cdot n)u, u)_{L^2(\Gamma)}, \end{aligned}$$

we can derive the following energy inequality, which expresses the continuous dependence of the solution of (2) on the data,

$$\begin{aligned} &\|u(t)\|^2_{L^2(\Omega)} + \int_0^t e^{t-\tau} \int_{\Gamma_+} (\beta(x) \cdot n(x)) u(x, \tau)^2 \, dx \, d\tau \\ &\leq e^t \|u_0\|^2_{L^2(\Omega)} + \int_0^t e^{t-\tau} \|f(\tau)\|^2_{L^2(\Omega)} \, d\tau, \quad t \in [0, T]. \end{aligned}$$

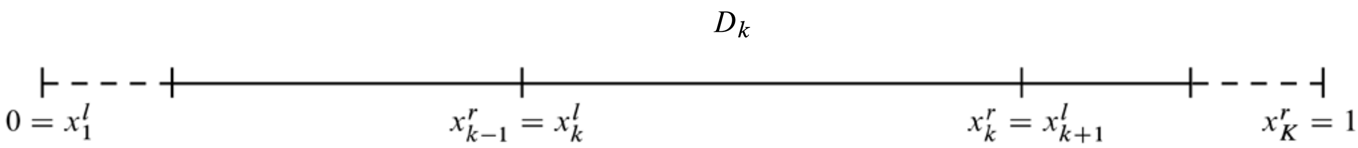

Figure 1.— Partition of the computational domain in one dimension

The proof of the uniqueness of solution follows from the above inequality. Further results on the well-posedness of (2), namely the existence of solution, can be found in [19].

3 The discrete setting

We introduce some key ideas behind the DG finite element method in a simple case, considering the scalar wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in (0,1) = \Omega,\ t \in (0,T], \qquad (3)$$

with $a > 0$, subject to the initial condition $u(x,0) = u_0(x)$ and the inflow boundary condition $u(0,t) = 0$.

Assume that the computational domain Ω is partitioned into K nonoverlapping elements D_k such that $\overline{\Omega} = \cup_k D_k$, as illustrated in the Figure 1. On each element D_k, the solution is approximated by polynomials of degree less than or equal to $N = N_p - 1$,

$$\tilde{u}_k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\varphi_n(x),$$

where φ_n, $n = 1,\dots,N_p$, form the local polynomial basis. The global solution $u(x,t)$ is then assumed to be approximated by the piecewise N order polynomials defined as the direct sum of the K local polynomial solutions

$$u(x,t) \simeq \tilde{u}(x,t) = \bigoplus_{k=1}^{K} \tilde{u}_k(x,t).$$

In order to deduce the method, we start by multiplying equation (3) by test functions φ_n. Spatial integration by parts over each element D_k yields

$$\int_{D_k}\left(\frac{\partial \tilde{u}_k}{\partial t}\varphi_n - a\tilde{u}_k\frac{\partial \varphi_n}{\partial x}\right)dx = -\left[a\tilde{u}_k\varphi_n\right]_{x_k^l}^{x_k^r}$$
$$= -\int_{\partial D_k} n\cdot a\tilde{u}_k\varphi_n\,dx, \quad 1 \le n \le N_p,$$

where n represents the local outward pointing normal. The next step is to substitute in the resulting contour integral the flux by a numerical flux $(a\tilde{u})^*$, which will be specified later. Reversing the integration by parts yields

$$\int_{D_k}\left(\frac{\partial \tilde{u}_k}{\partial t}\varphi_n + a\frac{\partial \tilde{u}_k}{\partial x}\varphi_n\right)dx$$
$$= \int_{\partial D_k} n\cdot\left(a\tilde{u}_k - (a\tilde{u})^*\right)\varphi_n\,dx, \quad 1 \le n \le N_p.$$

The approximate solution is allowed to be discontinuous across elements boundaries. In this way, we introduce the notation of average $\{\!\{\tilde{u}\}\!\} = \frac{\tilde{u}^- + \tilde{u}^+}{2}$ and of the jumps of the solution values across the interfaces of the elements, $[\tilde{u}] = \tilde{u}^- - \tilde{u}^+$, where the superscript "+" denotes the neighbouring element and the superscript "−" refers to the local element. The coupling between elements is introduced via the numerical flux

$$(a\tilde{u})^* = \{\!\{a\tilde{u}\}\!\} + a\frac{1-\alpha}{2}n\cdot[\tilde{u}], \quad 0 \le \alpha \le 1.$$

If $\alpha = 1$ the numerical flux is called central flux being the average of two solutions. The case $\alpha = 0$, corresponds to the upwind flux which takes into account the direction of the flux.

Figure 2 (see next page) shows the computed solution of equation (3), considering $a = 2$, $u_0(x) = \sin(\pi x)$, at time $t = 0.1$, obtained by means of the DG method with upwind flux, for different values of N and K.

The flexibility of DG methods allows us to easily change basis functions. For instance, we could use Lagrange polynomials or other polynomials satisfying a desired orthogonality property. One possible choice is to consider the orthonormal basis

$$\varphi_j(r) = \frac{P_j(r)}{\sqrt{\gamma_j}},$$

where P_j are the Legendre polynomials of order j and $\gamma_j = \frac{2}{2j+1}$. This basis can be computed through the recurrence

$$a_{j+1}\varphi_{j+1}(r) = r\varphi_j(r) - a_j\varphi_{j-1}(r),$$

$$a_j = \sqrt{\frac{j^2}{(2j+1)(2j-1)}},$$

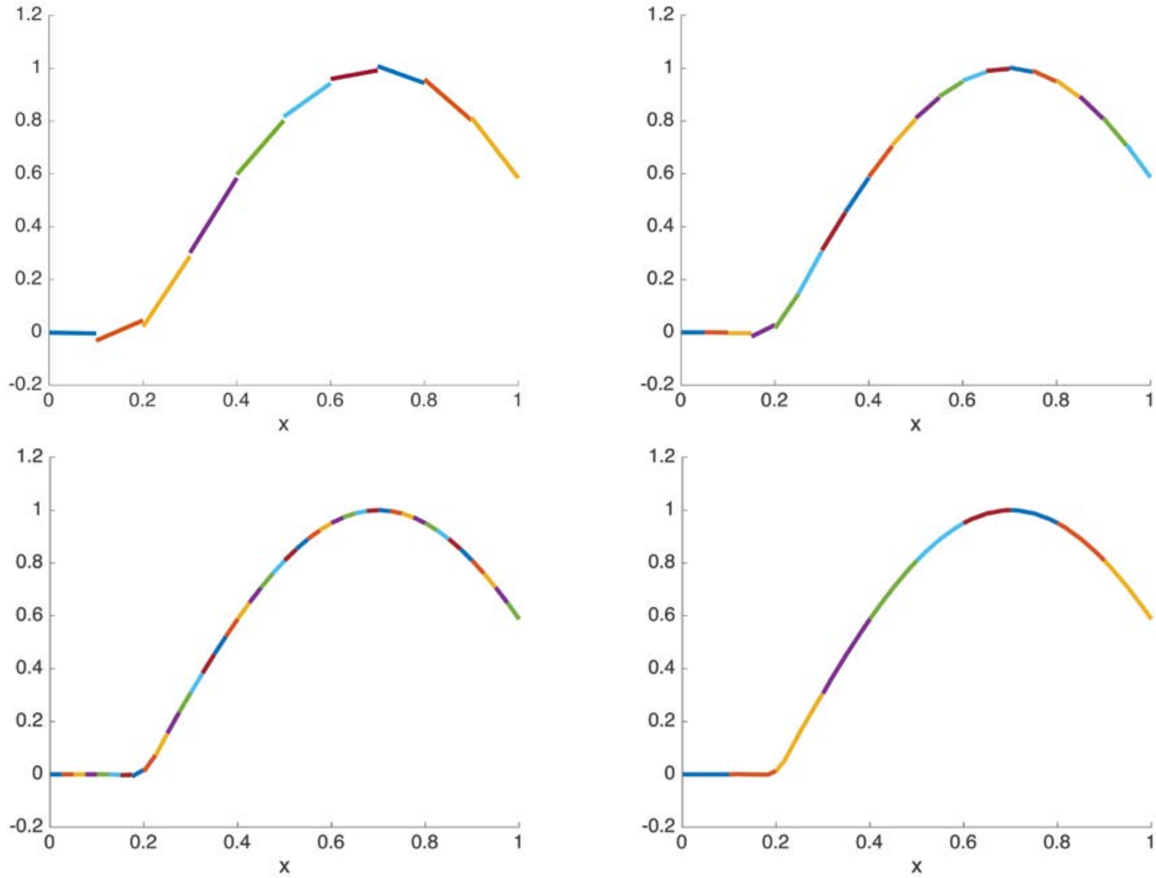

Figure 2.— Numerical solution of the wave equation. Top left: *N*=1, *K*=10. Top right: *N*=1, *K*=20. Bottom left: *N*=1, *K*=40. Bottom right: *N*=4, *K*=10

with $\varphi_0(r) = \frac{1}{\sqrt{2}}$, $\quad \varphi_1(r) = \sqrt{\frac{3}{2}}r$. The affine mapping

$$x(r) = x_k^l + \frac{1+r}{2}(x_k^r - x_k^l),$$

relates $x \in D_k$ with the reference variable $r \in [-1, 1]$.

We now go back to the more general problem (1) in two or three space dimensions. We will present the discrete setting in both time and space based on DG in time-space discretizations. We will also present a result for the error analysis.

4 Semi-discretization in time

We start by decomposing the time interval $I = (0, T]$ into disjoint subintervals $I_n = (t_{n-1}, t_n]$, where $n = 1, \ldots, N$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. We use the notation $\tau_n = t_n - t_{n-1}$.

The approximate solution is a piecewise polynomial with respect to time, locally defined on the space

$$P_k(I_n, V) = \{w : I_n \to V, w(t) = \sum_{j=0}^{k} W^j t^j, \forall t \in I_n, W^j \in V, \quad \forall j\}.$$

The space $P_k(I_n, L^2(\Omega))$ is defined analogously, with V replaced by $L^2(\Omega)$. The jump of w_τ at t_n is defined as

$$[w_\tau]_n = w_\tau(t_n^+) - w_\tau(t_n),$$

where $w_\tau(t_n^+) = \lim_{t \to t_n^+} w_\tau(t)$. Using the known value $u_\tau(t_{n-1})$ from the previous time interval and u_0 for $n = 1$, the local problem on I_n reads: find $u_\tau|_{I_n} \in P_k(I_n, V)$ such that

$$\int_{I_n} (\frac{\partial u_\tau}{\partial t} + Au_\tau, v_\tau)_{L^2(\Omega)}\, dt + \left([u_\tau]_{n-1}, v_\tau(t_{n-1}^+)\right)_{L^2(\Omega)} = Q_n\left((f, v_\tau)_{L^2(\Omega)}\right),$$

$\forall v_\tau \in P_k(I_n, L^2(\Omega))$. The right-hand side is evaluated by means of some numerical integration formula

$$Q_n\left((f, v_\tau)_{L^2(\Omega)}\right) \simeq \int_{I_n} (f, v_\tau)_{L^2(\Omega)}\, dt.$$

5 Space discretization

Let $\mathcal{T}_h$ be a shape-regular mesh of Ω which is assumed to have a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary. By h we denote the mesh diameter. Let V_h be the space of piecewise polynomials of order less or equal to r. The mesh

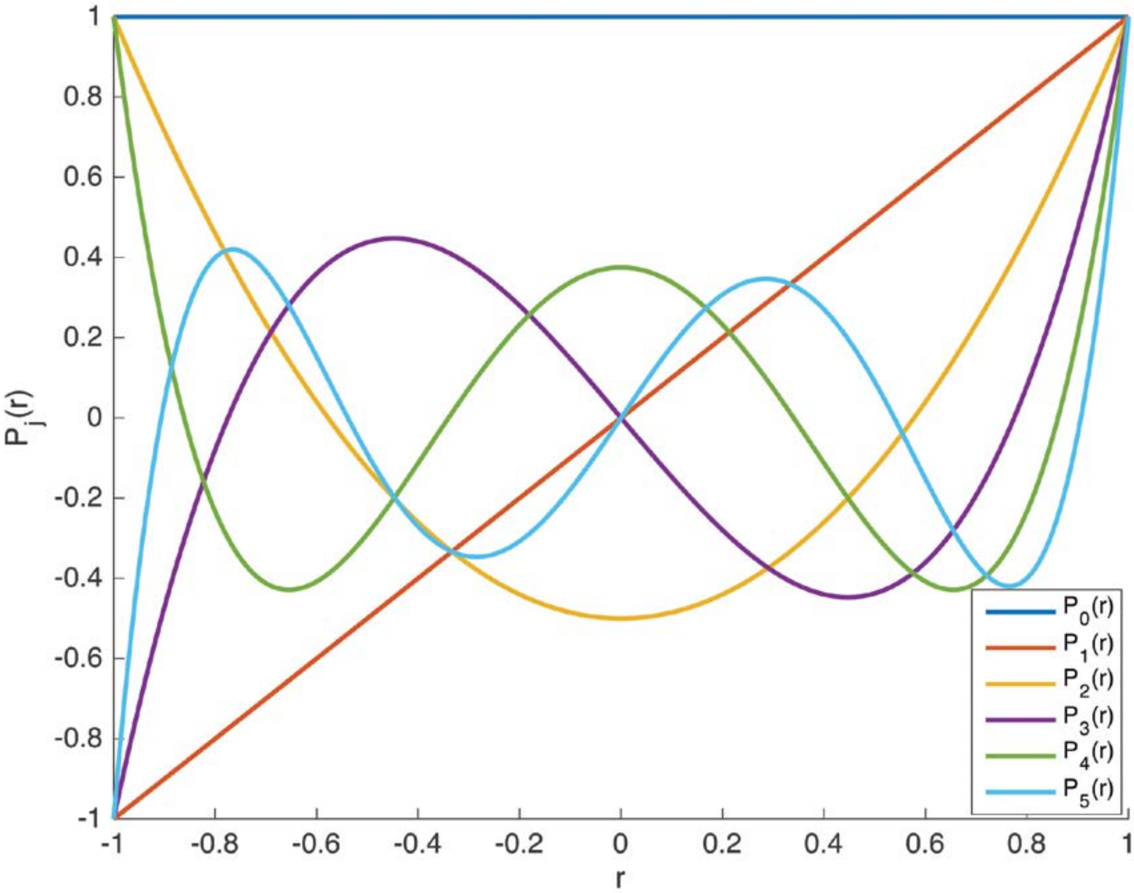

Figure 3.— Legendre polynomials

edges or faces (cases $d = 2$ and $d = 3$, respectively) are collected in the set $\mathcal{F}_h$, split into the set of the ones belonging to the interior, $\mathcal{F}_h^{int}$, and boundary, $\mathcal{F}_h^{ext}$.

The discrete operator which defines the DG method in time, A_h, defined for all $v \in H^1(\Omega) \cup V_h$ and $w_h \in V_h$, is given by

$$\begin{aligned}(A_h v, w_h)_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h} (\sigma v + \beta \cdot \nabla v, w_h)_{L^2(T)} \\ &+ \sum_{F\in\mathcal{F}_h^{ext,inflow}} \left((\beta \cdot n)v, w_h\right)_{L^2(F)} \\ &- \sum_{F\in\mathcal{F}_h^{int}} \left((\beta \cdot n)[v], \{\!\{w_h\}\!\}\right)_{L^2(F)} \\ &+ \sum_{F\in\mathcal{F}_h^{int}} \left(\frac{1}{2}|\beta \cdot n|[v], [w_h]\right)_{L^2(F)}.\end{aligned}$$

This operator verifies the following important properties.

- Consistency: Let $P_h : L^2(\Omega) \to V_h$ be the L^2-orthogonal projector onto V_h. Then

$$A_h w = P_h A w, \quad \forall w \in H^1(\Omega).$$

- Discrete coercivity: Let us consider the mesh-dependent norm

$$\begin{aligned}|||v_h|||^2 = \mu_0 \|v\|^2_{L^2(\Omega)} &+ \sum_{F\in\mathcal{F}_h^{ext}} \||\beta \cdot n|^{1/2} v\|^2_{L^2(F)} \\ &+ \frac{1}{2} \sum_{F\in\mathcal{F}_h^{int}} \||\beta \cdot n|^{1/2}[v]\|^2_{L^2(F)}.\end{aligned}$$

Then $\exists C > 0$ such that

$$C|||v_h|||^2 \leq (A_h v_h, v_h)_{L^2(\Omega)},$$

$\forall v_h \in V_h$.

6 Full space-time discretization

Putting all together, we now derive the fully discrete method.

We consider the finite element space V_h^n resulting from the mesh $\mathcal{T}_h^n$ which can change from one time interval to the next. The local problem in I_n reads: find $u_{\tau h}|_{I_n} \in P_k(I_n, V_h^n)$ such that, for all $v_{\tau h} \in P_k(I_n, V_h^n)$,

$$\begin{aligned}\int_{I_n} &(\frac{\partial u_{\tau h}}{\partial t} + A_h u_{\tau h}, v_{\tau h})_{L^2(\Omega)}\, dt \\ &+ \left([u_{\tau h}]_{n-1}, v_{\tau h}(t_{n-1}^+)\right)_{L^2(\Omega)} \\ &= Q_n\left((f, v_{\tau h})_{L^2(\Omega)}\right).\end{aligned}$$

This method, which was analysed in [7], is unconditionally stable and convergent. The error bound in the following result shows that the method is of arbitrary high order in time and in space.

THEOREM 1.— Let u be the exact solution of (2), which is assumed to be enough regular, and let $u_{\tau h}$ be the fully discrete solution of the DG method. Assume that $k \geq 1$ and $\tau_n \leq 1$, for all $n = 1, \ldots, N$. Then the following error bound holds for all $m = 1, \ldots, N$,

$$\|u(t_m) - u_{\tau h}(t_m)\|^2_{L^2(\Omega)} \leq C\left((E_0)^2 + t_m \max_{1\leq n\leq m}\left\{C_n^T(u)\tau_n^{2(k+2)} + C_n^S(u)h^{2r+1}\right\} + C'_m(u)h^{2(r+1)}\right),$$

with $E_0 = \|P_h u(0) - u_{\tau h}(0)\|_{L^2(\Omega)}$,

$$C_n^T(u) = |u|^2_{C^{k+3}(\bar{I}_n, L^2(\Omega))} + |u|^2_{C^{k+2}(\bar{I}_n, V)},$$
$$C_n^S(u) = \|u\|^2_{C^1(\bar{I}_n, H^{r+1}(\Omega))}$$

and,

$$C'_m(u) = |u(t_m)|^2_{H^{r+1}(\Omega)}.$$

The error bound point out not only the influence of the mesh size but also the dependence on the choice of the degree of the polynomials used in the construction of the finite element space, making possible to balance accuracy and computational efficiency.

7 Outlook

The demand for modelling intricate systems often involving multiscales and multiphysics around complex geometries has been a source of motivation for great progress in the field of computational mathematics. High order methods for solving partial differential equations, such as finite element methods or spectral methods, are attractive due to the need of great accuracy on realistic models. Nevertheless a number of challenges still exist not only in the development of new mathematical tools but also in translating academic progresses into engineering practice.

There is a truly need of a formulation and analysis of new multiscale, multiphysics, scalable, parallel efficient methods for treating multiple time and spatial scales that arise in modelling complex phenomena. The arising of new methods demands developments in their analysis and investigators are engaged to seek results on the well-posedness of the models, *a priori* and *a posteriori* error estimators, stability and convergence aspects. Another important issue to address is reliability of computer predictions due to uncertainty. Physical phenomena can rarely be modelled with complete fidelity even under the best of circumstances, even though they often support life-and-death decisions in different fields. The uncertainty may occur in all phases of the predictive process, from model selection and choice of the parameters to the observation data. Mathematicians are driven forward to investigate uncertainty quantification and error estimators.

In the particular topic of the present article, there are still important questions to be addressed. First, the investigation of the theoretical aspects of the DG time-stepping method, as the convergence properties, is far from being closed. The existent literature does not encompasses all models. The introduction of nonlinearities or the change of the boundary conditions, often needed to model real applications, entail subtleties and often the analysis is not straightforward from the existent results. Another challenge appears when applying the DG time-stepping method in practice and we are faced with the task of solving big linear systems at each time-step possible defined by matrixes with large condition numbers. The drawback in the computational cost can be tamed using efficient solvers. There has been a great interest in investigating strategies like multigrid methods, domain decomposition methods and to develop robust and efficient preconditioners. An additional aspect which deserves attention is how to deal efficiently with the quadrature rules, which involve sums on the quadrature points, in the case of high order methods.

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The Willmore Conjecture: a Celebration of Mathematics

by Áurea Quintino*

Cum enim Mundi universi fabrica sit perfectissima, atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat[1]
— Leonhard Euler

1 Introduction

Established in 1961, the Oswald Veblen Prize in Geometry is an award granted by the American Mathematical Society in recognition of a notable research memoir in geometry and topology. Presented every three years, this year's edition of the prize distinguished the joint work of the Brazilian mathematician Fernando Codá Marques (Princeton University) and the Portuguese mathematician André Neves (Imperial College London), for their landmark achievement and major contribution to the use of variational methods in differential geometry, with a special highlight for the proof of the long-standing Willmore Conjecture.

Proposed in 1965 by the English geometer Thomas J. Willmore, the Willmore Conjecture concerned the quest for the torus with the lowest bending energy of all and predicted the equilibrium state of such curved surfaces. The problem has resisted proof for many years and inspired many mathematicians over time, borrowing ideas from several distinct areas from partial differential equations to algebraic geometry, conformal geometry, geometric measure theory and minimal surfaces. Willmore died on February 20, 2005, seven years before Marques and Neves posted a preprint of their 96-page proof on the arXiv, on February 27, 2012.

This article is dedicated to an overview of the history of the conjecture and its proof, in celebration of pursuit and achievement, through the works of Willmore, of Marques and Neves and of all those involved in this half century quest, as well as those of their precursors.

2 Willmore energy and the Willmore Conjecture

A central theme in Mathematics is the search for the optimal representative within a certain class of objects, often driven by the minimization of some energy, reflecting what occurs in many physical processes. From the early 1960s, Thomas Willmore devoted particular attention to the quest for the optimal immersion of a compact surface in Euclidean 3-space regarding the minimization of some natural energy motivated by questions on the elasticity of membranes and the energetic cost associated with membrane bending deformations.

We can characterize how much a membrane is bent at a particular point on the membrane by means of the curvature of the osculating circles of the planar curves obtained as perpendicular cross sections through the point (see Figure 2). The curvature of these circles consists of the inverse of their radii, with a positive or negative sign depending on whether the membrane curves upwards or downwards, respectively. The minimal and maximal values of the radii of the osculating circles associated with a particular point on the membrane define the principal curvatures, k_1 and k_2, and, from these, the mean curvature, $H = (k_1 + k_2)/2$, and the Gaussian curvature, $K = k_1 k_2$, at the point.

In modern literature on the elasticity of membranes (see, for example, [11] and [31]), a weighed sum $a \int H^2 + b \int K$, of the total squared mean curvature and the total Gaussian curvature, is considered as the elastic bending energy of a membrane. Having in consideration the Gauss-

[1] Leonhard Euler, *Methodus inveniendi lineas curvas Maximi Minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti*, Lausannae & Genevae: Apud Marcum-Michaelem Bousquet & Socios (1744), p. 245.

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Figure 1.—Thomas J. Willmore. Portrait by Christine Choa (1999)

Bonnet theorem, according to which the total Gaussian curvature is a topological invariant and, therefore, negligible in deformations conserving the topological type, Willmore defined

$$\mathscr{W} = \int_\Sigma H^2 d\Sigma$$

as the Willmore [bending] energy of a compact, oriented [Riemannian] surface Σ [isometrically] immersed in $\mathbb{R}^3$.

The Willmore energy had already made its appearance early in the nineteenth century, through the works of Marie-Sophie Germain [6] and Siméon Denis Poisson [22] and their pioneering studies on elasticity and vibrating properties of thin plates. This energy had also appeared in the 1920s, in the works of Wilhelm Blaschke [4] and Gerhard Thomsen [27], but their findings were forgotten and only brought to light after the increased interest on the subject motivated by the work of Thomas Willmore.

A very interesting fact about the Willmore energy is that it is scale-invariant: if one dilates the surface by any factor, the Willmore energy remains the same. Think of a round sphere in $\mathbb{R}^3$ as an example: if one increases the radius, the surface becomes flatter and its squared mean curvature H^2 decreases, but, at the same time, its area gets larger, which increases the value of the integral in $\mathscr{W}$. One can show that these two phenomena counterbalance each other on any surface. In fact, the Willmore energy has the remarkable property of being invariant under any conformal transformation of $\mathbb{R}^3$, as established in the paper of White [32] and, actually, already known to Blaschke and Thomsen.

In view of the scale-invariance of the Willmore energy, the energy of round spheres coincides with the surface area of the round sphere of radius 1: 4π. Note, on the other hand, that $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$, so that $H^2 \geq K$, with equality at umbilical points ($k_1 = k_2$). By the Gauss-Bonnet theorem, it follows that

$$\int_\Sigma H^2 d\Sigma \geq \int_\Sigma K d\Sigma = 4\pi(1-g),$$

where g denotes the genus of the surface. In particular, for surfaces of genus zero, we get $\int_\Sigma H^2 d\Sigma \geq 4\pi$, with equality only for the totally umbilical surfaces of $\mathbb{R}^3$. We conclude that round spheres are the minimizers of the Willmore energy among all topological spheres. Willmore showed, furthermore, that 4π is the absolute minimum of energy among all compact surfaces in $\mathbb{R}^3$:

Theorem 1 (Willmore [33, 35]).— Let Σ be a compact surface in $\mathbb{R}^3$. Then

$$\mathscr{W}(\Sigma) \geq 4\pi,$$

with equality if and only if Σ is a round sphere.

Having found the compact surfaces with least possible energy, and, with these, the energy minimizers within the class of surfaces of genus zero, Willmore embarked on the quest for the energy-minimizing shape among all topological tori. It seems reasonable that no obvious candidate stands out a priori. In order to develop some intuition on the problem, Willmore considered a particular type of torus: he fixed a circle of radius R on a plane and considered tubes Σ_r of constant radius $r < R$ around that circle. When r is very small, Σ_r is a very thin tube and so $\mathscr{W}(\Sigma_r)$ is very large. As we keep increasing the value of r, the hole of the torus decreases and eventually disappears, for $r = R$. Thus the function $r \mapsto \mathscr{W}(\Sigma_r)$ must reach an absolute minimum for some $r \in]0, R[$. Willmore [33] computed this minimum to be $2\pi^2$ and showed that, up to scaling, the optimal torus in this class has $r = 1$ and $R = \sqrt{2}$. Willmore conjectured

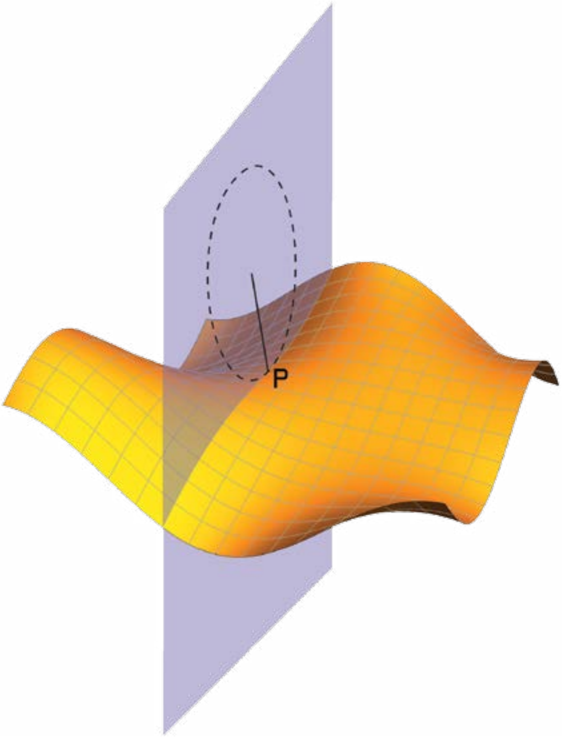

Figure 2.—Osculating circle to a surface at a point *P* on the surface

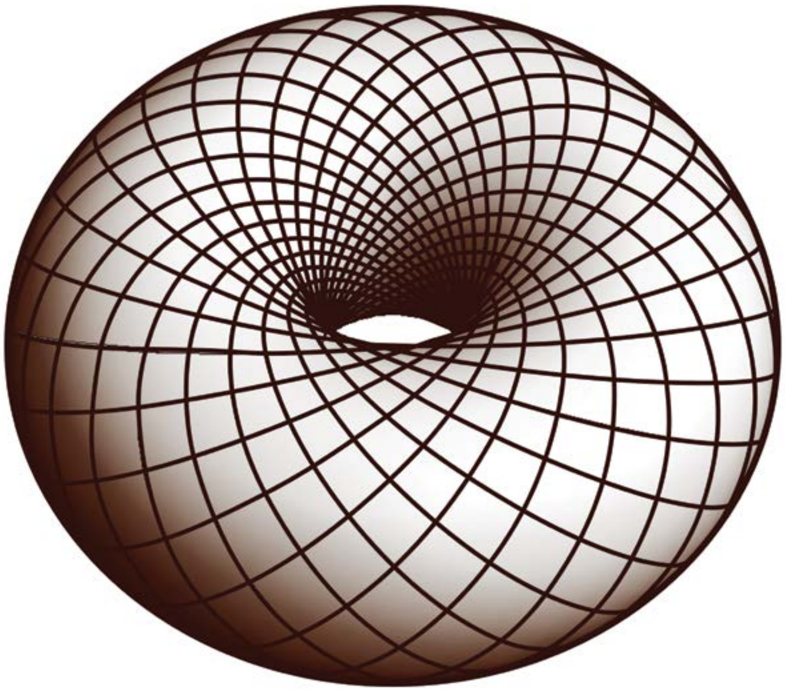

Figure 3 .—Torus of revolution with $W=2\pi^2$

that this torus of revolution should minimize the Willmore energy among all tori:

Willmore Conjecture

(Willmore [33]) *Let Σ be a compact surface of genus one in $\mathbb{R}^3$. Then*

$$\mathscr{W}(\Sigma) \geq 2\pi^2.$$

3 On the quest for the optimal torus

The Willmore Conjecture has been verified in many special cases. Willmore himself [34] and, independently, Katsuhiro Shiohama and Ryoichi Takagi [25] proved it when the torus is a tube of constant radius around an arbitrary space curve in $\mathbb{R}^3$. Over the decades, more and more classes of tori were proven to have bending energy greater than or equal to $2\pi^2$, through the works of Rémi Langevin and Harold Rosenberg [9], Bang-Yen Chen [5], Joel Langer and David Singer [8], Peter Li and Shing-Tung Yau [10], Sebastián Montiel and Antonio Ros [18, 23, 24] and Peter Topping [28, 29]. In 1991, the biophysicists David Bensimon and Michael Mutz [3] have experimentally verified the conjecture in membranes of toroidal vesicles produced in laboratory. In 1993, Leon Simon [26] established the existence of a torus that minimizes the Willmore energy. An overview of partial results can be found in [14]. We select the following, which, in particular, reduced the verification of the Willmore Conjecture to embedded tori:

Theorem 2 (Li-Yau [10]).— Compact surfaces with self-intersections have Willmore energy greater than or equal to 8π.

A key to the proof of the Willmore Conjecture was moving the problem from $\mathbb{R}^3$ to the unit 3-sphere $S^3 \subset \mathbb{R}^4$, having in mind that the two are conformally related by stereographic projection. The torus found by Willmore is mapped onto the Clifford torus $S^1\left(\frac{1}{\sqrt{2}}\right)\times S^1\left(\frac{1}{\sqrt{2}}\right)$, which is a classical example of a minimal surface in S^3.

Minimal surfaces are defined variationally as the stationary configurations for the area functional, surfaces that locally minimize the area. In general, these surfaces admit ambient deformations that can decrease their area and are, therefore, not (globally) area-minimizing.

Minimal surfaces were first considered by Joseph-Louis Lagrange [7], in 1762, who raised the question of existence of surfaces of least area among all those spanning a given closed curve in Euclidean 3-space as the boundary. Earlier, in a work published in 1744,[2] Leonhard Euler had already discussed minimizing properties of the surface now known as the catenoid, although he only considered variations within a certain class of surfaces. The problem raised

[2] Ibidem.

by Lagrange became known as the Plateau's Problem, referring to Joseph Antoine Ferdinand Plateau, who first experimented with soap films [21].

A physical model of a minimal surface can be obtained by dipping a wire frame into a soap solution. The resulting soap film is minimal in the sense that it always tries to organize itself so that its surface area is as small as possible whilst spanning the wire contour. This minimal surface area is, naturally, reached for the flat position,[3] which happens to be a position of vanishing mean curvature. This does not come as a particular feature of this rather simple example of minimal surface. In fact, the Euler-Lagrange equation of the variational problem underlying minimal surfaces turns out to be precisely the zero mean curvature equation, as discovered by Jean Baptiste Meusnier [17].

With the characterization of minimal surfaces by identically vanishing mean curvature, the theory of minimal submanifolds has been developed and extended to other ambient geometries and ended up playing a crucial role in the understanding of the Willmore energy.

On the sphere, the Willmore energy becomes area plus the total squared mean curvature: if $\pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ denotes the stereographic projection, then

$$\int_\Sigma H^2 d\Sigma = \int_{\tilde{\Sigma}} (1 + \tilde{H}^2) d\tilde{\Sigma},$$

for $\tilde{H}$ the mean curvature of $\tilde{\Sigma} := \pi^{-1}(\Sigma) \subset S^3$ (with respect to the standard metric on S^3). In particular, the Willmore energy of a minimal surface in S^3 coincides with its area.

Crucially, Marques and Neves reduced the quest for an optimal embedding in S^3 to the class of minimal embeddings in S^3:

Theorem 3 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded closed surface with positive genus. Then there exists an embedded closed minimal surface $\tilde{\Sigma} \subset S^3$ such that $\mathcal{W}(\Sigma) \geq \text{area}(\tilde{\Sigma})$.

Next they established the Clifford torus as a surface of least area among all minimal embeddings of closed surfaces in S^3 with genus (at least) one:

Theorem 4 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded closed minimal surface with positive genus. Then $\text{area}(\Sigma) \geq 2\pi^2$, and equality holds if and only if Σ is the Clifford torus, up to isometries of S^3.

With Theorems 3 and 4, Marques and Neves established, in particular, the following:

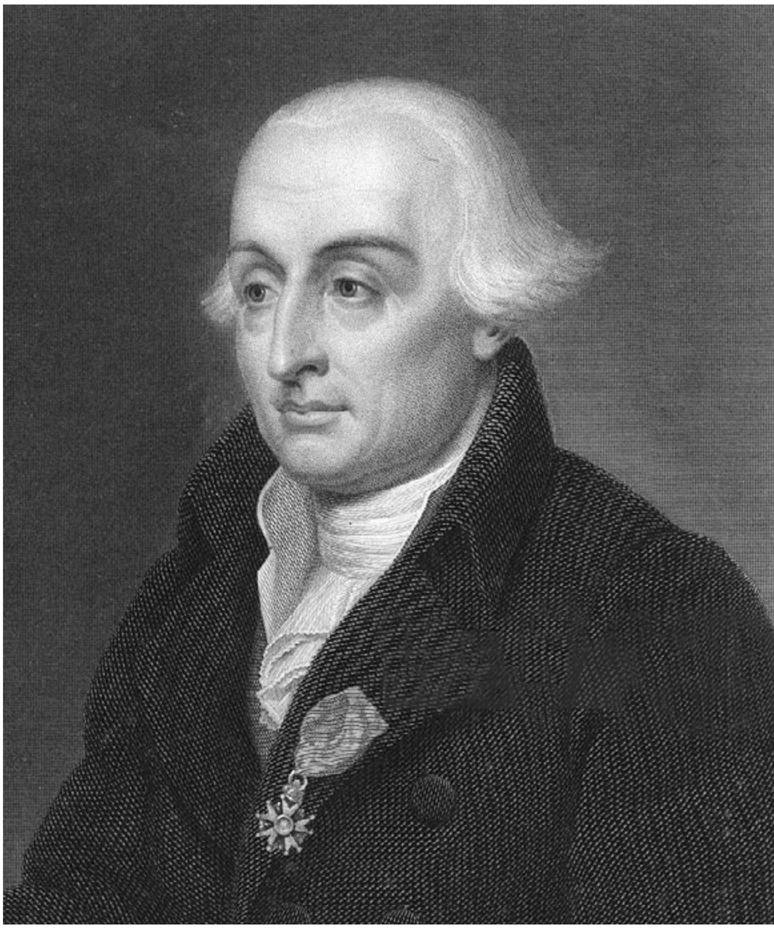

Figure 4.—Joseph Louis Lagrange. Engraving by Robert Hart (ca. 1834-1837), from a bust in the Library of the Institute of France

Theorem 5 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded compact surface with positive genus. Then $\mathcal{W}(\Sigma) \geq 2\pi^2$, and the equality holds if and only if Σ is the Clifford torus, up to conformal transformations of S^3.

With this, and in the light of Theorem 2, Fernando Codá Marques and André Neves have proved the Willmore Conjecture:

Corollary 6.— The Willmore Conjecture holds.

The milestone step achieved in Theorem 3 comes as an aplication of the Min-max Theory developed by Frederick Almgren [1] and Jon Pitts [20]. Driven by the problem of existence of minimal submanifolds of dimension higher than 2, Almgren introduced the notion of *varifold* and developed a general scheme to produce minimal manifolds in Riemannian manifolds. The question of regularity of these objects was later treated by Pitts, in the case of codimension one. Their combined works established, remarkably, the existence of an embedded, closed minimal hypersurface for any

[3] This is also the position in which the membrane is the most relaxed. In fact, minimal surfaces are examples of *Willmore surfaces*, surfaces that satisfy the equation $\Delta H + 2(H^2 - K)H = 0$ which, in the particular case of compact surfaces, characterizes the stationary configurations for the Willmore functional (see, for example, [35]).
Unlike flat soap films, soap bubbles exist under a certain surface tension, in an equilibrium where slightly greater pressure

Figure 5.—Fernando Codá Marques

Figure 6.—André Neves

given n-dimensional compact Riemannian manifold, with $3 \leq n \leq 6$, cf. [20].

As with many groundbreaking results in Mathematics, the work of Marques and Neves has provided new insights and suggested new approaches to other significant questions. Their contribution includes, in particular, two sequels of a similar spirit, namely, the proof of the Freedman-He-Wang conjecture for links [2], jointly with Ian Agol, and the proof of Yau's conjecture on the existence of infinitely many minimal hypersurfaces in manifolds of positive Ricci curvature [16] (see also [12, 13, 15, 19]).

4 The recipients of the 2016 Oswald Veblen Prize in Geometry

Fernando Codá Marques was born in São Carlos, Brazil, in 1979. He received a BS from the Federal University of Alagoas and an MS from IMPA, both in 1999, and his PhD from Cornell University in 2003. He became a Professor at IMPA in 2010 and, four years later, a Professor at Princeton University. In 2012, he was distinguished with the TWAS (The World Academy of Sciences for the advancement of science in developing countries) Prize in Mathematics, the Ramanujan Prize and the UMALCA (Unión Matemática de América Latina y el Caribe) Prize.

André Neves was born in Lisbon, Portugal, in 1975. He received his first degree from Instituto Superior Técnico in 1999 and his PhD from Stanford University in 2005. He has held positions at Princeton University from 2005 to 2009, the year he moved to Imperial College London, where he became a Professor in 2013. He received the Philip Leverhulme Prize in 2012, the LMS Whitehead Prize in 2013, the Royal Society Wolfson Merit Award in 2015 and the New Horizons Prize in Mathematics, also in 2015.

inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are area-minimizing surfaces under the constraint of volume enclosed. These are surfaces of (non-zero) constant mean curvature and, therefore, examples of *constrained Willmore surfaces*, the generalization of Willmore surfaces that arises when we restrict to infinitesimally conformal variations (for more details, see, for example, [30]).

Figure 7.— Leonhard Euler. Portrait by Jakob Emanuel Handmann (1753)

Fernando Codá Marques and André Neves were awarded the 2016 Oswald Veblen Prize in Geometry at the 122nd Annual Meeting of the American Mathematical Society in Seattle, Washington, on January 7, 2016.

"It is an honor and an immense pleasure to be a recipient, together with my friend André, of the prestigious Oswald Veblen Prize in Geometry.

I am thankful to the committee for this recognition of our work. I am grateful to my family, especially my parents, Severino and Dilze, my wife Ana, and my siblings Gustavo and Clarissa. I am sure that without their love and support I would not be here today. I also look forward to meeting my baby son, Pedro, who is joining us.

I thank also my late advisor, José Fernando Escobar (Chepe), who was always kind and supportive of me, and Richard Schoen, whose influence has been fundamental in my career. The year I spent with Rick was decisive and helped shape my vision of what is important in mathematics. I thank all my teachers, especially Professor Manfredo do Carmo. His lessons inspired me to choose the beautiful field of geometry. I am also grateful to Harold Rosenberg for the many mathematical discussions and to my students, who provide further motivation in my life. The collaboration and friendship with André has been a constant source of joy to me over the last ten years.

The study of minimal varieties is an old subject that began with the work of Lagrange on the foundations of the calculus of variations. The solution of the Plateau problem for mappings of the disk (Douglas and Rado, 1930) and for rectifiable currents (Federer and Fleming, 1960) are milestones of the field. But the question of existence of closed minimal varieties in general compact Riemannian manifolds is not a problem of minimization. This inspired Almgren (1965) to develop a deep min-max theory for the area functional. His work was improved by his PhD student J. Pitts (1981), but remained largely untouched until the last few years.

André and I were extremely delighted when we discovered that this old theory would play a major role in the solution of the Willmore conjecture. This required a change of perspective: instead of trying to minimize the conformally invariant Willmore functional, as originally proposed, we used conformal transformations to convert the problem into a question of minimizing the *maximum of the area of certain five-parameter families of surfaces in the three-sphere. Our work was done mainly while we were both visiting Stanford University at the end of 2011, and the main breakthrough came when we realized how to prove such families are topologically nontrivial. We were very amazed. A few months later we wrote a paper with Ian Agol in which we used similar ideas to solve a conjecture of Freedman, He, and Wang on the Moebius energy of links. Then we turned our attention to the general min-max theory and used it to prove Yau's conjecture about the existence of infinitely many minimal hypersurfaces in the positive Ricci curvature setting. The ideas of Gromov and a paper of Guth on multiparameter sweepouts were very influential. There have been several articles on min-max theory recently, especially by young people, and this makes us very happy. Major questions remain open, such as understanding the index, topology, and multiplicity of these minimal varieties. We hope to contribute further to the field."*[4] (Fernando Codá Marques)

"It is a great honor to receive the Oswald Veblen Prize in Geometry along with my dear friend Fernando.

Working and developing min-max theory together with Fernando has been a tremendous experience: it started with an academic interest in conformal deformations of surfaces, but soon we realized that we were discovering some new rich topology in the space of all surfaces. Coupling that with principles of Morse theory and ideas from minimal surfaces theory, we were able to answer some long-standing open questions in geometry. Since its beginnings, variational methods have had great influence in geometry, and I am delighted that our work made some contributions on that front. This is a beautiful subject, and I hope that its contributions will keep increasing for many years to come.

I consider myself very fortunate to have had Richard Schoen — one of the pioneers of geometric analysis — as my PhD advisor. His mathematical work and sharp intuition have been a towering influence on my research. I would also like to thank my collaborators and friends, from whom I have undoubtedly learned a lot, and my colleague Sir Simon Donaldson for all his support and encouragement throughout my career.

Finally, none of this would have been possible without the constant love and unyielding support of my parents Nelsa and Custódio, my wife Filipa, and our two adorable children, Eva and Tomás. In one way or another, they have all made sacrifices for the pursuit of my career."[5] (André Neves)

Acknowledgments

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Topological Methods in Dynamics with Applications in Ergodic Theory

Old and New

Paulo Varandas*

1 A brief introduction

After the notion of uniform hyperbolicity was coined in the seventies by Smale [26], it became the paradigm of chaotic dynamics. If, on the one hand, the local dynamics of a uniformly hyperbolic diffeomorphism is simple and conjugated to the one exhibited by linear saddles, on the other hand the global dynamics presents an unpredictable character due to sensibility to initial conditions, denseness of periodic trajectories and orbits with transitive and (ir)regular behavior. The geometric theory developed for uniformly hyperbolic dynamics guarantees the existence of local immersed submanifolds (called stable and unstable manifolds) that are invariant by the dynamics and that constitute a crucial ingredient in the construction of finite Markov partitions. In consequence, subshifts of finite type can be used as combinatorial models for the dynamics and the powerfull techniques and ideas from statistical mechanics extend to this context. The geometric and functional analytic approaches (via construction of Markov partitions and the description of the spectrum of transfer operators, respectively) play a key role in the construction of physical, Sinai-Ruelle-Bowen and equilibrium measures and the study of their statistical properties. We refer the interested reader to e.g. [5, 24] for a more complete account.

The aim of this text is to survey and to be an invitation to the use of topological methods in dynamics, as a valid and handy alternative to the aforementioned geometric and functional analytic approaches. The starting point is the notion of specification proposed by Rufus Bowen which consists of the ability of the dynamics to recreate with sharp proximity true orbits from any given number of finite pieces of orbits (see e.g. [11]). The relation between shadowing and specification, the description of the latter concept and its extensions, and its importance as a tool in ergodic theory will guide the exposition in the remaining sections. There is evidence that these topological methods may be used to describe partial hyperbolic dynamics and dynamics of group actions, contexts in which topological and functional analytic methods are still unavailable, contributing to one of the most important leading research directions and challenges in dynamical systems.

2 Basics on topological dynamics

Throughout this article we assume that M is a compact Riemannian manifold. Let f be a continuous map on M. Some of the main concerns of the characterization of the dynamics from a topological viewpoint involve the description of periodic and transitive behavior, and chaoticity. Let $\Omega(f) \subset M$ be the set of non-wandering points of f, that is, the points $x \in M$ so that every open neighborhood of x intersects a positive iterate of itself by the dynamics. A point $x \in M$ is *periodic* (of period n) if there exists $n \in \mathbb{N}$ so that $f^n(x) = x$. Let $\mathrm{Per}(f)$ denote the set of all periodic points of f. Recall that f is called *transitive* if there exists a point $x \in M$ so that its orbit $\{f^n(x) : n \in \mathbb{N}\}$ is dense in M, and it is called *topologically mixing* if for any pair of open sets U, V there exists $N \in \mathbb{N}$ so that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$. An intricate challenge that goes back to the sixties was to propose suitable mathematical notions of *chaos*. Historically, one refers to chaotic dynamics the ones that exhibit at least one of the following properties: sensitive dependence to initial conditions, expansiveness, strong recurrence and mixing conditions, shadowing, specification, exponential growth of periodic points or positive topological entropy (see e.g. [15]).

Our main interest here concerns chaotic dynamics in the sense that pieces of true orbits or pseudo-orbits can be well approximated by true orbits of the dynamical system. The first notion that we shall consider is that of shadowing, which we now describe. Given a metric space M, a continuous map $f : M \to M$ and $\delta > 0$, a sequence of points $(x_n)_{n\geq 0}$

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is a δ-*pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for every $n \geq 0$. We say that $[x_i, t_i]_{i\in\mathbb{Z}}$ is a (δ, T)-*pseudo-orbit* for a flow $(X^t)_t$ if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A continuous map has the *shadowing property* if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any δ-pseudo-orbit $(x_n)_{n\geq 0}$ there exists $x \in M$ satisfying $d(f^n(x), x_n) < \varepsilon$ for every $n \geq 0$. In the case of homeomorphisms, pseudo-orbits and shadowing points are defined for both negative and positive iterates of the dynamics. A continuous flow $(X^t)_t$ satisfies the *shadowing property* if, for any $\varepsilon > 0$ and $T > 1$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that for any (δ, T)-pseudo-orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ there is $\tilde{x} \in \Lambda$ and a reparametrization $\tau \in \mathrm{R}(\varepsilon)$ (cf. 1) such that

$$d(X^{\tau(t)}(\tilde{x}), x_0 \star t) < \varepsilon, \text{ for every } t \in \mathbb{R}.$$

where for $t \in \mathbb{R}$, $x_0 \star t = X^{t-\sigma(i)}(x_i)$ if $\sigma(i) \leq t < \sigma(i+1)$.

3 Basics on ergodic theory

3.1 Invariant measures

The purpose of ergodic theory is to describe the asymptotic behavior of the orbits of 'almost every point' with respect to relevant measures for the dynamics. Given a σ-algebra $\mathscr{B}$ and a measurable map f on M, we say that a probability measure μ is f-*invariant* if $\mu(f^{-1}(A)) = \mu(A)$ for every $A \in \mathscr{B}$. We denote by $\mathscr{M}_1(f)$ the space of f-invariant probability measures. A set $A \in \mathscr{B}$ is f-*invariant* if $\mu(f^{-1}(A)\triangle A) = 0$. An invariant probability measure μ is *ergodic* if $\mu(A) \in \{0, 1\}$ for every f-invariant set A. By ergodic decomposition, the space $\mathscr{M}_1(f)$ is the convex hull of the space $\mathscr{M}_e(f)$ of f-invariant and ergodic probability measures (see e.g.[33]).

In what follows let μ be an f-invariant probability measure. Two pillars in ergodic theory are due to Poincaré (1890), and to von Neumann and Birkhoff (1931-1932). First, if $\mu(A) > 0$ then Poincaré recurrence theorem asserts that μ-almost every $x \in A$ is recurrent: there exists $n \geq 1$ so that $f^n(x) \in A$. Later, the ergodic theorems of von Neumann and Birkhoff brought the ideas present in Boltzman ergodic hypothesis in thermodynamics into the realm of dynamical systems. Birkhoff's ergodic theorem guarantees that if $\mu \in \mathscr{M}_1(f)$ is ergodic and $\phi \in L^1(\mu)$ then

$$\frac{1}{n}\sum_{j=0}^{n-1}\phi(f^j(x)) \longrightarrow \int \phi\, d\mu \quad \text{as } n \to \infty$$

for μ-almost every $x \in M$ and, thus, time averages of an observable for typical orbits coincide with the space average with respect to the underlying measure. von Neuman ergodic theorem guarantees the convergence of the previous time averages for observables ϕ on the Hilbert space $L^2(\mu)$.

3.2 Thermodynamic formalism

Some of the ideas of thermodynamic formalism, which aims the selection of invariant measures and the study of their statistical properties, where introduced from statistical mechanics into the realm of dynamical systems by pioneering contributions of Sinai, Bowen and Ruelle in the late seventies (see [11, 22] and references therein). Two particularly important classes of invariant measures are the so called equilibrium states and physical measures.

Given a potential $\phi \in C(M, \mathbb{R})$ the *topological pressure* $P_{\text{top}}(f, \phi)$ for f and ϕ can be defined by the variational principle

$$P_{\text{top}}(f, \phi) = \sup\Big\{h_\mu(f) + \int \phi\, d\mu :\ \mu \in \mathscr{M}_1(f)\Big\},$$

where $h_\mu(f)$ stands for the entropy of μ (see e.g. [33]). An invariant probability measure μ is called an *equilibrium state* for f with respect to ϕ if it attains the previous supremum. If $\phi \equiv 0$ the previous notion coincides with the topological entropy $h_{\text{top}}(f)$ of f. If there exists a unique equilibrium state then it is necessarily ergodic and we shall denote it by $\mu_{f,\phi}$. An f-invariant probability measure μ is a *physical measure* if its basin of attraction

$$B(\mu) = \Big\{x \in M :\ \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)} \to \mu \text{ as } n \to +\infty\Big\}$$

has positive Lebesgue measure. There are many examples where equilibrium and physical measures coincide and are absolutely continuous with respect to some reference measures with some weak Gibbs property. We say that a probability measure ν is a *weak Gibbs measure* for f and ϕ if for ν-almost every x there are constants $(K_n)_{n\geq 1}$ so that $\limsup_n \frac{1}{n}\log K_n(x) = 0$ and

$$K_n^{-1}(x) \leq \frac{\nu(B(x,n,\epsilon))}{e^{-nP(\phi)+S_n\phi(x)}} \leq K_n(x)$$

for every $n \geq 1$, where $S_n\phi = \sum_{j=0}^{n-1}\phi \circ f^j$ and the dynamical ball $B(x, n, \epsilon)$ is the set of points $y \in M$ such that $d(f^j(y), f^j(x)) < \varepsilon$ for all $0 \leq j \leq n$. We say that ν is a *Gibbs measure with respect to* ϕ if there exists $K > 0$ such that the previous property holds with $K_n = K$ (independent of n and x).

4 Uniform hyperbolicity

A compact f-invariant set $\Lambda \subset M$ is called *uniformly hyperbolic* for f if there exists a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0, 1)$ so that

$$\|Df^n(x)\mid_{E^s_x}\| \leq C\lambda^n \;\&\; \|Df^{-n}(x)\mid_{E^u_x}\| \leq C\lambda^n$$

for every $x \in \Lambda$ and $n \geq 1$. We say that f is an *Anosov* diffeomorphism if M is a uniformly hyperbolic set. Given a hyperbolic set Λ, a point $x \in \Lambda$ and $\varepsilon > 0$, the *ε-stable set* of x is defined by $W^s_\varepsilon(x) = \{y \in M : d(f^n(y), f^n(x)) \leq \varepsilon \text{ for all } n \geq 0\}$. Similarly, the set $W^u_\varepsilon(x)$ of points $y \in M$ so that $d(f^{-n}(y), f^{-n}(x)) \leq \varepsilon$ for all $n \geq 0$ is the *ε-unstable set* of x. Given a hyperbolic set Λ for f there exists a uniform $\varepsilon > 0$ so that the stable and unstable sets $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are C^r submanifolds tangent to E^s_x and E^u_x, respectively, for every $x \in \Lambda$. These are referred, respectively, as the *local stable* and *local unstable manifolds* at x of size ε. Uniform hyperbolicity is a C^1-open condition in the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms. We refer the reader to [24] for proofs.

Uniform hyperbolicity for flows is defined similarly. Given a C^1-flow $(X^t)_t$ on M and a compact $(X^t)_t$-invariant set $\Lambda \subseteq M$, we say that Λ is a *hyperbolic set* if there exists a DX^t-invariant and continuous splitting $T_\Lambda M = E^- \oplus E^0 \oplus E^+$ (E^0 subspace generated by the vector field $X(\cdot) = \frac{dX^t(\cdot)}{dt}\mid_{t=0}$) and constants $C > 0$ and $0 < \theta < 1$ such that

(i) $\|DX^t(x)\mid_{E^-_x}\| \leq C\theta^t$, and

(ii) $\|(DX^t(x)\mid_{E^+_x})^{-1}\| \leq C\theta^t$

for every $x \in M$ and $t \geq 0$. The flow $(X^t)_t$ is *Anosov* if the whole manifold M is a hyperbolic set. We refer the reader to [24] for more details on uniform hyperbolicity.

5 The notions: specification and gluing orbit properties

5.1 Discrete-time dynamics

A continuous map f on M satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any sequence of positive integers $n_1, \dots, n_k$ and $p_1, \dots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that $d(f^j(x), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d\left(f^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\right) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Among the maps that satisfy specification property one should refer topologically mixing subshifts of finite type, topologically mixing Anosov diffeomorphisms and topologically mixing continuous interval maps (see [10] and references therein). More flexible concepts include some measure theoretical non-uniform versions of the specification property that proved to hold for invariant measures with no zero Lyapunov exponents (cf. [18, 32]).

In the sequel we introduce two extensions of the notion of specification. Let μ be an f-invariant probability measure. We say that (f, μ) satisfies the *non-uniform specification property* if there exists $\delta > 0$ so that for μ-a.e. x and every $0 < \varepsilon < \delta$ there exists $p(x, n, \varepsilon) \in \mathbb{N}$ satisfying:

(i) $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} p(x, n, \varepsilon) = 0$

(ii) given $x_1, \dots, x_k$ in a full μ-measure set and positive integers $n_1, \dots, n_k$, if $p_i \geq p(x_i, n_i, \varepsilon)$ then there exists z that ε-shadows the orbits of each x_i during n_i iterates with a time lag of $p(x_i, n_i, \varepsilon)$ in between $f^{n_i}(x_i)$ and x_{i+1}; that is, $z \in B(x_1, n_1, \varepsilon)$ and

$$f^{n_1+p_1+\dots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$$

for every $2 \leq i \leq k$.

We say a continuous map f on M satisfies the *gluing orbit property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ so that for any points $x_1, x_2, \dots, x_k \in M$ and positive integers $n_1, \dots, n_k$, there are $p_1, \dots, p_k \leq N(\varepsilon)$ and $x \in M$ so that $d(f^j(x), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d\left(f^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\right) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. The latter property is satisfied e.g. by irrational rotations, which are far from having any mixing property and it is sometimes referred also as a transitive specification property [8, 34]. Similar flavored notions of linkability and closeability were introduced by Gelfert and Kwietniak [14]. We refer the reader to [17] and references therein for a more exhaustive description of the state of the art.

5.2 Continuous-time dynamics

In opposition to the discrete-time setting, the mixing properties of continuous-time dynamical systems are harder to analyze. For instance, while for uniformly hyperbolic diffeomorphisms every Hölder continuous potential admits a unique equilibrium state, which is a Gibbs measure and mixes exponentially fast, not all hyperbolic flows have exponential mixing (see e.g. [5]). Moreover, not all hyperbolic flows have the specification property, which is an indicator that a suitable notion should be more flexible to hold for a larger class of dynamics. Recall a continuous flow $(X^t)_{t\in\mathbb{R}}$ has the *specification property* on $\Lambda \subset M$ if for any $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that: given any finite colection τ of intervals $I_i = [a_i, b_i]$ ($i = 1 \dots m$) of the real line satisfying $a_{i+1} - b_i \geq T(\epsilon)$ for every i and every map $P : \bigcup_{I_i\in\tau} I_i \to \Lambda$ such that $X^{t_2}(P(t_1)) = X^{t_1}(P(t_2))$ for any $t_1, t_2 \in I_i$ there exists $x \in \Lambda$ so that $d(X^t(x), P(t)) < \epsilon$ for all $t \in \bigcup_i I_i$.

In continuous-time setting the shadowing property of the finite pieces of orbits should reflect the speed at which different points travel in their trajectories. For that reason let $\mathscr{R}$ be the set of all increasing homeomorphisms $\tau : \mathbb{R} \to \mathbb{R}$ so that $\tau(0) = 0$ and, given $\varepsilon > 0$, set

$$\mathscr{R}(\varepsilon) = \left\{\tau \in \mathscr{R} : \Big|\frac{\tau(t) - \tau(s)}{t - s} - 1\Big| < \varepsilon, s \neq t \in \mathbb{R}\right\}. \qquad (1)$$

We say that a continuous flow $(X^t)_t$ has the *reparametrized gluing property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \in$

$\mathbb{R}^+$ such that for any points $x_0, x_1, \dots, x_k \in M$ and times $t_0, t_1, \dots, t_k \geq 0$ there are $p_0, p_1, \dots, p_{k-1} \leq K(\varepsilon)$, a reparametrization $\tau \in \mathscr{R}(\varepsilon)$ and a point $y \in M$ so that

$$d(X^{\tau(t)}(y)), X^t(x_0)) < \varepsilon \quad \forall t \in [0, t_0]$$

and

$$d(X^{\tau(t+\sum_{j=0}^{i-1} p_j + t_j)}(y), X^t(x_i)) < \varepsilon \quad \forall t \in [0, t_i]$$

for every $1 \leq i \leq k$. If, in addition, the point y can be chosen periodic we say that $(X^t)_t$ satisfies the periodic reparametrized gluing orbit property. Criteria for (semi)flows to satisfy gluing orbit properties can be found in [8, 10].

6 Specification and gluing orbit properties: some consequences

In this section we shall focus on the analysis of continuous-time dynamics (since proofs are technically more demanding and results are in many cases harder to find in the literature) and on the comparison between continuous and discrete time dynamics.

6.1 Topological aspects

The space of homeomorphisms are often described in terms of topological classes, where we say that the homeomorphisms f and g are topologically conjugate if there exists a homeomorphism h so that $f \circ h = h \circ g$. Hence, the dynamics of homeomorphisms in the same topological class is the same up to a continuous change of coordinates. Similarly, flows are usually classified up to topological equivalence, that is, homeomorphisms that preserve orbits and their orientation but not necessarily the speed of the trajectories. If, on the one hand, it is not hard to check that the specification and the gluing orbit property are topological invariants, on the other hand topological equivalence may fail to preserve the gluing orbit properties for flows since these may affect the kind of reparametrizations that are considered at the shadowing process.

The strong contrast between discrete and continuous time dynamics is also present in the relation between shadowing and specification. While topologically mixing expansive continuous maps on compact metric spaces with shadowing property satisfy the specification property, this may not hold even for very simple Anosov flows. Moreover, minimal flows on $\mathbb{T}^2$ satisfy gluing orbit properties but fail to be topologically mixing. See [1, 8] for more details. Nevertheless, flows with the reparametrized gluing orbit property satisfy some 'weak mixing' conditions [8]. More precisely:

Theorem 1.— If $(X^t)_t$ satisfies the reparametrized gluing orbit property then $(X^t)_t$ has positive lower frequency $\tau(B_1, B_2)$ of visits to balls B_1, B_2 of radius ε given by

$$\liminf_{t\to+\infty} \frac{1}{t} \mathrm{Leb}(\{s \in [0,t] : B_1 \cap X_{-s}(B_2) \neq \emptyset\})$$

is strictly positive. Moreover, for all balls B_1, B_2 of radius ε centered at points with closed orbits there exists $C > 0$ such that $\tau(B_1, B_2) \geq C\varepsilon$.

We also note that if the flow $(X^t)_t$ is expansive then the topological entropy is bounded by the exponential growth rate of periodic orbits, a result which also holds in the context of semigroups of expanding maps (cf. [8, 12]).

6.2 Space of invariant measures

The push-forward $f_\sharp$ acting on the space of probability measures in M is defined by $(f_\sharp\mu)(A) = \mu(f^{-1}(A))$ for every $A \in \mathscr{B}$. This map inherits some of the the topological characteristics of the original dynamics. First, if f has a specification property then so does $f_\sharp$ and these are equivalent in the context of continuous interval maps (see e.g. [21]). Second, the simplex of invariant measures for maps with specification is the Poulsen simplex (see [25, 17]). Given a continuous flow $(X^t)_t$ we denote by $\mathscr{M}_1((X^t)_t)$ the space of $(X^t)_t$-invariant probabilities. In [8] one could recover part of the "richness" for the simplex of invariant probability measures for dynamics with gluing orbit properties. More precisely,

Theorem 2.— If $(X^t)_t$ satisfies the periodic reparametrized gluing orbit property then periodic measures are dense in $\mathscr{M}_1((X^t)_t)$, and the set of ergodic measures forms a residual subset of $\mathscr{M}_1((X^t)_t)$.

As continuous flows with shadowing and a dense set of periodic orbits satisfy the reparametrized gluing orbit property (cf. [7]) we obtain the following consequence:

Corollary 3.— Assume $(X^t)_t$ is a continuous and volume preserving flow. If $(X^t)_t$ satisfies the periodic shadowing property and the periodic points are dense in M then periodic measures are dense in $\mathscr{M}_1((X^t)_t)$.

6.3 Large deviations.

In the early nineties, L.-S. Young [35] addressed the question of the velocity of convergence of ergodic averages on Birkhoff's ergodic theorem in the case of Gibbs measures. Here, given a potential $\phi : M \to \mathbb{R}$ and a probability μ, we say that μ is *weak Gibbs* with respect to ϕ, with constant $P_\mu \in \mathbb{R}$, if for any $\varepsilon > 0$ there exists $K_t(\varepsilon)$ (depending only on ε and on the time t) so that $\lim_{t\to\infty} \frac{1}{t} \log K_t(\varepsilon) = 0$ and

$$\frac{1}{K_t(\varepsilon)} \leq \frac{\mu(B(x,t,\varepsilon))}{\exp\left[\int_0^t \phi(X^s(x))ds - tP_\mu\right]} \leq K_t(\varepsilon)$$

for μ-almost every $x \in M$ and every $t \geq 0$. A continuous observable $\psi : M \to \mathbb{R}$ is called of *tempered variation* if there is $\delta > 0$ such that $\lim_{t\to\infty} \frac{1}{t}\gamma(\psi, t, \delta) = 0$, where

$$\gamma(\psi, t, \delta) = \sup_{y\in B(x,t,\delta)} \left| \int_0^t \psi(X^s(x)) - \psi(X^s(y)) ds \right|.$$

Gluing orbit properties were first introduced in [10] with the motivation of obtaining large deviations principles for *all* hyperbolic flows:

THEOREM 4.— Assume the semiflow $(X^t)_{t\geq 0}$ satisfies the gluing orbit property, ϕ is a bounded potential with tempered variation and μ is a weak Gibbs probability with respect to ϕ. If $a < b$ and $\psi : M \to \mathbb{R}$ is a bounded observable with tempered variation then

$$\liminf_{t\to\infty} \frac{1}{t} \log \mu\Big(\frac{1}{t}\int_0^t \psi \circ X^s(\cdot) ds \in (a, b)\Big) \\ \geq -\inf\Big\{P_\mu - h_\nu(X^1) - \int \phi\, d\nu\Big\},$$

where the infimum is taken over all $(X^t)_t$-invariant probability measures ν so that $\int \psi\, d\nu \in (a, b)$. If, in addition, M is compact and $\psi : M \to \mathbb{R}$ is continuous then

$$\limsup_{t\to\infty} \frac{1}{t} \log \mu\Big(\frac{1}{t}\int_0^t \psi \circ X^s(\cdot) ds \in [a, b]\Big) \\ \leq -\inf\Big\{P_\mu - h_\nu(X^1) - \int \phi\, d\nu\Big\},$$

where the infimum is taken over all $(X^t)_t$-invariant probability measures ν so that $\int \psi\, d\nu \in [a, b]$.

A surprising connection between large derivations and multifractal analysis (cf. Subsection 6.4 below) allows to use the large deviations estimates to study the size of the level sets and irregular set in the multifractal decomposition [9].

6.4 Some other aspects

For shortness, in what follows we give a more direct and informal presentation of other important characterizations of dynamics with some gluing orbit property and their use as an important tool.

A characterization for uniform hyperbolicity

The relation between specification, gluing and uniform hyperbolicity among smooth dynamics is well understood. If the specification property holds in a C^1-open neighborhood of diffeomorphisms or vector fields then these are Anosov [23, 4]. Similarly any C^1-open subset of diffeomorphisms (resp. vector fields) with the gluing orbit property is formed by transitive Anosov diffeomorphisms (resp. Anosov flows) [34, 10]. So, from the C^1-robust viewpoint, uniform hyperbolicity, specification and the gluing orbit properties coincide. The picture is radically different beyond the scope of uniform hyperbolicity. Indeed, specification is rare even among partially hyperbolic diffeomorphisms [27, 28].

Thermodynamic formalism

Bowen [11] proved that expansive homeomorphisms with specification have a unique equilibrium state with respect to all continuous potentials with tempered variation. More recently, Climenhaga and Thompson extended Bowen's approach to deal with dynamical systems where the set of points with obstructions to either specification or expansiveness do not carry full topological pressure (we refer the reader to [13] for a precise formulation and applications). More recently, Pavlov [19] showed that expansive maps with non-uniform specification may have more than one equilibrium state.

Multifractal formalism

The general idea of multifractal analysis, that can be traced back to Besicovitch, is to decompose the phase space in subsets of points which have a similar dynamical behavior and to describe the size of each of such subsets from the dimensional or topological viewpoint. Given a continuous map f on M and $\phi : M \to \mathbb{R}$ continuous, decompose

$$M = \bigcup_{\alpha\in\mathbb{R}} M_\alpha \cup I_\phi(f)$$

where $M_\alpha = \{x \in M : \lim_n \frac{1}{n} S_n\phi(x) = \alpha\}$ are level sets of convergence for Birkhoff averages and the irregular set $I_\phi(f)$ is the set of points for which the Birkhoff averages for ϕ do not converge. The irregular set for continuous observables and maps with specification is either empty or carries full topological entropy. Moreover, the topological pressure of level sets can be characterized by the supremum for invariant measures supported on them (see [31] and references therein). A much harder situation is to describe the topological entropy of saturated sets. Given a subset $K \subset \mathscr{M}_1(f)$ of f-invariant probability measures, a saturated set in M is the subset $G_K \subset M$ of points $x \in M$ whose accumulation points $V_T(x)$, in the weak* topology, of the empirical measures

$$\mathscr{E}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)}$$

coincides with the prescribed subset K of invariant probability measures. Saturated sets can be used to describe convergence properties of Birkhoff averages with respect to every continuous observable. Clearly $V_T(x)$ is a singleton if and only if the Birkhoff averages of every continuous observable are convergent at x. Some extensions of the original notion of specification can be used to estimate the topological pressure of saturated sets for some non-uniformly hyperbolic maps [20, 29, 30].

7 A final invitation: some open questions

The use of topological methods in ergodic theory is nowadays a very active area of research. We will finish this short article with some open questions as an invitation for the reader to explore the underlying ideas presented here.

1. The relation between specification and the gluing orbit property is still not fully understood. Given the previous discussion it is natural to ask whether there exists a Baire residual subset $\mathscr{R}$ of the space of C^1-diffeomorphisms with the gluing orbit property so that every topologically mixing diffeomorphism $f \in \mathscr{R}$ satisfies the specification property.

2. Regarding the thermodynamic formalism of maps displaying some weak form of specification, it is natural to ask if an expansive map f with the gluing orbit property has a unique equilibrium state for every regular (e.g. Hölder continuous) potential. Are the related transfer operators quasi-compact on the Banach space of Hölder continuous observables? See [5] for definition of transfer operators. Similar question can be posed for flows with the reparametrized gluing orbit property.

3. The non-wandering set of a uniformly hyperbolic diffeomorphism can be decomposed in a finite number of pieces on which the dynamics acts as a subshift of finite type and each piece, up to an iterate of the dynamics, satisfies the specification property. On the converse direction, if f is a continuous expansive map with the gluing orbit property does there exist $N \geq 1$ and a disjoint union $M = \bigcup_{1\leq i\leq N} \Lambda_i$ of compact sets so that $f(\Lambda_i) = \Lambda_{i+1}$ for all $1 \leq i \leq N$ (with the convention that $\Lambda_{N+1} = \Lambda_1$) and the iterate $f^N : \Lambda_i \to \Lambda_i$ has the specification property? If so, which extra information can be given on each of the 'basic' pieces Λ_i?

4. The relation between specification and uniform hyperbolicity is well established (recall Subsection 6.4). However, much less is known on the relation between these topological concepts with the measure theoretical notions of non-uniform specification. Assume $\mathscr{U}$ is a C^1 open set of transitive diffeomorphisms on M so that all g-invariant measures satisfy the non-uniform specification property, for all $g \in \mathscr{U}$. Is $\mathscr{U}$ formed by maps with some gluing orbit property? We believe the C^1-robustness assumption should be crucial above.

5. The multifractal analysis of time averages for flows is much harder than for maps even when assuming the reparametrized gluing orbit property. In comparison with the discrete time setting, the difficulty relies on the fact that the reparametrization depends on the points that are being shadowed. Nevertheless we expect that if $(X^t)_t$ is a continuous flow with the reparametrized gluing orbit property and the Birkhoff irregular set of a continuous potential is non-empty then it should carry full topological entropy.

6. Geometric Lorenz attractors are among the simpler flows where regular orbits accumulate on singular orbits (see e.g. [2]). The coexistence of singular and regular orbits brings much complexity to the dynamics and imply, in particular, the absence of weak forms of shadowing for most geometric Lorenz attractors [16, 3]. In view of some criteria for non-uniform specification properties [10] it is natural to ask whether geometric Lorenz attractors enjoy a reparametrized gluing orbit property with respect to reparametrizations with a logarithmic singularity at the origin. This can be thought as a step in the direction of establishing a thermodynamic formalism for geometric Lorenz attractors.

7. Finally, the underlying ideas of the property of specification are expected to be applied in far more general situations. This property was proved to hold for C_0 semigroups on Banach spaces, including solutions of the hyperbolic heat equation and Black-Scholes equation (see e.g. [6] and references therein). Since most results addressed here require compactness as a crucial ingredient it is a challenge to understand up to which extent the ideas arising from multifractal formalism can extend to the context of partial differential equations and/or operators on infinite dimensional ambient spaces.

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International Conference on Semigroups and Automata (CSA 2016)
Celebrating the 60th Birthdays of Jorge Almeida and Gracinda Gomes

Mário Branco* and Pedro V. Silva**

The *International Conference on Semigroups and Automata* (CSA 2016) was held in *The Faculty of Sciences of the University of Lisboa*, from the 20th to the 24th June 2016. CIM assumed the role of host institution.

This conference was organized with the purpose of celebrating the 60th birthdays of Professors Jorge Almeida (Univ. Porto) and Gracinda Gomes (Univ. Lisboa) for their extraordinary role over the years in the development of semigroup theory, in Portugal and abroad. Recognition of their work by the international community contributed decisively to achieve a record number of 117 participants for a semigroup conference and to the high quality of the program.

Some financial support was available for students and young researchers, which were encouraged to participate. The program included 19 invited talks and 33 contributed talks. They illustrated recent trends of semigroup theory, and connections to automata theory and many other areas: category theory, combinatorics, computational algebra, dynamical systems, geometry, group theory, logic, operator theory, probability theory, ring theory, topology. The detailed program and the slides of many talks can be found on

http://ciencias.ulisboa.pt/en/conferencia/csa-2016.

We present next some basic information:

Scientific Committee: Karl Auinger, Peter Cameron, Volker Diekert, John Fountain, Mark Lawson, Stuart Margolis, John Meakin, Jean-Eric Pin, Benjamin Steinberg, Mikhail Volkov.

Organizing Committee: Mário Branco, Alfredo Costa, Manuel Delgado, Vítor Hugo Fernandes, António Malheiro, Ana Moura, Catarina Santa-Clara, Pedro Silva.

* FCUL/CEMAT
** FCUP/CMUP

Invited speakers: João Araújo, Karl Auinger, José Carlos Costa, Volker Diekert, Ruy Exel, Victoria Gould, Robert Gray, Mark Kambites, Ganna Kudryavtseva, Mark Lawson, Markus Lohrey, Volodymyr Mazorchuk, Jean-Eric Pin, John Rhodes, Emanuele Rodaro, Anne Schilling, Benjamin Steinberg, Mária Szendrei, Marc Zeitoun.

Sponsors: CIM, CEMAT, CMA (with NOVA.ID.FCT), CMUC, CMUP, FCT (with COMPETE, QREN, UE), FCT-UNL, FCUL, UP.

We include some comments from two members of the Scientific Committee which are experts in both semigroup and automata theory:

Jean-Eric Pin (CNRS/Univ. Paris-Diderot): "(. . .) Thanks to the efforts of a mixed team from Porto and Lisbon, the conference was perfectly organised and both the scientific and the social aspects were a great success. The conference attracted about 120 people from all over the world. The programme covered most of the major current research topic on semigroups and automata.

"The high quality of the scientific programme and the birthday celebrations of our colleagues made this conference a unique moment in the history of semigroup theory."

Mikhail Volkov (Ural Federal Univ.): "CSA 2016 was a clear scientific success. As a member of the Scientific Committee I had been involved in the process of selecting invited speakers and contributed talks, and the presentations held at the conference confirmed that the selection had been done properly. The conference was very representative in many senses: first, all leading groups in semigroup theory were involved, second, all major directions of the theory and its applications to various parts of mathematics and computer science were shown, and last but not least, both renowned specialists and young researchers were given the opportunity to present their achievements. (. . .)"

Recent Trends in Differential Equations
Aveiro June 27–29, 2016

Vasile Staicu*

The conference *Recent trends in Differential Equations* held in Aveiro (Portugal) from 27 to 29 June, 2016, celebrated the 75th birthday of Professor Arrigo Cellina from the University Milano-Bicocca, Italy and the 60th birthday of Professor Alberto Bressan from Penn State University, USA.

Organized by the *Functional Analysis and Applications* group of the *Center for Research and Development in Mathematics and Applications* (CIDMA), and hosted by the *Department of Mathematics of the University of Aveiro*, the conference brought together mathematicians engaged in research on Differential Equations, Differential Inclusions and Set Valued Maps, Calculus of Variations, Control Theory and Applications. All these are areas where Professors Arrigo Cellina and Alberto Bressan have both made major contributions.

Several participants presented their most recent results and illustrated recent trends in these areas. The conference was particularly addressed to young researchers and Portuguese and foreign PhD students, who had the opportunity to meet with world-renown mathematicians and start collaborations in these research areas. There were 5 Keynote talks of 50 minutes each, 15 Invited talks (of 30 minutes long) and 15 Contributed talks (of 20 minutes long), all of them of high scientific level.

The opening session was held in the auditorium of the Rectorate of the University of Aveiro, in the presence of Professor Manuel Antonio Assunção, Rector of the University of Aveiro; Professor José Francisco Rodrigues, President of the CIM (International Center for Mathematics), Professor Luís Filipe Pinheiro de Castro, Scientific Coordinator of CIDMA, and Prof. João Manuel da Silva Santos, Director of the Department of Mathematics. Afterwards, the conference was hosted in the Department of Mathematics and held in plenary session (the Keynote and the Invited talks) and parallel sessions (the Contributed talks).

The Keynote talks were given by Zvi Artstein from Weizmann Institute of Science in Rehovot (Israel), Alberto Bressan from Penn State University (USA), Gianni Dal Maso from the International Schol for Advanced studies (SISSA) in Trieste (Italy), Héctor J. Sussmann from Rutgers University (USA), and Richard Vinter from Imperial College in London (UK).

The Invited talks were given by professors Andrea Bacciotti (Polytechnic of Turin, Italy), Ugo Boscain (Ecole Polytechnique, Palaiseau Cedex France), Fabian Flores-Bazan (University of Concepcion, Chile), Rinaldo M. Colombo (University of Brescia, Italy), Graziano Crasta (Sapienza Università di Roma, Italy), Elsa Maria Marchini (Politecnico di Milano, Italy), Carlo Mariconda (Universita degli Studi di Padova, Italy), Manuel Monteiro Marques (University of Lisbon), Marco Mazzola (University Pierre et Marie Curie, Paris VI, France), Antonio Ornelas (University of Evora, Portugal), Michele Palladino (Penn State University, USA), Franco Rampazzo (University of Padova, Italy), Wen Shen (Penn State University, USA), Susana Terracini (University of Torino, Italy), and Giulia Treu (University of Padova, Italy).

The high scientific level of the conference has been ensured by the prestige of the two celebrated mathematicians, by the prestige of the Keynote and Invited speakers, as well as, by the high level of all contributions of the participants. In this occasion, the organizers took advantage of the best facilities offered by the university campus in Aveiro, providing a pleasant and fruitful environment. This was possible also thanks to the partial support from the *Portuguese Foundation for Science and Technology* (FCT) and the support received from CIM, here gratefully acknowledged.

Arrigo Cellina

Born in Varese (Italy) at 3 August 1941, graduated in Physics at the University of Milan in 1965. In 1968 he obtained the title of Doctor of Philosophy (Ph.D.) in Mathematics at the University of Maryland. In 1969 he obtained the *Libera Docenza* in Mathematical Analysis. He taught as a Lecturer and Assistant Professor at the University of Perugia and then at the University of Florence.

After one year on leave at the University of Paris IX, in 1974 he took service as *Extraordinary Professor* at the University of Padua. In 1979 he moved to S.I.S.S.A. (Trieste), where he served as Coordinator of the Sector of Functional Analysis and Applications and as Deputy Director of S.I.S.S.A.

In the Fall of 1997 he took service at the University of Milan, and the following year, at the University of Milano-Bicocca, where he was coordinator of the Doctoral program in Mathematics.

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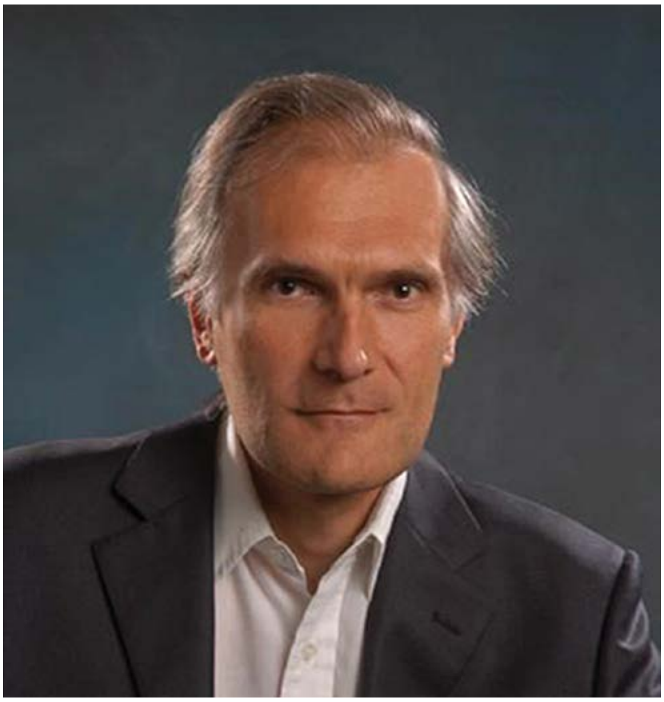
Arrigo Cellina



Alberto Bressan

In the fall of 2011 he was placed on leave as tenured professor. He continues to teach as an unpaid professor from 2011 to date, teaching courses of Higher Analysis and Calculus of Variations.

He has taught at various times at the University of Southern California Los Angeles and in Santa Barbara; He has been visiting professor at several institutions, including the universities of Paris, Montpellier and Limoges, the Courant Institute of Mathematical Sciences (NYU) and M.S.R.I. Berkeley. He was President of the Scientific Foundation C.I.M.E. and member of the Scientific Committee of CIM (1996-2000).

He is the author of more than a hundred mathematical publications and of a classical monograph on differential inclusions (with J.P.Aubin), published in the prestigious Springer's series on Grundlehren der Mathematischen Wissenschaften. He also edited the books Methods of Nonconvex Analysis and Optimal Shape Design published within Springer's series *Lecture Notes in Mathematics / C.I.M.E. Foundation Subseries*.

Alberto Bressan

Born in Venice at 15 June 1956, graduated in Mathematics in Padova in 1978, with a thesis on linear control processes, supervised by Roberto Conti. From February 1979 to January 1980 he had a C.N.R. research fellowship at University of Florence. He obtained the Ph.D. in Mathematics at the University of Colorado, Boulder, in 1982. He worked as Researcher at the University of Padova (Italy) from 1982 to 1986, Associate Professor at University of Colorado, Boulder (USA) from 1986 to 1991, Professor at the International School for Advanced Studies (SISSA) in Trieste, Italy, from 1991 to 2003. In 2003, he moved to Penn State University to assume a full professorship there — a position he still holds. In 2002 he was a Plenary speaker at the International Congress of Mathematicians, Beijing, 2002. In 2006 he was awarded the *Antonio Feltrinelli* prize for Mathematics, Mechanics and Applications of the Accademia Nazionale dei Lincei, Rome; in 2007 he won the *Analysis of Partial Differential Equations* prize of the Society for Industrial and Applied Mathematics, Phoenix (with Stefano Bianchini); in 2008 he was awarded the *M. Bôcher Memorial Prize* of the American Mathematical Society, San Diego. He was appointed to the Eberly Family Chair in Mathematics at Penn State in August 2008. In 2010 he received the *Luigi and Wanda Amerio* prize of the Accademia di Scienze e Lettere, Istituto Lombardo, Milan. In 2011 he became Member of the Royal Norwegian Society of Sciences and Letters, Trondheim, and Member of the Accademia di Scienze e Lettere, Istituto Lombardo, Milan, and was awarded the *Gaetano Fichera* prize for mathematical analysis by the Unione Matematica Italiana.

The research of Alberto Bressan covers several areas, including: Hyperbolic conservation laws and nonlinear wave equations; Modeling and optimization of traffic flow on networks; Optimal control, non-cooperative games, and applications to economics and finance; PDE models of controlled biological growth; Differential inclusions, dynamic blocking problems; Impulsive control of Lagrangian systems by means of active constraints.

He is author of three books, 8 book chapters, various lecture notes, and more than 170 research papers in international journals.

International Center for Mathematics

SOCIEDADE PORTUGUESA DE MATEMÁTICA

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

PEDRO NUNES LECTURES

JEAN-PIERRE **BOURGUIGNON** (CNRS-IHÉS)

Jean-Pierre Bourguignon holds an engineering degree from École Polytechnique and a PhD in mathematical sciences from the University Paris VII. A differential geometer by training, he has since pursued his interest in the mathematical aspects of theoretical physics.
He is the President of the European Research Council as of 1 January 2014.
He was the Director of the Institut des Hautes Études Scientifiques (IHÉS) from 1994 till 2013. This international research institute located near Paris, France, was built as the European counterpart of the Institute for Advanced Study in Princeton. He was also the first ERC Panel Chair in Mathematics, for Starting Grants.
He spent his whole career as a fellow of the Centre National de la Recherche Scientifique (CNRS). He held a Professor position at École polytechnique from 1986 to 2012. From 1990 to 1992, he was President of the Société Mathématique de France and President of the European Mathematical Society from 1995 to 1998. He is a former member of the Board of the EuroScience organisation (2002-2006) and served on EuroScience Open Forum (ESOF) committees since 2004.
Jean-Pierre Bourguignon received the Prix Paul Langevin in 1987 and the Prix du Rayonnement Français in Mathematical Sciences and Physics from the Académie des Sciences de Paris in 1997. He is a foreign member of the Royal Spanish Academy of Sciences. In 2005, he was elected honorary member of the London Mathematical Society and has been the secretary of the mathematics section of the Academia Europaea. In 2008, he was made Doctor Honoris Causa of Keio University, Japan, and, in 2011, Doctor Honoris Causa of Nankai University, China.

20 OCTOBER 2016
UNIVERSIDADE DE AVEIRO

15:30 — DEP. MATH, ROOM 11.1.3

PERSPECTIVES ON A GEOMETRY WHICH UNDERWENT SEVERAL REJUVENATIONS: FINSLER GEOMETRY

21 OCTOBER 2016
UNIVERSIDADE DO PORTO

14:30 — ROOM 0.07

GENERAL RELATIVITY AND GEOMETRY, INTERACTIONS AND MISSED OPPORTUNITIES

24 OCTOBER 2016
UNIVERSIDADE DE LISBOA

16:30 — SALÃO NOBRE DA REITORIA DA UNIVERSIDADE DE LISBOA

CONNECTING SPINOR FIELDS AND DIRAC OPERATORS TO GEOMETRY: THE GENERAL CONTEXT AND SOME RECENT DEVELOPMENTS

CIDMA — CENTRO DE I&D EM MATEMÁTICA E APLICAÇÕES / CENTER FOR R&D IN MATHEMATICS AND APPLICATIONS

CENTRO DE MATEMÁTICA UNIVERSIDADE DO PORTO

CMAF CIO — CENTRO DE MATEMÁTICA, APLICAÇÕES FUNDAMENTAIS E INVESTIGAÇÃO OPERACIONAL

CAMGSD