CIM

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Contents

Editorial

Due to some logistic and budget constraints, the editorial board of the bulletin decided to change the policy and publish a single issue per year.

In this number of the bulletin, we initiate a cycle of articles dedicated to honouring the work and dedication to mathematical research of distinguished Portuguese mathematicians who have recently retired. Since there are many such examples, the editorial board decided to start by those who were directly connected to the history of CIM. Hence, this issue features an interview to Ivette Gomes, who belonged to the first Scientific Council of CIM and had a prominent role in promoting and stimulating the Portuguese community of statisticians.

We present an article about the recent trends in the area of Semigroup Theory and its connection with Symbolic Dynamical Systems. We also exhibit a paper that gives a first insight into the world of coquaternions, which are elements of a four-dimensional hypercomplex real algebra generalising complex numbers. In order to celebrate the the international year of Mathematical Biology, we include an article on the mathematics of populations with particular insight about Epidemiology.

Inserted in the cycle of historical articles, we include a piece about the work developed by Daniel da Silva and, as usual, we publish several summaries and reports of some of the activities partially supported by CIM, of which we highlight the workshop about wild fires, which deserved special attention after their catastrophic impact during 2017.

We celebrate the success of another edition of the CIM-SPM's initiative Pedro Nunes' Lectures, which brought Alfio Quarteroni to the Academia das Ciências in Lisbon and to the universities of Coimbra and Porto. For that purpose we feature an interview with Alfio Quarteroni.

We recall that the bulletin welcomes the submission of review, feature, outreach and research articles in Mathematics and its applications.

Jorge Milhazes Freitas
Centro de Matemática & Faculdade de Ciências da Universidade do Porto
https://www.fc.up.pt/pessoas/jmfreita/

Matemática e Património Cultural

Academia das Ciências de Lisboa

19 December, 2018

15:00 h

Lisbon

Program

15:00 Azulejos e ciência
por Henrique Leitão (Universidade de Lisboa | CIUHCT)

15:15 Os azulejos euclideanos jesuítas
por António Leal Duarte (Universidade de Coimbra) e
Carlota Simões (Universidade de Coimbra)

15:30 Relógios de Sol
por Suzana Nápoles (Universidade de Lisboa)

15:45 Calçadas portuguesas
por Ana Cannas da Silva (ETH Zurich) e
Suzana Nápoles (Universidade de Lisboa)

16:00 Intervalo para café

16:20 Arquitetura e Matemática
por João Pedro Xavier (Universidade do Porto | Faculdade de Arquitectura)

16:35 A arte e a geometria de Almada Negreiros
por Pedro Jorge Freitas (Universidade de Lisboa | DHFC e CIUHCT) e
Simão Palmeirim Costa (Universidade de Lisboa | CIEBA)

16:50 Literatura e Matemática
por António Machiavelo (Universidade do Porto)

17:10 Concerto comentado: Matemática, processos composicionais e estratégias de preparação para a performance em obras para piano de Jaime Reis, João Madureira e Christopher Bochmann
por Ana Telles (Universidade de Évora | CESEM-UÉ)

18:00 Sessão de Encerramento
com a presença de Guilherme D'Oliveira Martins, Coordenador Nacional do Ano Europeu do Património Cultural

Organizing Committee

Henrique Oliveira (CIM)
Mário Branco (SPM)
Carlota Simões (MPT)

http://www.cim.pt/agenda/event/197


MATEMÁTICA E
PATRIMÓNIO CULTURAL
19 DEZ 2018 • 15:00 H
ACADEMIA DAS CIÊNCIAS DE LISBOA
Comissão Nacional da UNESCO Portugal
United Nations Educational, Scientific and Cultural Organization
Portuguese National Commission for UNESCO
2018 ANO EUROPEU DO PATRIMÓNIO CULTURAL
ORGANIZAÇÃO
CENTRO INTERNACIONAL DE MATEMÁTICA
MPT
spm
SOCIEDADE PORTUGUESA DE MATEMÁTICA
APOIO
ACADEMIA DAS CIÊNCIAS LISBOA
LISBOA
muhnac
MUSEU NACIONAL DE HISTÓRIA NATURAL E DA CIÊNCIA
MUSEU DA CIÊNCIA
UNIVERSIDADE DE COIMBRA

AN INTERVIEW WITH

Alfio Quarteroni

by **Gonçalo Pena***

Alfio Quarteroni (born 30 May 1952) is Professor of Numerical Analysis at Politecnico di Milano (Italy) and Director of MOX. He is the founder of MOX at Politecnico di Milano (2002), the founder of MATHICSE at EPFL, Lausanne (2010), the co-founder (and President) of MOXOFF, a spin-off company at Politecnico di Milano (2010). Co-founder of MATHESIA (2015) and of MATH&SPORT (2016).

He is author of 25 books, editor of 5 books, author of more than 300 papers published in international Scientific Journals and Conference Proceedings, member of the editorial board of 25 International Journals and Editor in Chief of two book series published by Springer.

Among his awards and honors are: the NASA Group Achievement Award for the pioneering work in Computational Fluid Dynamics in 1992, the Fanfullino della Riconoscenza 2006, Città di Lodi, the Premio Capo D'Orlando 2006, the Ghislieri prize 2013, the International Galileo Galilei prize for Sciences 2015, and the Euler Lecture 2017.

* Centro de Matemática & Faculdade de Ciências e Tecnologia da Univ. de Coimbra

What took you to choose Mathematics? Was it a calling or pure chance?

It was (almost) pure chance. After my high school diploma I was supposed to start with my first job of bank accountant. It was indeed the president of my diploma committee who urged me to continue studying. I didn't have any specific education in Math (my high school was a technical one) and I wanted to challenge myself with a tough subject I didn't know almost anything about. It was, indeed, a rather irrational move, but I had somehow the feeling that Math would have become important for Economics, the subject I really loved at that time.

When did you decide to study Numerical Analysis and Scientific Computing?

At the end of my third year at University I asked Enrico Magenes, a famous analyst in Pavia (my university) a subject for my Laurea (today we would call it master). Professor Magenes redirected me toward Franco Brezzi, a very young researcher although already quite famous in the field of finite element theory. That was my start with Numerical Analysis. My passion for scientific computing and applications came later on.

Whose work on (applied) mathematics do you admire the most? A book you read, a paper?

The book by Jacques Louis Lions *Optimal control of systems governed by partial differential equations*, Springer 1971 (not only for the optimal control theory) and that of Isaacson and Keller, *Analysis of Numerical Methods*, 1966 (my first book in numerical analysis).

You work on an area that bridges scientific computing, simulation and mathematics (which is quite a feat!) What are your thoughts on the difference between pure and applied mathematics, if there is any?

There is just one mathematics. There are, however, pure and applied mathematicians. The former go as deep as possible into the matter, to reveal the secret of mathematical structures. The latter face a real life problem and have to develop (or use) the best possible mathematics to solve it; simplification is admissible, provided however it does not change (or disregard) the essential feature of the solution.

Throughout your career, you worked in many interesting projects (like the Alinghi yacht or Solar Impulse).
I won't ask you to pick a favorite, but what attracted you to each of them?

I must admit that I was always warmly asked to work on those problems (and many others). I accepted those which fascinated

me, I left apart those that were less original (or challenging).

I know you travel quite a bit, especially when you were working in MOX and EPFL. How did you keep up with such a tight schedule?

No answer. It comes like that.

What hobbies do you have (besides travelling)? Favorite book?

I loved playing soccer and tennis when I was young. Now I am still a soccer fan. I do not practice any sport, unfortunately, apart from swimming (but not regularly). I walk a lot, I read reasonably often: novels, science-fiction, non-fiction, books on history. I love listening music, any kind of good music (rock and classic music).

This is not your first time in Portugal or Coimbra. Do you recall your first visit here? Can you tell us a bit about it?

I came in 1997 for an International Conference in Lisboa organized by Adelia Sequeira, then in 1999 in Coimbra for an internal summer school where I gave a series of lectures, then again many times in Lisboa, Coimbra, Porto and Madeira. My very first impressions were that there was a special light in the sky and full of colors in the building and a great similarity between Portuguese and Italians. Fantastic food, warm hospitality, a great cultural heritage, and a great talent in the young people. A place to be.

What do you find most interesting in our country and what do you like the least?

See above for the plus. On the negative side, it is sad to see how many Portuguese are not in their home country. This is a great loss for Portugal. (Italy is experiencing the same diaspora, especially in the last 20 years)

When I was a student at the University of Coimbra in 2000 (The World Mathematical Year), you came here to talk at a conference addressed to a general public and I was very impressed with everything you presented. Do you realize the impact you have, not only on mathematics, but on the people you talk to at conferences? Have you ever felt that you also have a kind of mission in the way you communicate mathematics?

I still remember that event. Paula Oliveira invited me for the conference *Teias matematicas*. I believe mathematics has not the place it deserves, especially among the students at primary and secondary schools. I am often invited to make presentations to high school students. I want to convince them that Math is not less important than Physics, Philosophy or Chemistry, say, and does have a great impact on our lives. There are too

many brilliant students who do not enroll in Math just because they have never been exposed to this kind of arguments, even though they are (potentially) very good in Math. I like presenting major math achievements in order to convince them that they can be a mathematician and be proud of that.

I've also been one your PhD students and noticed the strong group bonds that were built. What do you think is the key to a good work environment?

I believe that the major responsibility of a PhD advisor is to provide constant source of new and interesting problems to his/her students (and to show the horizon). Also, to create a warm and stimulating work environment, and the right opportunity for everyone for knowledge exchange.

Which advice would you give a young research starting their PhD now?

Being a researcher is the best possible job in the world. This is absolutely true: I am old enough and experienced enough to guarantee that this statement is true. Competition, however, is tough. You must be motivated and ready to experience frustration (when results do not come as expected: there is no low-hanging fruit in our business) but also excitement. You need to be patient: it takes time and sacrifice before the right job opportunity materializes. It is also important to work on a challenging and timely subject.

It's clear Mathematics helps to shape the world and it's a fundamental tool in its development. Which challenges do you think lie ahead for the next 20 years?

Providing a sound, rigorous and general mathematical theory to AI and machine learning, helping doctors to improve personalized healthcare through mathematics and improving our mathematical understanding of climate change and the social effect of people's migration

If you were not a mathematician, what do you see yourself doing in your professional life?

A farmer. My very first job, long ago.

140th European Study Group with Industry

by **Raquel Barreira***

The 140th European Study Group with Industry (ESGI140) was held at Barreiro, from the 4th to the 8th of June 2018, and it was organized by the Barreiro School of Technology of the Polytechnic Institute of Setúbal along with PT-MATHS-IN. This has been the 12th time that this successful European instrument for cooperation between Mathematics and the Industry took place in Portugal.

During a week, nearly 40 participants, from several nacionalities and areas of Mathematics, worked collaboratively in challenges posed by this year participant companies: Infraquinta and Lap2go.

Infraquinta manages the water and the wastewater services of a well-known tourist place in Algarve, known as Quinta do Lago. Tourism increase trend and climate change scenarios can be identified as two predominant drivers which strongly influence Infraquinta's activity and the company presented two of their challenges.

The first one was related to determining which is the desirable balance between fixed and variable revenues, considering the costs and financial structure of the company as well as the consumption uncertainty. The group that has worked in this challenge developed an optimiza-

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tion model that minimizes the maximum price increase while satisfying the recommendations of ERSAR, the Portuguese Water and Waste Services Regulation Authority, also taking into account the financial sustainability of the company. Based on that model several scenarios have been tested, characterized by sales decrease due to climate changes and also changes in the consumer's behaviour.

A second challenge posed by Infraquinta was the evaluation of water meter performance by analysing historical data from water consumption. The goal was to antecipate the need of replacing the water meters before their breakpoint. The main challenge was the decomposition of the monthly time series provided by the company into seasonality, trend and irregular components. A combination of two existing methodologies was able to detect the water meter break points which, if incorporated in a system that automatically analyses the data, might lead to automatic detection of the need of replacing the meters.

Lap2go is a timekeeping company for sports events and the challenge they presented was related to the management of road running events with thousands of participants. In this type of events it is usual to use a strategy of starting waves in order to avoid congestions and runners falling down. The goal was to come up with a model that could take into account each participant running pace, the total number of participants and, if possible, the topography and width of the road, to avoid the aforementioned problems that may be caused by a less suitable starting wave strategy. The group proposed a model based on a system of ordinary differential equations and performed some numerical simulations using also the data provided by the company for a particular race. The model is able to simulate race conditions on arbitrary tracks and can help the organizers to decide the best way to distribute runners into waves and when to release each wave.

The companies' representatives had the chance to work along with groups during the week and were quite satisfied during the final presentations due to the quality of the achieved results. A set of suggestions for future developments has been presented by the groups and some further collaboration between some of the participants and the companies has been envisioned.

The ESGI140 was funded by the COST Action TD1409, Mathematics for Industry Network (MI-NET), by the participating companies, by the research centres and institutions involved in the organization and had also the kind support of CIM.

More information about ESGI140 in www.esgi140.ips.pt.

Daniel Augusto da Silva, Poet of Mathematics

by **Carlos Florentino***

Dedicated to the memory of Jaime de Lima Mascarenhas

Daniel da Silva was a remarkable Scientist and Mathematician of the mid 19th century. Working in Portugal, isolated from the main scientific centers of the time, his investigations in pure mathematics had almost no impact. Apart from giving a short biography of his life and work, this article makes the case for considering him an unavoidable character in the History of Science and one of the founders of Discrete Mathematics, through his introduction of a key method in Enumerative Combinatorics: the *Principle of Inclusion-Exclusion*.

> *Ninguem está authorisado a capitular quaesquer theorias mathematicas como destituidas de applicação vantajosa, como um mero recreio de elevadas intelligencias, e como inuteis trabalhos em relação à verdadeira sciencia.*
>
> —Daniel da Silva

1 Introduction

A Daniel Augusto da Silva (1814–1878) is one of the greatest Portuguese scientists of the 19th Century. His contributions are remarkable, for their quality and originality, not only in the context of Portuguese Science, but also internationally. Even though his life and mathematical work is documented by historians and some researchers, his important contributions have not yet received the deserved recognition from the mathematical community.

Daniel da Silva produced only a few manuscripts of scientific nature, most of them published by the Lisbon Academy of Sciences between the years 1851 and 1876. These articles belong to the fields of Statics (more precisely, what we call today Geometric or Rational Mechanics), Physics/Chemistry, Statistics and Actuarial Sciences, and Number Theory.

The book by Francisco Gomes Teixeira (1851–1933) [T1], published in 1934, and considered to be the most important reference about History of Mathematics in Portugal up to the end of the 19th century, singles out the four most important mathematical characters, according to the author: Pedro Nunes (1502–1578), Anastácio da Cunha (1744–1787), Monteiro da Rocha (1734–1819), and Daniel da Silva. Gomes Teixeira refers to Daniel in this way:

> Daniel da Silva, poet of mathematics, searched in these sciences what they have of beautiful; [. . .] he gave to the world of numbers his Statics, without worrying with the applications of this chapter of rational mechanics, that others later did, and gave it also his beautiful investigations about binomial congruences.

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This work was partially supported by CMAFcIO of the University of Lisbon, FCT, Portugal

Figure 1.—Daniel Augusto da Silva

Figure 2.— Dom Pedro II, Brasil's emperor, docks at Lisbon's Navy Arsenal in 1876

As happened with other great names of Science, the work of Daniel da Silva did not take place without unfortunate moments and drama. Indeed, two facts played a major role: his enormous creative capabilities ended up being constrained by a major illness he suffered for several years; and the fact that he always used the Portuguese language made it very difficult for the recognition of his contributions in other European countries with much more solid and developed scientific communities.

In this short article, we are going to summarise his biography and his work, concentrating on his most important innovations in Pure Mathematics, area which represented, in his own words, his great passion (sections 2 and 3). For their relevance in today's mathematics, we finish by analyzing with further detail (section 4) da Silva's contributions to Discrete Mathematics and Number Theory, which deserve to be widely known.

Acknowledgements

I would like to thank many friends and colleagues who encouraged me to write this article. I thank especially José Francisco Rodrigues, Luís Saraiva, Helena Mascarenhas, Cristina Casimiro and Pedro J. Freitas.

2 Education and Professional Life

Daniel Augusto da Silva was born in Lisbon on the 16th of May of 1814, being the second son of Roberto José da Silva and Maria do Patrocínio. Roberto Silva was a merchant although his specific business does not seem to be documented.

Daniel's formative years took place against a difficult background of tensions and wars in Portugal. In fact, historians agree that the whole first half of the 19th century was not auspicious for the development of scientific culture in the country: this period included the Napoleon invasions, between 1807 and 1811, the violent campaigns opposing liberals and absolutists, and a civil war between 1828 and 1834.

At the age of 15, Daniel enrolled in the Royal Navy Academy (Academia Real da Marinha, ARM), and took mathematics courses ranging from Arithmetics to Calculus, as well as some courses on Mechanics, Physics and Navigation. He also attended courses at the Lisbon Royal Naval Observatory. He immediately showed special talent for mathematics and was awarded a distinction in each of the three years there. In 1832, he entered, by merit, the Royal Academy of Marine Guards (Academia Real dos Guardas-Marinhas, ARGM), an academy typically reserved for sons of officials, and was appointed Navy Officer in 1833. As he became interested in mathematics, after finishing the ARGM degree in 1835, he asked for permission and for a small fellowship, to enrol in the Mathematics Faculty of the Coimbra University (the only Portuguese University at the time). Being approved by the Navy, he moved to Coimbra, and his performance in the University was no less brilliant than in both Academies: many years later some of his old professors could still remember the brightness of da Silva as a student.

Having finished his studies in Coimbra in 1839, he immediately returned to Lisbon, and followed a career in the Navy. He was promoted to Brigadier on 1840, and, later that year, to Second Lieutenant of the Navy. Following the French trend of *Grands Écoles*, in 1845, the ARGM was transformed into the Navy School (Escola Naval), and Daniel was appointed as a professor there. He taught Mechanics; Astronomy and Optics; Artillery and Fortification, and Geography and Hydrography. Initially, he was hired as *Lente Substituto* (Assistant Professor), and became *Lente Proprietario* (Full Professor) in 1848.

There were two unfortunate moments when da Silva lost the opportunity of becoming a Professor at the blossoming Polytechnic School of Lisbon. This School had just been created in 1837, in the context of a higher education reform, by a Royal Decree, to replace the ARM. In 1839 he applied to a teaching position through a competitive process but unfortunately, for health reasons, Daniel da Silva could not be present in a kind of interview/examination. Even after justifying his absence, the panel decided to cancel the placement, afraid of a possible impugnation. The second occasion was in 1848 when the Directing Body of the Polytechnic School of Lisbon, acknowledging Daniel da Silva's value, directly asked the Government to authorise his appointment, something rejected on the grounds that, by law, all places should be filled by public competition.

Nonetheless, while in the Navy School, it is between 1848 and 1852 that Daniel da Silva experiences his first main creative period with the completion of his 3 first manuscripts, sent for publication by the Royal Lisbon Academy of Sciences (Academia Real das Ciências, ARC[1]). In 1851, he becomes a *free member* of this Society, and is elected as full member the following year.

In late 1852 his health problems became so severe, and magnified by his overwork and his great dedication to research, that he applied for a leave and went to Madeira, hoping to recover there. However, his poor health persisted and he was unable to carry out his duties until in 1859 the Naval Health Board classified him "unfit for active duty."

This same year, he was ellected *honorary member* of the ARC, and married Zefferina d'Aguiar (1825–1913) from the town of Funchal. Daniel and Zefferina had a single child, Júlio Daniel da Silva who was born in 1866. Sadly, Júlio would die at the age of 25 without descendants.

Even without teaching duties, he continued to hold his Navy position until retiring in 1868. In his latest years, worried that his passion by Pure Mathematics would worsen his health condition, Daniel continued to do research, but dedicated himself to more applied Sciences, publishing works in actuarial sciences and on the theory of the flame. In his words:

> The passion for the study of mathematics, that was in me greatly disordered by excess, many years now has been reduced to the modest proportions of a platonic love.

In 1871, the young Francisco Gomes Teixeira, a third year student in Coimbra heard his professor José Queirós mention da Silva's theory of couples in Statics with high praises, recalling his brightness as a student, more than 30 years before. An excellent mathematician himself, Teixeira became acquainted with Daniel's work, and decided to write an essay on continued fractions, a subject of Daniel's incomplete chapter 10 of [dS]. He then wrote a letter to Daniel including this essay, and this started an excellent and joyful relationship between the two. Da Silva soon invited Gomes Teixeira to become a member of the ARC and tried to get him a position in the Lisbon Astronomic Observatory. After Daniel died, Teixeira presented his eulogy to the ARC and he would become da Silva's most complete biographer [T2]. Further accounts on Daniel da Silva life and work can be found in [Di, Du, Ma, O, Sa1, Va].

3 Scientific Work

In the period 1849–51, Daniel da Silva wrote 3 manuscripts concerning investigations on geometric methods in statics of rigid bodies, and on number theory. These show that he was an avid reader of the classics, being inspired by names such as Euler (1707–83), Lagrange (1736–1813), Legendre (1752–1833), Gauss (1777–1855) and Poinsot (1777–1859). He would obtain mathematical articles published in European Academies of Sciences, especially from the one in Paris.

Daniel's first paper *On the transformation and reduction of binaries of forces* was written before 1850, but only published in 1856 by the ARC. Closely following an article of Louis Poinsot on the same theme, this article contains no original results, but presents a new treatment and some simplified proofs.

His second paper, *Memoir on the rotation of forces about their points of application*, was read to the Academy in 1850 and published the following year. Here, da Silva considers a system of forces turning around their points of application, but maintaining their relative angles during the rotation. The article was written without knowledge of the results of A. F. Möbius (1790–1868) on this subject. Möbius had incorrectly stated that if a system is in equilibrium in four different orientations, then it is in equilibrium in all possible positions. In this memoir of da Silva describes correctly the equilibrium properties of a system of forces, and

[1]Nowadays called *Academia das Ciências de Lisboa*.

also proves that, in general, there are only four equilibrium positions. The third of da Silva's memoirs, on number theory, was read to the ARC on March 1852, but his illness prevented the completion of the published version [dS] (see below). We dedicate section 4 to an exposition (of the initial part) of this article, and its important achievements. About it, Teixeira writes:

> The main subject he considered was the resolution of binomial congruences, a theory which belongs simultaneously to the domain of higher arithmetic and higher algebra, and he enriched it with such important and general results that his name deserves to be included in the list of those who founded it. It was indeed Daniel da Silva who first gave a method to solve systems of linear congruences, an honour which has been unduly attributed to the distinguished English arithmetician Henry Smith,[2] who only in 1861 dealt with this subject, and was also the one who first undertook the general study of binomial congruences.[T1]

As mentioned, da Silva's health prevented him from doing any serious mathematical research for quite some time. Only many years later, his health did improve a little and he again undertook research. This new period started in 1866, with a very short article on Statics and, shortly after, two articles on Statistics and Actuarial Sciences. Daniel proposed mathematical models of demography and applied them to the financial structuring of pension funds, and in particular to one of the oldest portuguese Welfare Institutions called Montepio Geral (founded in 1840). These two articles, *Average annual amortization of pensions in the main Portuguese Welfare Institutes*, and *Contribution to the comparative study of population dynamics in Portugal* are described in detail in a recent PhD thesis [Ma], which is also a very important source of information on Daniel da Silva, and on Actuarial Calculus and the Navy Schools, at the epoch.

In 1872, he published *On several new formulae of Analytic Geometry relative to oblique coordinate axes*, which generalizes certain well known formulae in Analytic Geometry to a setup based on non-orthogonal frames.

He also carried out studies in the area of Physics and Chemistry, that could have been motivated by his previous lectures in the Navy School. He performed several experiments, with the collaboration of António Augusto Aguiar (1838–1887), professor at the Polithecnic School of Lisbon, and studied the speed of transmission of a gas flame in its blueish and brightest part in the 1873 article *Considerations and experiments about the flame*.

In 1877, the last year of his life, Daniel received some unpleasant news from France. As he had worked in scientific isolation, and used the portuguese language, J. G. Darboux (1842–1917) had just published results very similar to Daniel's own investigations on Statics (including the same correction of Möbius' mistake) without acknowledging da Silva's work. His letter to Teixeira denotes his disappointment:

Figure 3.—Logo of *Academia das Ciências de Lisboa*

> Almost all propositions of Darboux are published twenty six years ago in the Memoirs of the Lisbon Academy of Sciences, in my work on the rotation of forces about their points of application! [. . .] My memoir, which contains many other things, besides those considered by Möbius, including a correction of one mistake he did, the same one which Darboux proudly claims correcting, lies ignored, for nearly twenty six years, in the Libraries of almost all Academies of the world! What worth it is writing in portuguese!

Daniel da Silva knew of Darboux's article [Da] in the French journal "Les Mondes", and sent to it a "Réclamation de Priorité"; this reclamation was published in this periodical on March that same year, but, as L. Saraiva writes in [Sa1]:

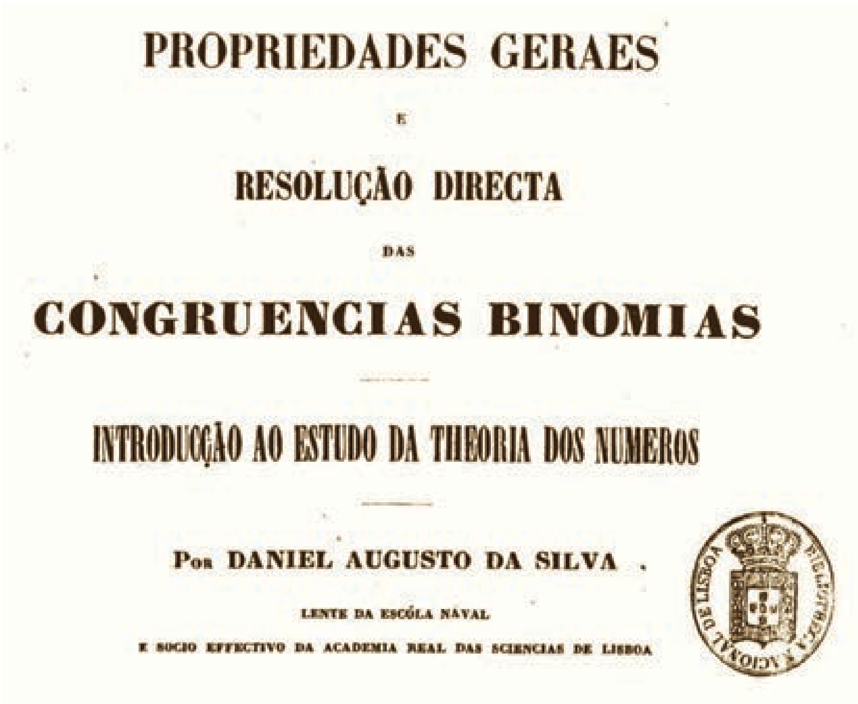
> It certainly would have been better if he had written directly to the French Academy of Sciences, where his work could have been more widely discussed.

After Daniel's death, this whole story had such a profound influence on Gomes Teixeira, that he became one of the first portuguese scientists and mathematicians to continuously promote the interaction of portuguese academics with foreign researchers. He founded the first mathematical journal, independent of any academic institution, printed in the Iberian Peninsula (Jornal de Sciencias Mathematicas e Astronomicas, see [R]) which substantially contributed to the dissemination of portuguese research. And he also stimulated the analysis of Daniel's work in the international scene, inviting some of his students and collaborators to review and to continue da Silva's work.

For a comparative analysis of the results of da Silva on Statics and those of A. F. Möbius and F. Minding (1806–1885), see F. A. Vasconcelos [Va], who was encouraged by Gomes Teixeira to perform this detailed study.

Daniel's research on the propagation of the flame was rediscovered by the German chemist and professor in Zurich, Karl Heumann (1850–1894) who, recongnizing the priority of Daniel da Silva in some of these investigations, wrote him a letter in 1878. Unfortunately, Daniel would never read it, as the letter arrived right after he passed away. For a complete list of da Silva's publications, see [DdS].

[2] *Henry J. S. Smith (1826–1883), see [Sm].*



PROPRIEDADES GERAES

E

RESOLUÇÃO DIRECTA

DAS

CONGRUENCIAS BINOMIAS

INTRODUCÇÃO AO ESTUDO DA THEORIA DOS NUMEROS

Por DANIEL AUGUSTO DA SILVA

LENTE DA ESCÓLA NAVAL

E SOCIO EFFECTIVO DA ACADEMIA REAL DAS SCIENCIAS DE LISBOA

Figure 4.—Cover page of Daniel da Silva's memoir on Congruences

4 Discrete Mathematics and Number Theory

The memoir entitled *General properties and direct resolution of binomial congruences* [dS] was presented to the Lisbon Academy of Sciences in 1852 and published two years later. There are many reasons to consider this as Daniel da Silva's masterpiece. Right in the first pages, da Silva develops a case for the importance of Pure Mathematics and its relationship with Applications, for the relevance and elegance of Number Theory, citing and praising several famous mathematicians such as Fermat (1601–65), Euler, Lagrange, Legendre, Poinsot and Gauss, instead of going directly to the results as in his other articles. Even though it includes many original results, this memoir appears also to have a pedagogical goal, as indicated by the subtitle *Introduction to the study of number theory*. This (as well as the expression *General Properties*) hints that Daniel da Silva had in mind the foundations of a whole new theory of mathematics, naturally abstract, but that could provide, in his opinion, numerous applications in many contexts.

A second reason is that a big portion of the basics of what we call today Discrete Mathematics are literally present in this work, constituting a great advance for the epoch. A typical syllabus for a freshman Discrete Mathematics course includes:

- Some Logic and Set Theory (including operations with sets (intersections, unions, etc), cardinality, examples such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$ etc);
- Basics of Number Theory (including modular arithmetic, divisibility, the Euler φ function, the theorems of Fermat/Euler, etc);
- Some Enumerative Combinatorics (including the binomial formula, the principle of inclusion-exclusion, generating functions, etc).

It is remarkable that da Silva's memoir from 1854 treats, in a clear, elegant and modern way, most of the above list of topics and subtopics. Moreover, some of the methods used bear a striking coincidence with those of textbooks for first year courses of Discrete/Finite Mathematics adopted nowadays in Colleges and Universities around the world. This happens, e.g, in Daniel's proof of the Euler's formula for $\varphi(n)$, the function that counts the number of positive integers less than $n \in \mathbb{N}$ and prime with it (see below).

A final argument for considering [dS] as Daniel's masterpiece are the original results and their relevance today. Indeed, two very important results introduced here are unanimously attributed to da Silva: the famous Principle of Inclusion-Exclusion, and a formula for congruences that generalises the well-known formula of Euler involving his φ function. Let us recall this material, from a modern perspective, and compare with the way Daniel introduces it.

4.1 The Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion (abbreviated PIE) is one of the essential counting methods in Combinatorics, allowing a multitude of applications. It is ubiqui-

De (7) conclue-se

$$(8)\quad \begin{cases} {}^{b,a}S = {}^{b}\,{}^{a}S = S\,[1-{}_a]\,[1-{}_b] \\ {}^{c,b,a}S = {}^{c}\,{}^{b,a}S = S\,[1-{}_a]\,[1-{}_b]\,[1-{}_c] \end{cases}$$

isto é, em geral

$$(9)\quad {}^{\cdots,c,b,a}S = S\,[1-{}_a]\,[1-{}_b]\,[1-{}_c]\cdots$$

entendendo-se sempre que os productos dos numeros *a*, *b*, *c*, etc. passam a indices compostos das series respectivas, e que qualquer indice com-

Figure 5.—Part of page 10 of [dS] where Daniel's symbolic formula is written

tous in most books dedicated to this area.[3] The PIE generalises the formula for the cardinality of the union of two finite sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where $|A|$ denotes the cardinality of A. This is also widely used in Probability or Measure Theory since, with appropriate interpretation, we can replace cardinality by probability or by measure. To state the PIE in modern terms, for $i = 1, \cdots, N$ ($N \in \mathbb{N}$) consider finite subsets $A_1, \cdots, A_N$ of a given finite set X, and let $A = \cup_{i=1}^{N} A_i$ be their union inside X, then the PIE is the formula:

$$|A| = \sum_i |A_j| - \sum_{i<j} |A_{ij}| + \sum_{i<j<k} |A_{ijk}| - \cdots + (-1)^{N-1}|A_{12\cdots N}| \quad (1)$$

where $A_{ij} := A_i \cap A_j$, $A_{ijk} := A_i \cap A_j \cap A_k$, etc. Equivalently, the PIE determines the cardinality of the complement $A^c := X \setminus A$ of A, as:

$$\begin{aligned} |A^c| &= |X| - |A| = \\ &= |X| - \textstyle\sum_i |A_i| + \sum_{i<j} |A_{ij}| + \cdots + (-1)^N |A_{12\cdots N}| \end{aligned} \quad (2)$$

For us, the most interesting aspect of da Silva's proof of (1) is that it requires the introduction of the *notion of set*, at least of a finite one, a couple of decades before the foundations of set theory laid by George Cantor (1845–1918)! Even though Daniel believes the concept of set is of major importance, he refers to this just as a convenient "notation". In da Silva's own words:

> To prove this formula we will employ a notation, that may advantageously serve in other cases. Suppose that in a sequence S of numbers (that we consider *united* and not *summed*, since if even some of them could be negative, there wouldn't result any *subtraction*) one asks which are the ones that satisfy some property a; we will denote by S_a the *reunion* of those numbers; Similarly S_b, $S_{b,c}$, $S_{a,b,c}$ etc, the reunion of those terms of S verifying property b, or simultaneously the properties b, c, etc.[4]

using also the notation ${}^{\cdots,c,b,a}S$ for the elements of S that do not possess any of the properties $a, b, c, \ldots$, da Silva's presents his *symbolic formula* as follows:

$${}^{\cdots,c,b,a}S = S[1-{}_a][1-{}_b][1-{}_c]\cdots \quad (3)$$

and he clarifies that, on the rigth hand side, "in such a product the letters a, b etc., become indices, and any composed index such as a_{b_c} becomes a simple index a, b, c" *since* "it is easy to see that $S_{a_{b_c}} = S_{a,b,c}$" and similarly for upper left indices.

Daniel continues, using ψ for the cardinality of a set:

> The same formula also immediately gives us the number of numbers contained in ${}^{\cdots,c,b,a}S$; denoting this number by $\psi^{\cdots,c,b,a}S$, and letting

[3]One standard textbook [St] explicitly mentions the PIE more that 50 times.

[4]We reproduced here Daniel's own emphasis in the 4 words: *united*, *summed*, *subtraction* and *reunion*.

the sign ψ have an analogous meaning applied to the additive and subtractive series in 3 it is clear that we will have:

$$\psi^{\cdots,c,b,a}S = \psi S[1-_a][1-_b][1-_c]\cdots. \tag{4}$$

One of the most common modern proofs of PIE uses the characteristic function of a subset $A \subset X$, defined by:

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

It is easy to see that characteristic functions are multiplicative for intersections, and additive for disjoint unions:

$$\chi_{A\cap B} = \chi_A \chi_B, \quad \chi_{A\sqcup B} = \chi_A + \chi_B.$$

This implies, upon observing that $A^c = \cap_{i=1}^N A_i^c$, when $A = \cup_{i=1}^N A_i$:

$$\begin{aligned} \chi_{A^c} &= \prod_{i=1}^N \chi_{A_i^c} = \prod_{i=1}^N (\mathbb{1} - \chi_{A_i}) = \\ &= \mathbb{1} - \sum_i \chi_{A_i} + \sum_{i<j} \chi_{A_{ij}} - \cdots + (-1)^N A_{12\cdots N}, \end{aligned} \tag{5}$$

where $\mathbb{1} = \chi_X$ (the constant function 1 defined on X). Noting that cardinality is just given by summing over X (recall this is a finite set):

$$|A| = \sum_{x\in X} \chi_A(x),$$

equation (5) is transformed into the PIE, in the form (2). By replacing a with χ_{A_1}, b with χ_{A_2} etc., the similarity of this proof with Daniel's formulae (3) and (4) is manifest!

Next, Daniel applies his formula (4) to deduce the formula of Euler

$$\varphi(n) = n\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\cdots\left(1-\frac{1}{c}\right), \tag{6}$$

when $n = a^\alpha b^\beta \cdots c^\gamma$ is the prime factorization of $n \in \mathbb{N}$, and this is done in *the same way* as in many modern books: see, for example, the standard textbook [Bi], from 2005, that derives the formula (6) precisely as da Silva does, using PIE.

4.2 Euler's Theorem and Bézout identity

Euler used the function φ in his celebrated theorem:

$$a^{\varphi(n)} \equiv 1 \pmod n, \tag{7}$$

for $a \in \mathbb{Z}$ relatively prime to $n \in \mathbb{N}$. This formula generalizes Fermat's theorem:

$$a^{p-1} \equiv 1 \pmod p,$$

for p a prime and a not multiple of p.

The following elegant and interesting generalization of (7) was proved by Daniel da Silva. Let $n = a_1 a_2 \cdots a_k$ where all factors ($k > 1$ in number) are pairwise relatively prime. Then:

$$a_1^{\varphi(n/a_1)} + a_2^{\varphi(n/a_2)} + \cdots + a_k^{\varphi(n/a_k)} \equiv k-1 \pmod n. \tag{8}$$

The proof can be found in the original memoir of Daniel da Silva [dS] or in [CRS, p. 211].

The particular case $k = 2$ of this formula connects beautifully with the Bézout identity. This identity (in a simplified form) states that, given two relatively prime natural numbers $a, b \in \mathbb{N}$, there are solutions $x, y \in \mathbb{Z}$ to:

$$ax + by = 1,$$

and it is well known that this can be solved by an ancestral method: the (extended) Euclidean algorithm. From (8), we see that Daniel's formula provides a direct solution:

$$x = a^{\varphi(b)-1}, \quad y = b^{\varphi(a)-1},$$

for the "congruence version" of Bézout's identity:

$$ax + by \equiv 1 \pmod{ab}$$

(under the same assumptions on a, b).

The monograph goes on with many interesting applications of these formulas and related questions. Among these, Daniel provides direct resolutions for linear congruences, for the chinese remainder theorem, and for many congruences of the form

$$ax^n \equiv b \pmod N.$$

Daniel's health problems intensified as he was approaching the end of his monograph: he wasn't able to revise neither the preface nor the final part (see [Sa2]). For the same reason, the last two chapters are incomplete: in the 9th some theorems he would like to add are missing; and sections 4 and 5 of the 10th chapter have only their title. The last section was supposed to be a study of continued fractions, the theme of Teixeira's first letter to Daniel, 20 years later.

Without any impact whatsoever at the time it was written, this memoir was discovered almost by accident, half a century later, by the italian mathematician Cristoforo Alasia (1864–1918) who, suprised by its depth, dedicated three articles to Daniel's work between 1903 and 1914 (all in italian, the first being [A]). However, as far as we know, only one conference proceedings (written in portuguese) has addressed the concept of set in da Silva's work [dC].[5]

We finish with a quote from [dS], a wonderful illustration of Daniel da Silva's passion for mathematics and his opinion on the importance of pure mathematics and its role in science:

[5] I thank J. F. Rodrigues for the indication of this reference.

In general, it can be said that nobody is authorized to capitulate any mathematical theory as deprived from advantageous applications, as a mere recreation of elevated minds, and as useless work towards true science. All acquired truths are that many elements of acumulated intelectual wealth. Soon or late, the day will come when concrete science will have to search within this vast arsenal the necessary tools for great discoveries, which in this way will pass from speculative theorems to the category of practical truths. Every day, one or another area of mathematical-physics or celestial or industrial mechanics, is observed to suddenly stop its development to implore assistance from the further improvements in pure analysis, without which those very important sciences cannot progress.

Daniel da Silva could have hardly guessed that this phrase would be, more than a century later, so appropriate to the subject of his own article. In fact, the algorithms that preserve the security of data across the internet, such as the famous RSA (Rivest–Shamir–Adleman) criptographic system that we use (even without noticing) on a daily basis, depend crucially on Euler's formula (7).

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ECMTB2018

LISBON

11th European Conference on Mathematical and Theoretical Biology

by **Maíra Aguiar***; **Carlos Braumann****; and **Nico Stollenwerk**

Biomathematics is a fast growing subject in Europe and to celebrate the importance of applications of mathematics to biology and life sciences, the Year of Mathematical Biology 2018 (YMB) was declared, a joint initiative of the European Mathematical Society (EMS) and the European Society for Mathematical and Theoretical Biology (ESMTB). With many events organized, including thematic programs and conferences and workshops, the 11th European Conference on Mathematical and Theoretical Biology (ECMTB 2018) (http://www.ecmtb2018.org) was the main event of the YMB.

Traditionally organized by the ESMTB, the ECMTB 2018 was this time (and for the first time) also organized by the EMS, having the Portuguese Mathematical Society (SPM) as co-organizer. The conference venue was the Faculty of Sciences of the University of Lisbon, in Portugal (FCUL), hosted by its research centre, Centro de Matemática, Aplicações Fundamentais e Investigação Operacional (CMAFcIO–Centre for Mathematics, Fundamental Applications and Operational Research). The ECMTB 2018 had the patronage of His Excellency the President of the Republic of Portugal and was granted the UNESCO seal by the National UNESCO Committee. Besides the sponsorship of the three organizing societies, it has been sponsored by several Portuguese research centres (CMAFcIO, CMA, CIMA, CEAUL) and the Portuguese Foundation for Science and Technology (FCT), Instituto Gulbenkian de Ciência, international publishers (Springer, MDPI, PLOS One, Elsevier, EMS-PH, IOP Publishing. Oxford University Press, Wiley), the Bernoulli Society, the Portuguese Statistical Society, the Centro Internacional de Matemática and other organizations and companies.

From July 23 (July 22 for those attending the early registration and cocktail welcoming party) to July 27, the city of Lisbon (Portugal) welcomed the more than seven hundred participants from 80 countries with a pleasant weather (it was cooler than the usual hot weather typical of this time of the year, a blessing for participants from Central and Northern Europe coming from unusually high temperatures in their home countries). The record number of participants (only beaten by joint ESMTB- SMB Conferences) is a sure sign of the growing importance of Mathematical Biology.

After the opening ceremony, there was a Tribute to Karl Peter Hadeler by Odo Diekmann, immediately followed by the opening plenary conference, a Bernoulli Society-European Mathematical Society Joint Lecture, by Samuel Kou (Harvard University, USA), on the exciting topic of *Big data, Google and disease detection: A statistical adventure*. The other plenary conferences were on equally exciting topics and were given by the eminent scientists Helen Byrne (Oxford University, UK, *Mathematical approaches to modelling*

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and remodelling biological tissues), Antonio DeSimone (SISSA, Italy, *Biological and bio-inspired motility at microscopic scales: locomotion by shape control*), Eva Kisdi (University of Helsinlki, Finland, *Adaptive dynamics and the evolution of diversity*), Mirjam Kretzschmar (University Medical Centre Utrecht, The Netherlands, *Modelling the waning and boosting of immunity*), Eva Löcherbach (Cergy-Pontoise University, France, *Modeling interacting networks of neurons as processes with variable length*), Andrea Pugliese (University of Trento, Italy, *Epidemic models structured by parasite load and immune level*), Eörs Szathmáry (Eötvös Loránd University, Hungary, *Models of learning and evolution: what do they have in common?*) and Kees Weijer (University of Dundee, UK, *Analysis of collective cell behaviours underlying primitive streak formation in the chick embryo*). One of the recent winners of the Reinhart Heinrich Best Ph. D. Thesis Award, Jochen Kursawe, was able to come and give the traditional winner talk on *Quantitative approaches to investigating epithelial morphogenesis*.

ECMTB 2018 had also a very rich programme of 36 cutting-edge Mini-symposia, 60 Contributed Talk parallel sessions and 2 Poster sessions (in a nice get together coffee+cocktail break), totalling 455 oral communications and 119 posters covering all areas of Mathematical and Theoretical Biology. There were 4 poster prizes given by a jury and sponsored by publishers (Elsevier, MDPI, Springer). The reader can take a look at the Programme (http://www.ecmtb2018.org/files/files/ecmtb_booklet_a4.pdf or the short version http://www.ecmtb2018.org/ActMap) and the Book of Abstracts with ISBN: http://www.ecmtb2018.org/files/files/BookOfAbstracts_ECMTB2018_inclusions.pdf

Moreover, the ECMTB Mentorship Programme was set up to facilitate research and career interactions between junior and more senior scientists attending the meeting.

The General Assembly of the ESMTB took place on July 26 and it was opened to members and non-members of the Society. Several issues concerning the working of the ESMTB and the development of Mathematical and Theoretical Biology were discussed, followed by further discussions over a wine tasting event.

The social programme provided ample opportunities for scientific exchange and personal contacts. Besides the already mentioned social activities and the coffee and the lunch breaks, there were excursions and a conference dinner that started with a Tuna (which is a not a fish, but a typical Portuguese University playing and singing student group) and was followed by dancing.

The ESMTB, to celebrate the Year of Mathematical Biology 2018 and wishing to extend its membership to other researchers in the area, invited every participant registered for the ECMTB 2018, that was not yet an ESMTB member, to become a member. The Society welcomes those accepting such invitation by exempting them of the first year membership fee. Note that the invitation is still standing (see detailed information on http://dev.ecmtb2018.org/RegRules).

On behalf of the Organizing Committee, we thank the organizing societies for their trust, the Scientific Committee, the sponsors, the plenary speakers, the organizers of the Mini-symposia, the session chairs, the mentors and mentees, the jury of the poster prizes, the student helpers and the hard-working and skilful members of the Secretariat (Ana Rita Ferrer, Ana Isabel Figueiredo, Joana Guia). We are especially grateful to all the participants, for whom this Conference was organized, for having made it a memorable event and a landmark in the growing path of Mathematical and Theoretical Biology.

On the Mathematics of Populations

by **M. Gabriela M. Gomes***

Selection acting on unobserved heterogeneity is a fundamental issue in the mathematics of populations. As recognised in disciplines as diverse as demography [11, 23, 24], ecology [10, 9], evolution [21] and epidemiology [1, 4, 5], in any population, individuals differ in many characteristics and it is essential that researchers understand which of these are under selection and how selection processes operate. Here I describe conceptual and methodological developments in demography and ecology, and discuss the importance of adopting similar approaches in evolution and epidemiology.

1 Demography

Unobserved heterogeneity can result from many interacting genetic and environmental factors. When operating on individual survival, it modifies the composition of cohorts as they age [11, 23]. Frail individuals die younger, leaving the cohort progressively composed of those who are more robust and have a propensity to live longer. This form of selection acting on longevity operates within cohorts and distorts patterns of age-specific survival [24]. It may also affect other life-history traits via correlations with longevity. For example, if those individuals who live longer have lower fecundity, then selection operating on individual longevity will reduce fecundity at the population level.

2 Ecology

Demographic heterogeneity has been contrasted with demographic stochasticity in ecology [10, 9]. Demographic heterogeneity, defined as variation among individuals in life-history characteristics, has been addressed in ecology by structured population models [3, 19]. The approach consists of incorporating the most important differences into the individual state of a matrix model. A major challenge is to reconcile those individual differences that matter (given the question under study) with those that can actually be measured. A natural tendency is to account for characteristics that can be measured most easily, such as age, size, or major developmental states, and collapse other important sources such as genetic variation, spatial heterogeneity in the environment, maternal effects, and differential exposure to stressors, under some form of unmeasured stochasticity. Recent research has elucidated, however, that demographic heterogeneity and demographic stochasticity have opposing effects in population dynamics and cannot be modelled interchangeably [3, 19].

3 Evolution

Recent research has begun to emphasise the importance of considering individual variation in non-heritable fitness components when interpreting the results of evolutionary studies [21]. By accommodating explicitly for individual variation in non-heritable fitness components, we have shown how common proxies for genotype fitness may be affected by a form of selection that is invisible to evolution and how this may explain observed trends when fitness is measured across stress gradients [6]. We then propose that unaccounted phenotypic variation within genotypes is capable of stabilising coexistence of multiple lineages and unexpectedly affect patterns of genetic variation, especially when levels of stress fluctuate.

Understanding the selective forces that shape variation is at the heart of evolutionary theory. Selection acting on non-heritable characteristics has not been given full attention, either because it is not directly measurable or because it was believed to be inconsequential for evolution. In [6] we oppose these commodities and argue that selection on

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non-heritable fitness components is essential both as a natural step in theory development and as a necessity for accurate interpretation of population and quantitative genetic data. More practically, we propose concrete experimental designs across stress gradients for informing models of variation and selection.

4 Epidemiology

Variation in individual characteristics has a generally recognised impact on patterns of occurrences in populations, and occurrence of disease is no exception. In infectious diseases, the focus has been on heterogeneities in transmission through their nonlinear effects on the basic reproduction number, R_0, in ways which are unique to these systems [15, 25]. The need to account for heterogeneity in disease risk, however, is not unfamiliar in epidemiology at large, where frailty terms are more generally included in linear models to improve data interpretation [1].

The premise is that variation in the risk of disease (whether infectious or not) goes beyond what is captured by measured factors, and a distribution of unobserved heterogeneity can be inferred from incidence patterns in a holistic manner. Such distributions are needed for eliminating biases in interpretation and prediction, and can be utilised in conjunction with more common reductionist approaches, which are required when there is desire to target interventions at individuals with specific characteristics.

To contrast different forms of heterogeneity, we consider four versions of a *Susceptible-Infected* (*SI*) model in a population of constant size.

Homogeneous *SI* model:

$$\frac{dS}{dt} = \mu - \beta I S - \mu S \tag{1}$$

$$\frac{dI}{dt} = \beta I S - \mu I, \tag{2}$$

where S and I are the proportions of susceptible (uninfected) and infected (and infectious) individuals, respectively, β is a transmission coefficient (effective contact rate) and μ accounts for birth and death. The force of infection upon uninfected individuals is $\lambda = \beta I$ and the basic reproduction number is:

$$R_0 = \frac{\beta}{\mu}. \tag{3}$$

SI model with heterogeneous susceptibility:

$$\frac{dS(x)}{dt} = q(x)\mu - \beta \int_0^\infty I(u)du\, x S(x) - \mu S(x) \tag{4}$$

$$\frac{dI(x)}{dt} = \beta \int_0^\infty I(u)du\, x S(x) - \mu I(x), \tag{5}$$

where x is an individual susceptibility factor taking values on a continuum, $q(x)$ is a probability density function, $S(x)$ and $I(x)$ represent the densities of susceptible and infected individuals, respectively, and β and μ are parameters governing transmission and demographic processes as before. The force of infection upon the average uninfected individual is $\lambda = \beta \int_0^\infty I(u)du$, which is then affected by a factor x to conform with individual susceptibilities. The basic reproduction number for this system is:

$$R_0 = \frac{\langle x \rangle \beta}{\mu}, \tag{6}$$

where $\langle x \rangle$ is the first moment of the susceptibility distribution (or mean susceptibility, i.e. $\langle x \rangle = \int_0^\infty x q(x) dx$). All model solutions presented here assume $\langle x \rangle = 1$, in which case the R_0 expression of the heterogeneous susceptibility model coincides with the homogeneous.

Figure 1 shows the prevalence of infection over time as governed by models (1)-(2) and (4)-(5), as well as density plots (frequency in the homogeneous case) for the susceptible and infected compartments at three time points: before the start of the epidemic; at the endemic equilibrium corresponding to 20% prevalence; after 100 years of control. The control measure simulated in Figure 1B is the 90-90-90 treatment as prevention cascade advocated by UNAIDS for HIV, which stipulates that 90% of infected individuals are detected, 90% of those detected enter antiretroviral therapy, and 90% of those entering treatment achieve viral suppression, becoming effectively non-infectious. As a result of this cascade, transmission is reduced by approximately 73%.

In comparison with the homogeneous model, a higher R_0 is required to reach the same endemic level, and the same control measure has lower impact under heterogeneity (this is irrespective of the R_0 expressions (3) and (6) being the same). This is evidenced by comparison of the solid (homogeneous) and dashed (heterogeneous) trajectories, and explained by the density plots. As the infection spreads in the population, more susceptible individuals are likely to be infected earlier (red distributions, with higher mean - red dotted lines) and, consequently, those who remain uninfected are being selected for lower susceptibility (blue distributions, with lower mean - white dotted lines). This process slows down the epidemic since the mean susceptibility among those at risk effectively decreases as the epidemic progresses. As a result, the heterogeneous model requires higher values of R_0 to attain the same endemic level as its homogeneous counterpart, and becomes more resilient to interventions designed to reduce transmission.

Figure 1 was generated assuming a gamma distribution with mean 1 and variance 10 for the heterogeneous susceptibility model, but the effects described above are generally manifested with a strength that increases

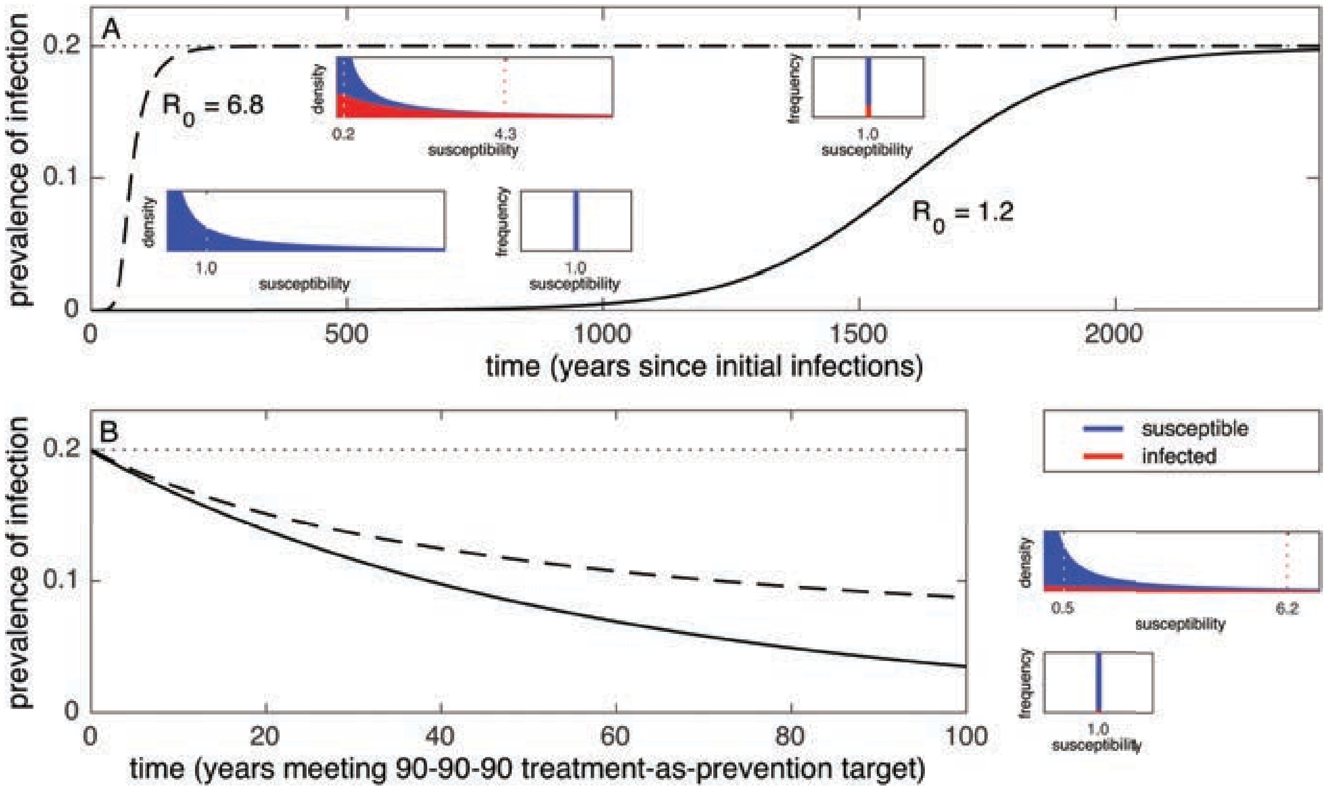

Figure 1.—Prevalence trajectories under models (1)–(2) and (4)–(5). Two susceptibility distributions are simulated: homogeneous (solid black trajectories); gamma distributed susceptibility to infection (dashed black trajectories). Insets depict the gamma distribution with mean 1 and variance 10 utilised in the heterogeneous implementation side-by-side with its variance 0 counterpart implicit in the homogeneous model. The vertical white and red dotted lines added to the gamma distribution plots mark mean values for the susceptible (blue) and infected (red) subpopulations, respectivelly.

with the variance (or coefficient of variation).

SI model with heterogeneous infectiousness:

$$\frac{dS(x)}{dt} = q(x)\mu - \beta\frac{\int_0^\infty uI(u)du}{\langle u\rangle}S(x) - \mu S(x) \quad (7)$$

$$\frac{dI(x)}{dt} = \beta\frac{\int_0^\infty uI(u)du}{\langle u\rangle}S(x) - \mu I(x), \quad (8)$$

where x is an individual infectiousness factor taking values on a continuum, and $q(x)$, $S(x)$, $I(x)$, β and μ represent densities and parameters as before. The force of infection upon uninfected individuals is $\lambda = \beta \int_0^\infty uI(u)du/\langle u\rangle$, and the basic reproduction number is the same as in the homogeneous implementation.

SI model with heterogeneous connectedess:

$$\frac{dS(x)}{dt} = q(x)\mu - \beta\frac{\int_0^\infty uI(u)du}{\langle u\rangle}xS(x) - \mu S(x) \quad (9)$$

$$\frac{dI(x)}{dt} = \beta\frac{\int_0^\infty uI(u)du}{\langle u\rangle}xS(x) - \mu I(x), \quad (10)$$

where x is again a factor taking values on a continuum but now affecting both dispositions for acquiring infection and for infecting others, due to variable connectedess. Everything else is defined as before, and the basic reproduction number is:

$$R_0 = \frac{\langle x^2\rangle\beta}{\langle x\rangle\mu}, \quad (11)$$

where $\langle x^2\rangle$ is the second moment of the connectedess distribution (i.e. $\langle x^2\rangle = \int_0^\infty x^2q(x)dx$). The expression for R_0 arising from this model is typically different from the other implementations, even when $\langle x\rangle = 1$.

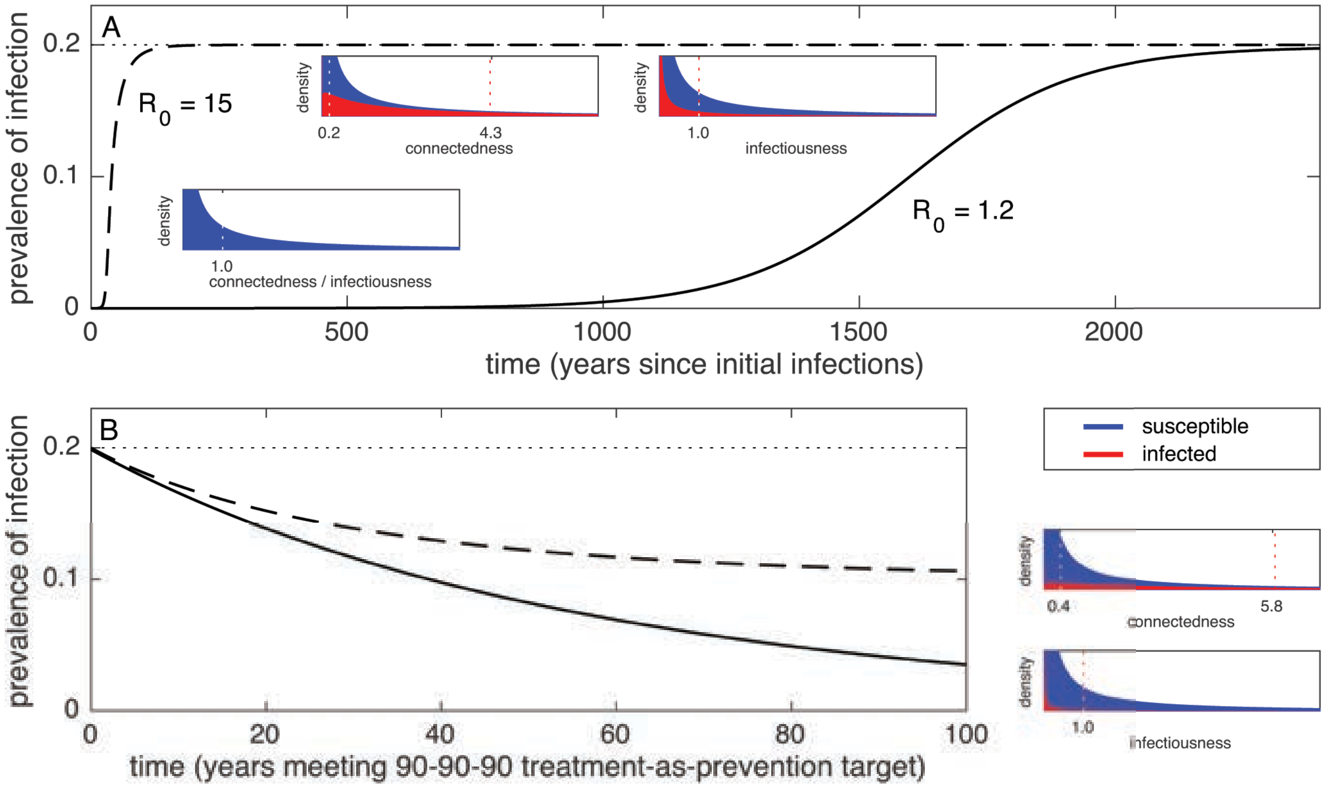

Figure 2.—Prevalence trajectories under models (7)–(8) and (9)–(10). Two distribution implementations are simulated: gamma distributed infectiousness (solid black trajectories); gamma distributed connectedess (dashed black trajectories). Insets depict the gamma distribution with mean 1 and variance 10 utilised in the distributed connectedess implementation side-by-side with the distributed infectiousness implementation. The vertical white and red dotted lines added to the gamma distribution plots mark mean values for the susceptible (blue) and infected (red) subpopulations, respectivelly.

Figure 2 shows the prevalence of infection over time as governed by models (7)-(8) and (9)-(10), accompanied by density plots for the susceptible and infected compartments before the start of the epidemic, at the 20% prevalence equilibrium, and after 100 years of control as above.

The model with heterogeneity in infectiousness produces identical outputs to the homogeneous model for the same parameter values because infectiousness is not under selection by the force of infection. Heterogeneity in connectedness implies a positive correlation between the dispositions to acquire infection and transmit to others. Effectively, this results in infectiousness being selected indirectly via acquisition of infection, enhancing the effects observed under heterogeneous susceptibility alone.

It is evident from this exercise that knowing the extent of variation in susceptibility present in a population is essential if models are to be predictive. Variation in infectiousness does not affect predictability unless it is correlated with susceptibility, such as in the case of heterogeneity in connectedness. Susceptibility involves a probability of response to a stimulus (i.e. become infected given a pathogen challenge) and therefore cannot be measured directly. This obstacle, which may be part of the reason behind the widespread adoption of homogeneous models, is starting to be overcome by specific study designs that recognise the need for unpacking exposure gradients [7, 20, 12, 5, 17, 13, 8] as explicit experimental or observational dimensions.

Co-circulation of multiple pathogen lineages:

Many pathogens appear structured into multiple genetic lineages which are simultaneously maintained within host populations.

Mathematical models, typically tied to lineages being homogeneous static entities [14], have invoked interactions

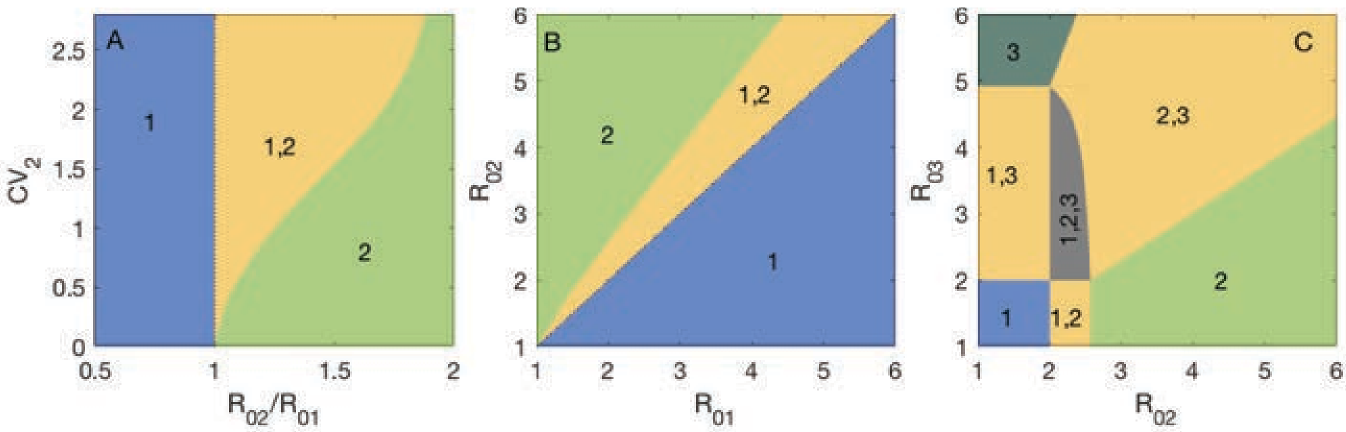

Figure 3.—Stable coexistence of microbial lineages colonising a host population. Model (14)–(15) was solved analytically with two (A, B) and three (C) lineages. Yellow regions represent conditions for 2-lineage stable coexistence as indicated, while 3-lineage coexistence is found in the grey zone (C). Other parameters: (A,B,C) CV_1=0; (B,C) CV_2=1; (C) CV_3=2 and R_{01}=2.

between strains for stabilising coexistence. Here I argue that unobserved host heterogeneity in susceptibility within lineages alone can stabilise coexistence of multiple lineages.

Consider a discretised version of the heterogeneous susceptibility SI model:

$$\frac{dS_i}{dt} = q_i\mu - \beta I x_i S_i - \mu S_i \quad (12)$$

$$\frac{dI}{dt} = \beta I x_i S_i - \mu I, \quad (13)$$

where x_i, for $i = 1, \dots, n$, are the susceptibility factors of hosts S_i that enter the system as a fraction q_i of all births, purporting a distribution with mean $\langle x\rangle = \sum_i q_i x_i = 1$, variance $\langle (x-1)^2\rangle = \sum_i q_i(x_i-1)^2$, and coefficient of variation $CV = \sqrt{\langle (x-1)^2\rangle}$, which will be treated as a varying parameter. The basic reproduction number is $R_0 = \beta/\mu$.

The extension of the model to N pathogen lineages, each characterised by an independent susceptibility distribution, is straightforward although the notation becomes cumbersome:

$$\frac{dS_{i_1 i_2 \cdots i_N}}{dt} = \prod_{j=1}^{N} q_{j i_j}\mu - \sum_{j=1}^{N} \beta_j I_j x_{j i_j} S_{i_1 i_2 \cdots i_N} - \mu S_{i_1 i_2 \cdots i_N} \quad (14)$$

$$\frac{dI_j}{dt} = \beta_j I_j \sum_{i_j=1}^{n} x_{j i_j} \sum_{i_1=1}^{n} \cdots \sum_{i_{j-1}=1}^{n} \sum_{i_{j+1}=1}^{n} \cdots \sum_{i_N=1}^{n} S_{i_1 i_2 \cdots i_N} - \mu I_j. \quad (15)$$

where β_j, for $j = 1, \dots, N$, is the effective contact rate between hosts infected by species j and susceptible hosts, $x_{j i_j}$, for $i_j = 1, \dots, n_j$, are the susceptibility factors of hosts $S_{\dots i_j \dots}$, who enter the system as fractions $q_{j i_j}$ of all births, purporting distributions with mean $\langle x_j\rangle = \sum_{i_j} q_{j i_j} x_{j i_j} = 1$, variance $\langle (x_j-1)^2\rangle = \sum_{i_j} q_{j i_j}(x_{j i_j}-1)^2$, and coefficients of variation $CV_j = \sqrt{\langle (x_j-1)^2\rangle}$ treated as varying parameters. The lineage-specific basic reproduction numbers are $R_{0j} = \beta_j/\mu$. This system accommodates an N-lineage coexistence region with all $R_{0j} > 1$. This region has a simple geometry in the R_{0j} space. In the special case where the host population is homogeneously susceptible to lineage 1, it is bounded by the hyperplanes $R_{0j} = R_{01}$, for $j = 2, \dots, N$, and by a hypersurface that can be obtained by setting to zero the abundance of 1 (shown in Figure 3 for 2 and 3 lineages). This coexistence region persists when we allow for heterogeneous susceptibility to lineage 1 as well.

5 Outlook

Selection within cohorts is ubiquitous in living systems, with manifold manifestations in any study that involves counting the individuals that constitute a population over time, across environments or experimental conditions. Whether we refer to populations of animals, microbes, or cells, the idea that in every observational or experimental study there is always a degree of unobserved heterogeneity that can reverse the direction of our conclusions is unsettling, but the issue can be tackled by general mathematical formalisms that account for it [1, 4, 10, 11, 16, 17, 22, 23] combined with study designs that enable its estimation [7, 8, 9, 12, 13, 20]. Collectively, the phenomenon appears to explain a wide range of reported discrepancies between studies and contribute to resolve decade-long debates, such as why vaccines appear less efficacious where disease burdens are high [5] and whether niche mechanisms need to be invoked to explain the levels of biodiversity observed in nature [6].

In addition to its omnipresence in studies that deal with populations explicitly, selection within cohorts may also play a central part in much debated issues surrounding accuracy and reproducibility of biological results more generally. Among aspects of research reproducibility discussed in the literature, those that pertain to methodology have focussed on how sample sizes must be sufficiently large to ensure a satisfactory level of certainty on the conclusions [2] and how shuffling is necessary to randomise conditions [18]. Additional problems, however, may result from overlooking forms of selection that may be occurring throughout the experiment. Any count of responses to a stimulus is affected by selection bias (unless all individuals, or cells, or other units, in the experiment were perfectly identical, which they are not). This is particularly problematic when different treatments are setup with the intent of comparing how differently individuals respond in one experimental condition vs another. Since the levels of selection bias will generally differ between treatments, comparisons of direct counts are not accurate representations of how differently individuals respond. Similar arguments apply to comparisons between different runs of entire experiments and compromise reproducibility. This is a problem of accuracy which cannot be resolved by increasing sample size or randomisation, but rather by adding dimensions to the experimental design [7]. This methodological issue can induce not only quantitative deviations in experimental results, but also invert the conclusions altogether [12].

This paper conveys how a wide variety of phenomena can be alternatively described by population thinking (individuals are different and selection operates) or individual thinking (all individuals are average and additional processes must be invoked) reaching contrasting conclusions. It is thus imperative to understand which frame is most appropriate is each case, and this implies understanding which individual characteristics may be subject to selection and how to obtain realistic descriptions of their variability in a population. The concerns presented here are pervasive across life and social sciences and can only be tackled in tight alliance with the mathematical sciences.

Acknowledgments

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REPORT

LxDS Spring School 2018

by João Lopes Dias*; José Pedro Gavião**; Pedro Miguel Duarte***; and Telmo Peixe****

The LxDS-Lisbon Dynamical Systems Group, the Department of Mathematics of ISEG, CEMAPRE, REM and CMAFCIO organised a spring school on the days 28-30 of May in dynamical systems, which took place at ISEG / ULisboa. The school consisted of three mini-courses in dynamical systems, which were given by specialists of recognised international merit. Namely,

- KAM theory for ultra-differentiable HamiltoniansAbed Bounemoura, Université Paris Dauphine,
- SRB measures and inducing schemesJosé Ferreira Alves, University of Porto,
- Dynamics of geodesic flowsMark Pollicott, University of Warwick.

The school had 27 participants, two of whom were international PhD students. In addition to the mini-courses the school also had a session of brief oral presentations, in which some of the PhD students presented their most recent work. There was also a poster session.Due to financial support provided by CIM, the organisation managed to support the participation of six PhD students (4 national and 2 international), covering their lodging and meals during the school days.

* ULisboa, ISEG, CEMAPRE; ** ULisboa, ISEG, CEMAPRE; *** ULisboa, FCUL, CMAFCIO;
**** ULisboa, ISEG, CEMAPRE

Dynamics with Coquaternions

by **Maria Irene Falcão***; **Fernando Miranda****; **Ricardo Severino*****; and **Maria Joana Soares******

1 Introduction

Coquaternions, also known in the literature as split quaternions, are elements of a four-dimensional hypercomplex real algebra generalising complex numbers. This algebra was introduced in 1849 by the English mathematician James Cockle [4], only six years after the famous discovery by Hamilton of the algebra of quaternions [12].

Although coquaternions are not as popular as quaternions, in recent years one can observe an emerging interest among mathematicians and physicists on the study of these hypercomplex numbers. In fact, they have been considered in several papers by different authors and various applications have been developed; see e.g. [1, 3, 6, 7, 8, 9, 10, 11, 16, 17, 18, 21].

The dynamics of the quadratic map in the complex plane has been intensively studied in the last decades and can now be considered a well-established theory. This map exhibits a rich dynamical behaviour and has given birth to extraordinarily beautiful pictures which have passed into the popular domain.

In this note, we give a first insight into the world of coquaternions, reflecting the recent interests of the authors. In particular, we recall some results on the zeros of coquaternionic polynomials [8] and discuss several aspects of the dynamics of one family of quadratic maps on coquaternions [6].

The nature of the algebra under consideration leads to results which can be considered as even richer and more interesting than the ones obtained in the complex or quaternionic cases.

2 The algebra of coquaternions

2.1 Basic results

The algebra of real coquaternions is an associative but non-commutative algebra over $\mathbb{R}$ defined as the set $\mathbb{H}_{\rm coq} = \{q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, with the operations of addition and scalar multiplication defined component-wise and where the so-called imaginary units $\mathsf{i}, \mathsf{j}, \mathsf{k}$ satisfy

$$\mathsf{i}^2 = -1, \;\; \mathsf{j}^2 = \mathsf{k}^2 = 1, \;\; \mathsf{ijk} = 1.$$

The expression for the product of two coquaternions follows easily from the above multiplication rules; in particular, $\mathsf{ij} = -\mathsf{ji} = \mathsf{k}$, $\mathsf{jk} = -\mathsf{kj} = -\mathsf{i}$, $\mathsf{ki} = -\mathsf{ik} = \mathsf{j}$ and therefore for $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ we have

$$\mathsf{q}^2 = q_0^2 - q_1^2 + q_2^2 + q_3^2 + 2q_0\left(q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}\right).$$

Given a coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} := q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k}$; the number q_0 is called the *real part* of q and denoted by $\operatorname{Re}\mathsf{q}$ and the *vector part* of q, denoted by $\operatorname{Vec}\mathsf{q}$, is given by $\operatorname{Vec}\mathsf{q} := q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$.

We identify the set of coquaternions with null vector part with the set $\mathbb{R}$ of real numbers. For geometric purposes, we also identify the coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ with the element (q_0, q_1, q_2, q_3) in $\mathbb{R}^4$.

It is easy to see that the algebra of coquaternions is isomorphic to $\mathcal{M}_2(\mathbb{R})$, the algebra of real 2×2 matrices, with the map $\mathbb{H}_{\rm coq} \to \mathcal{M}_2(\mathbb{R})$ defined by

$$\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \;\mapsto Q = \begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}$$

establishing the isomorphism. Keeping this in mind, we call *trace* of q, which we denote by $\operatorname{tr}\mathsf{q}$, to the quantity given by $\operatorname{tr}\mathsf{q} := 2q_0 = 2\operatorname{Re}\mathsf{q} = \mathsf{q} + \overline{\mathsf{q}}$ and call *determinant* of q to the quantity, denoted by $\det\mathsf{q}$, given by

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$\det \mathsf{q} := q_0^2 + q_1^2 - q_2^2 - q_3^2 = \mathsf{q}\overline{\mathsf{q}}$. The result contained in the following lemma can be shown by a simple verification.

LEMMA 1.— For any coquaternion $\mathsf{q} \in \mathbb{H}_{\mathrm{coq}}$, we have

$$\mathsf{q}^2 = (\operatorname{tr}\mathsf{q})\mathsf{q} - \det \mathsf{q}.$$

Naturally, some of the results for coquaternions can be established by invoking the aforementioned isomorphism and making use of known results for matrices. For example, one can use this approach to conclude that, unlike $\mathbb{C}$ and $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$ is not a division algebra. In fact, a coquaternion q is invertible if and only if $\det \mathsf{q} \neq 0$. In that case, we have $\mathsf{q}^{-1} = (\det \mathsf{q})^{-1}\overline{\mathsf{q}}$.

For our future purposes it is useful to recall now the following concept: we say that a coquaternion q is *similar* to a coquaternion p, and write $\mathsf{q} \sim \mathsf{p}$, if there exists an invertible coquaternion h such that $\mathsf{p} = \mathsf{h}\mathsf{q}\mathsf{h}^{-1}$. This is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$, partitioning $\mathbb{H}_{\mathrm{coq}}$ in the so-called *similarity classes*. As usual, we denote by [q] the similarity class containing q. The following result can be easily proved (see [6] and [16]).

LEMMA 2.— Let $\mathsf{q} = q_0 + \operatorname{Vec}\mathsf{q}$ be a coquaternion and let $r = \det(\operatorname{Vec}\mathsf{q})$. If q is real, then $[\mathsf{q}] = \{q_0\}$; if q is non-real, then $[\mathsf{q}] = [\mathsf{q}_s]$, where

$$\mathsf{q}_s = q_0 + \sqrt{r}\,\mathsf{i}, \qquad \text{if} \quad r > 0, \tag{1a}$$
$$\mathsf{q}_s = q_0 + \sqrt{-r}\,\mathsf{j}, \qquad \text{if} \quad r < 0, \tag{1b}$$
$$\mathsf{q}_s = q_0 + \mathsf{i} + \mathsf{j}, \qquad \text{if} \quad r = 0. \tag{1c}$$

Since similar coquaternions have the same determinant, the previous lemma completely characterizes the similarity classes in $\mathbb{H}_{\mathrm{coq}}$. This means that two non-real coquaternions p and q are similar if and only if

$$\operatorname{Re}\mathsf{p} = \operatorname{Re}\mathsf{q} \quad \text{and} \quad \det(\operatorname{Vec}\mathsf{p}) = \det(\operatorname{Vec}\mathsf{q}). \tag{2}$$

The coquaternion q_s will be referred to as the *standard representative* of [q]. Lemma 2 says that the standard representative of [q] is either a *complex number*, a *perplex number* (number of the form $a+b\mathsf{j}$) or a *dual number* (number of the form $a+b(\mathsf{i}+\mathsf{j})$). Associated with these numbers we will consider three important subspaces of dimension two of $\mathbb{H}_{\mathrm{coq}}$, the so-called *canonical planes* or *cycle planes*: the complex plane $\mathbb{C}$, the *Minkowski plane* $\mathbb{P}$ of perplex numbers and the *Laguerre plane* $\mathbb{D}$ of dual numbers.

Two coquaternions p and q (whether or not real) satisfying (2) are called *quasi-similar*. Naturally, quasi-similarity is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$; the corresponding equivalence class of q, i.e. the set

$$\{x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = r\},$$

is called the *quasi-similarity class* of q and denoted by $[\![\mathsf{q}]\!]$. Here, as before, q_0 and r denote respectively the real part and the determinant of the vector part of q. We can identify $[\![\mathsf{q}]\!]$ with an hyperboloid in the hyperplane $x_0 = q_0$, which will be:

- a hyperboloid of one sheet or a hyperboloid of two sheets, if $r < 0$ or $r > 0$, respectively; in such cases $[\![\mathsf{q}]\!] = [\mathsf{q}]$;
- a degenerate hyperboloid (i.e. a cone), if $r = 0$; in this case, $[\![\mathsf{q}]\!] = [q_0 + \mathsf{i} + \mathsf{j}] \cup \{q_0\}$.

2.2 SOME REMARKS ON COQUATERNIONIC POLYNOMIALS

In contrast to the case of quaternionic polynomials, the problem of finding the zeros of polynomials defined over the algebra $\mathbb{H}_{\mathrm{coq}}$ only drew the attention of researchers quite recently; see [5, 8, 13, 14, 15, 19].

A complete characterisation of the zero set of left unilateral polynomials over coquaternions, i.e. of polynomials whose coefficients are coquaternions located on the left-hand side of the variable, can be found in [8]. In particular, it is proved that the zeros of monic polynomials of degree n belong to, at most, $n(2n-1)$ quasi-similarity classes; each of these classes can either contain a unique zero (*isolated zero*) or be totally made up of zeros (*hyperboloidal zero*) or contain a straight line of zeros (*linear zero*). We point out that there is no analogue of the Fundamental Theorem of Algebra, as there are coquaternionic polynomials with no zeros.

To offer a glimpse of the diversity of behaviours that the zero sets of coquaternionic polynomials may have, we now present some examples. An algorithm to compute and classify all the zeros of a coquaternionic polynomial is available in [8] and can be used to check the following statements:

1. $P(x) = x^2 - \mathsf{j}$ has no zeros;
2. $P(x) = x^2 + (3+\mathsf{i}+\mathsf{j}+\mathsf{k})x + 3 + \mathsf{i} + \mathsf{j} + 3\mathsf{k}$ has only one isolated zero, $\mathsf{z} = -1 + \frac{1}{2}\mathsf{i} - \mathsf{j} - \frac{1}{2}\mathsf{k}$;
3. $P(x) = x^2 - \mathsf{j}x - 1 - \mathsf{i}$ has six isolated zeros (the maximum number of zeros a quadratic polynomial can have), namely

$$\mathsf{z}_1 = \mathsf{k},$$
$$\mathsf{z}_2 = \mathsf{j} + \mathsf{k},$$
$$\mathsf{z}_{3,4} = \pm\left(\tfrac{1+\sqrt{2}}{2} + \tfrac{1}{2}\mathsf{i}\right) + \tfrac{1}{2}\mathsf{j} + \tfrac{1-\sqrt{2}}{2}\mathsf{k},$$
$$\mathsf{z}_{5,6} = \pm\left(\tfrac{1-\sqrt{2}}{2} + \tfrac{1}{2}\mathsf{i}\right) + \tfrac{1}{2}\mathsf{j} + \tfrac{1+\sqrt{2}}{2}\mathsf{k};$$

4. $P(x) = x^2 + 1$ has two isolated zeros, $\mathsf{z}_{1,2} = \pm 1$, and the hyperboloidal zero, $H = [\![\mathsf{j}]\!]$ (which can be identified with an hyperboloid of one sheet in the hyperplane $x_0 = 0$);
5. $P(x) = x^2 - \mathsf{j}x - 1 - \mathsf{j}$ has two isolated zeros, $\mathsf{z}_1 = -1$ and

$z_2 = 1 + j$, and two linear zeros,

$$L_1 = \left\{-\frac{1}{2} + \alpha i - \frac{1}{2}j + \alpha k \,:\, \alpha \in \mathbb{R}\right\},$$
$$L_2 = \left\{\frac{1}{2} + \alpha i + \frac{3}{2}j - \alpha k \,:\, \alpha \in \mathbb{R}\right\}$$

(which can be identified with straight lines).

It is worth mentioning that a *Mathematica* notebook to classify and determine the nth roots of a coquaternion q (the zeros of $x^n - q$) is available at the webpage `http://w3.math.uminho.pt/CoquaternionsRoots`, where one can also obtain a *Mathematica* add-on application implementing the algebra of coquaternions.

3 Coquaternionic Quadratic Map

We now consider the quadratic map

$$\begin{aligned} f_c \,:\, \mathbb{H}_{coq} &\to \mathbb{H}_{coq} \\ q &\mapsto q^2 + c \end{aligned}$$

where c is a fixed parameter in $\mathbb{H}_{coq}$.

When the parameter $c \in \mathbb{C}$, we will use f_c to denote the complex map obtained by restricting f_c to the complex plane, i.e. $f_c := f_c|_{\mathbb{C}}$.

3.1 Preliminary results

We first recall several basic definitions and present some results which will play an important role in the remaining part of the paper.

For $k \in \mathbb{N}$, we shall denote by f_c^k the k-th iterate of f_c, inductively defined by $f_c^0 = \mathrm{id}_{\mathbb{H}_{coq}}$ and $f_c^k = f_c \circ f_c^{k-1}$. For a given initial point $q_0 \in \mathbb{H}_{coq}$, the *orbit* of q_0 under f_c is the sequence $\left(f_c^k(q_0)\right)_{k\in\mathbb{N}_0}$. A point $q \in \mathbb{H}_{coq}$ is said to be a *periodic point of* f_c *with period* $n \in \mathbb{N}$, if $f_c^n(q) = q$, with $f_c^k(q) \neq q$ for $0 < k < n$; in this case, we say that the set $\mathcal{C} = \{q, f_c(q), \dots, f_c^{n-1}(q)\}$ is an *n-cycle* for f_c, usually written as $\mathcal{C} \,:\, q_0 \overset{f_c}{\to} q_1 \overset{f_c}{\to} \cdots \overset{f_c}{\to} q_{n-1}$ with $q_i = f_c^i(q)$. Periodic points of period one are called *fixed points*.

It follows from the result in Lemma 1 that the orbit of any coquaternion q lies in the subspace $\mathrm{span}_{\mathbb{R}}(1, q, c)$ of $\mathbb{H}_{coq}$. The following result is also simple to establish.

Lemma 3.— For any invertible coquaternion h, let ϕ_h be the map defined by $\phi_h(q) = h^{-1}qh$. Then, the dynamical system $(\mathbb{H}_{coq}, f_c)$ is dynamically equivalent to the dynamical system $(\mathbb{H}_{coq}, f_{\phi_h(c)})$.

As a consequence of the two previous lemmas, we immediately conclude that to study the dynamics of the quadratic map $f_c(q) = q^2 + c$ there is no loss of generality in assuming that c is either real or has one of the standard forms (1).

3.2 Fixed points of f_c

Let $q = q_0 + q_1 i + q_2 j + q_3 k$ and $c = c_0 + c_1 i + c_2 j + c_3 k$ be coquaternions. From Lemma 1 we see that q is a fixed point of f_c if and only if it satisfies the equation

$$(2q_0 - 1)q - \det q = -c. \tag{3}$$

Next, we consider separately $q_0 \neq 1/2$ and $q_0 = 1/2$.

3.2.1 Case $q_0 \neq 1/2$

We first note that it follows from (3) that, if $q_0 \neq 1/2$, then $q \in \mathrm{span}_{\mathbb{R}}(1, c)$. In particular, if c is chosen in one of the cycle planes, then q belongs to the same plane.

(i) For $c = c_0 + c_1 i$, with $c_1 \geq 0$, we are simply considering the case of the complex quadratic map f_c; hence, the fixed points of f_c are, as is well-known, given by

$$q_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4c}\right).$$

Note that, for $c = c_0 \in \mathbb{R}$, $c_0 \geq 1/4$, the corresponding fixed points do not satisfy the condition we are considering here, $q_0 \neq 1/2$.

(ii) For $c = c_0 + c_2 j$, with $c_2 > 0$, the dynamics is restricted to the cycle plane $\mathbb{P}$. Here, it is convenient to use the so-called *dual basis* (e_1, e_2) with $e_1 = (1 + j)/2$ and $e_2 = (1 - j)/2$, which satisfies

$$e_1^2 = e_1, \quad e_2^2 = e_2 \quad \text{and} \quad e_1 e_2 = e_2 e_1 = 0.$$

Expressing q and c in this basis, we get $q = x e_1 + y e_2$ and $c = a e_1 + b e_2$, where

$$x = q_0 + q_2,\ y = q_0 - q_2,\ a = c_0 + c_2,\ b = c_0 - c_2.$$

Hence,

$$q^2 + c = (x^2 + a)e_1 + (y^2 + b)e_2.$$

This shows that f_c has fixed points if and only if $c_0 + c_2 < 1/4$ and $c_0 - c_2 < 1/4$, which are

$$q_{1,2} = \frac{1}{2} \pm \frac{1}{4}\left(A + B + (A - B)j\right),$$
$$q_{3,4} = \frac{1}{2} \pm \frac{1}{4}\left(A - B + (A + B)j\right),$$

where A and B are given by

$$A = \sqrt{1 - 4(c_0 + c_2)} \text{ and } B = \sqrt{1 - 4(c_0 - c_2)}.$$

(iii) For $c = c_0 + i + j$, we have $q = q_0 + a(i + j)$ and so

$$q^2 + c = (q_0^2 + c_0) + (2q_0 a + 1)(i + j).$$

Thus, f_c has fixed points if and only if $c_0 < 1/4$, which are given by

$$q_1 = \frac{1}{2}\left(1 - \sqrt{1 - 4c_0}\right) + \frac{1}{\sqrt{1-4c_0}}(i + j),$$
$$q_2 = \frac{1}{2}\left(1 + \sqrt{1 - 4c_0}\right) - \frac{1}{\sqrt{1-4c_0}}(i + j).$$

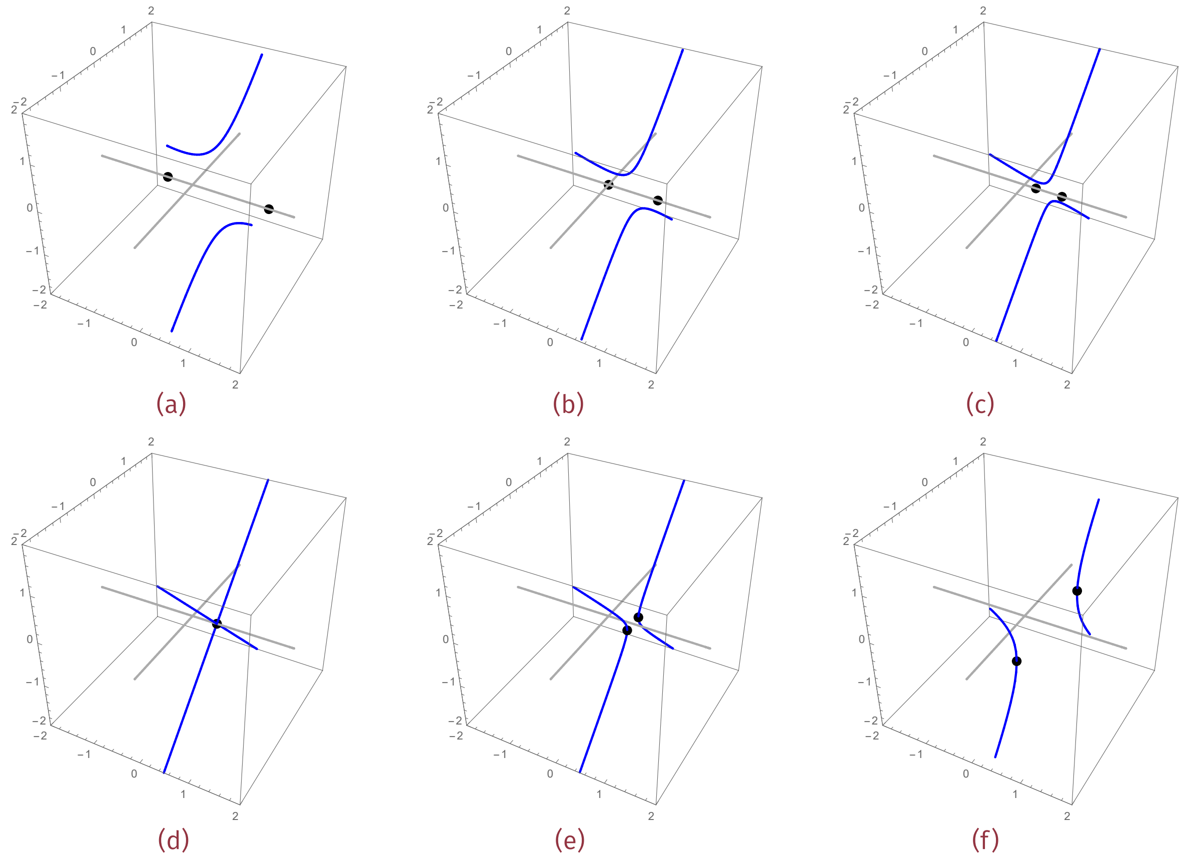

Figure 1.—Plots, in the hyperplane $q_3 = 0$, of the sets of fixed points corresponding to the case $c = c_0 \in \mathbb{R}$; the real and complex fixed points are identified with black points and the fixed points not in $\mathbb{C}$ form the blue lines; the gray lines represent the real and imaginary axes. (a) $c_0 = -0.8$; (b) $c_0 = -0.0$; (c) $c_0 = -0.18$; (d) $c_0 = -0.25$; (e) $c_0 = -0.3$; (f) $c_0 = -1.7$

3.2.2 Case $q_0 = 1/2$

In this case, Eq. (3) reduces to $\det \mathrm{q} = \mathrm{c}$. Since $\det \mathrm{q}$ is a real number, we conclude that $\mathrm{f_c}$ has no fixed points, unless $\mathrm{c} = c_0 \in \mathbb{R}$.

The map f_{c_0} has one real fixed point $\mathrm{q} = q_0 = 1/2$, for $c_0 = 1/4$. We now discuss the non-real fixed points of f_{c_0}. Since a real number commutes with any coquaternion, we have, for any invertible $\mathrm{h} \in \mathbb{H}_{\mathrm{coq}}$,

$$\begin{aligned}\mathrm{h}^{-1}\mathrm{f}_{c_0}(\mathrm{q})\mathrm{h} &= \mathrm{h}^{-1}\mathrm{q}^2\mathrm{h} + \mathrm{h}^{-1}c_0\mathrm{h}\\ &= (\mathrm{h}^{-1}\mathrm{q}\mathrm{h})^2 + c_0 = \mathrm{f}_{c_0}(\mathrm{h}^{-1}\mathrm{q}\mathrm{h}).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{f}_{c_0}(\mathrm{q}) = \mathrm{q} \iff& \mathrm{h}^{-1}\mathrm{f}_{c_0}(\mathrm{q})\mathrm{h} = \mathrm{h}^{-1}\mathrm{q}\mathrm{h}\\ \iff& \mathrm{f}_{c_0}(\mathrm{h}^{-1}\mathrm{q}\mathrm{h}) = \mathrm{h}^{-1}\mathrm{q}\mathrm{h}\end{aligned}$$

which shows that to determine the non-real fixed points of the coquaternionic map f_{c_0} we only have to identify the fixed points of this map with any of the three special forms (1) and to construct the corresponding similarity classes.

As it is well-known, there is only one fixed point of the form (1a), which occurs for $c_0 > 1/4$, the point $\mathrm{q_s} = 1/2 + (\sqrt{4c_0 - 1}/2)\mathrm{i}$. Also, it is simple to verify that the only fixed point of f_{c_0} of the form (1b) is given by $\mathrm{q_s} = 1/2 + (\sqrt{1 - 4c_0}/2)\mathrm{j}$, for $c_0 < 1/4$, whereas $\mathrm{q_s} = 1/2 + \mathrm{i} + \mathrm{j}$ is the only fixed point of the form (1c) and occurs when $c_0 = 1/4$. In summary, we have the following three sets of fixed points, depending on the value of c_0:

$$\begin{aligned}\mathcal{F}_1 &= \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4c_0}\,\mathrm{j}\right], && \text{if } c_0 < 1/4,\\ \mathcal{F}_2 &= \left[\tfrac{1}{2} + \mathrm{i} + \mathrm{j}\right] \cup \left\{\tfrac{1}{2}\right\}, && \text{if } c_0 = 1/4,\\ \mathcal{F}_3 &= \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{4c_0-1}\,\mathrm{i}\right], && \text{if } c_0 > 1/4.\end{aligned}$$

Having in mind the relation between similarity and quasi-similarity classes referred to in Sec. 2.1, it is clear that any of the above sets can be identified with an hyperboloid in the hyperplane $q_0 = 1/2$.

In Fig. 1 we present plots obtained by fixing $q_3 = 0$, and considering several values of the parameter c_0. The known fixed points of the dynamics in $\mathbb{C}$ are identified with black points and the fixed points not in $\mathbb{C}$ are given by blue lines (hyperbolas resulting from the intersection of the hyperboloids $\mathcal{F}_i$ with the hyperplane $H_3 = \{(q_0, q_1, q_2, q_3) \in \mathbb{R}^4 : q_3 = 0\}$); the real and imaginary axis are identified with gray lines.

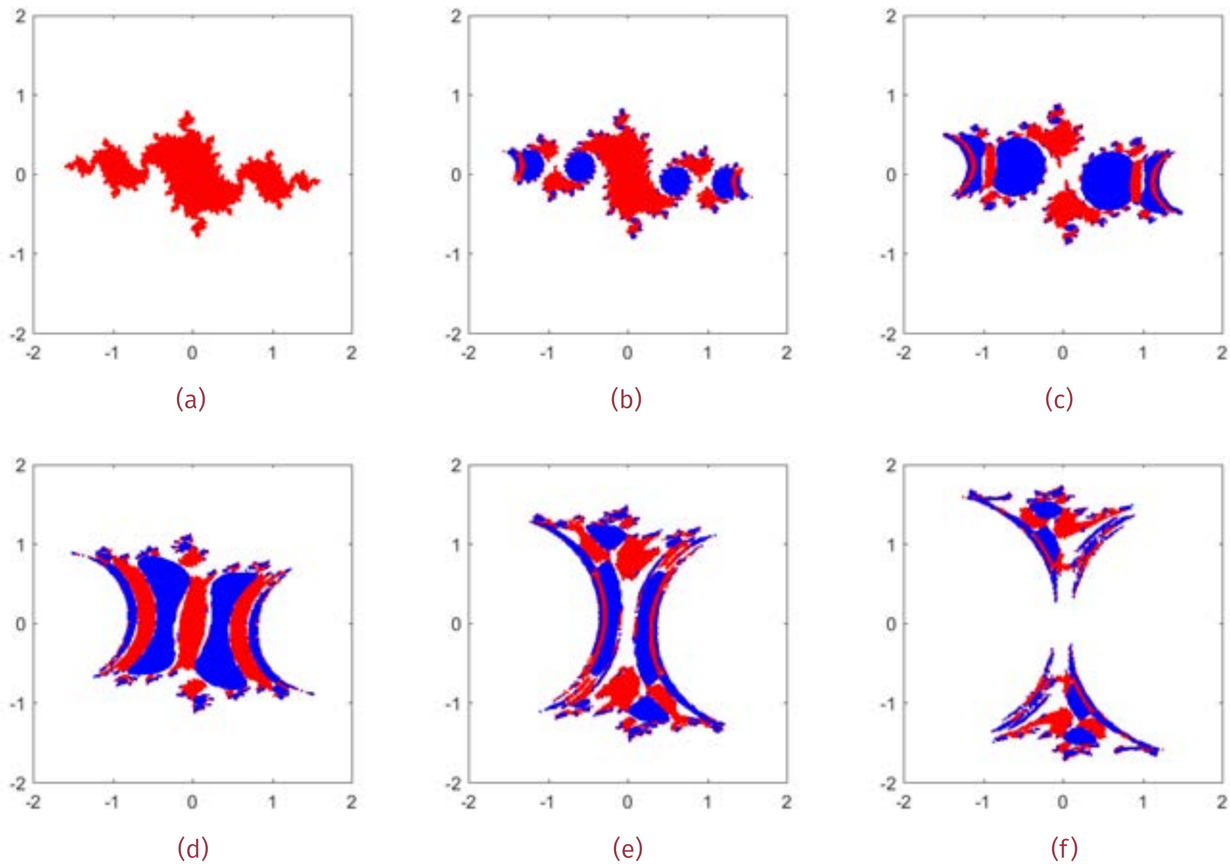

Figure 2.—Plots, in different planes parallel to the complex plane, of the basin of attraction of the 2-cycle C_2 (red) and the (1,2)-set cycle $C_{1,2}$ (blue). (a) $q_2 = 0$; (b) $q_2 = 0.2$; (c) $q_2 = 0.4$; (d) $q_2 = 0.8$; (e) $q_2 = 1.25$; (f) $q_2 = 1.55$

3.3 Cycles of sets

The results of the previous section show that we now have a situation not present in the classical case of the real/complex quadratic maps: the existence of fixed points forming sets of non-isolated points. The same may be true for periodic points of other periods; see, e.g. [6]. This motivates us to introduce a definition of cycles of sets.

Definition 4.— We say that the sets $\mathcal{S}_0, \dots, \mathcal{S}_{m-1}$ form an *(m,n)-set cycle $\mathcal{C}_{m,n}$* for the map $\mathrm{f_c}$, and write

$$\mathcal{C}_{m,n} : \mathcal{S}_0 \xrightarrow{\mathrm{f_c}} \mathcal{S}_1 \xrightarrow{\mathrm{f_c}} \cdots \xrightarrow{\mathrm{f_c}} \mathcal{S}_{m-1},$$

if:

(i) each of the sets $\mathcal{S}_i$; $i = 0, \dots, m-1$, is formed by periodic points of period n of $\mathrm{f_c}$;

(ii) $\mathcal{S}_i = \mathrm{f_c}(\mathcal{S}_{i-1})$, $i = 1, \dots, m-1$, and $\mathrm{f_c}(\mathcal{S}_{m-1}) = \mathcal{S}_0$;

(iii) the sets $\mathcal{S}_0, \dots, \mathcal{S}_{m-1}$ are pairwise separated by ε-neighborhoods.

Note that if $\mathcal{C}_{m,n}$ is an (m,n)-set cycle, then n must be a multiple of m. When $m = n$, we simply call the cycle an n-set cycle and denote it by $\mathcal{C}_n$.

As shown in [6], for $\mathrm{c} = c_0 + c_1\mathrm{i}$, with $c_1 > 0$ and c_0, c_1 satisfying $c_1^2 > 4c_0 + 3$, the set

$$\mathcal{P} = \left\{ -\tfrac{1}{2} + \tfrac{c_1}{2}\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} \;\middle|\; q_2^2 + q_3^2 = \tfrac{c_1^2 - 4c_0 - 3}{4} \right\} \qquad (4)$$

is made up of periodic points of period two of the map $\mathrm{f_c}$ and, if $\mathrm{q} \in \mathcal{P}$, then $\mathrm{q} = \mathrm{f_c}(\mathrm{p})$ where $\mathrm{p} = -1/2 + (c_1/2)\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k} \in \mathcal{P}$. Hence,

$$\mathcal{C}_{1,2} : \mathcal{P} \circlearrowleft_{\mathrm{f_c}} \qquad (5)$$

is a $(1,2)$-set cycle.

Other examples of set cycles for the quadratic coquaternionic map can be found in [6].

We would like to remark that some results for the quadratic map on the algebra $\mathcal{M}_2(\mathbb{R})$—which can naturally, be translated to the coquaternionic formalism—were obtained in [2] and [20].

3.4 Basins of attraction

Due to the appearance of set cycles, we now have to adapt the usual notion of basin of attraction. We propose to use the following definition.

Definition 5.— Let $\mathcal{C}_{m,n} : \mathcal{S}_0 \xrightarrow{f_c} \mathcal{S}_1 \xrightarrow{f_c} \cdots \xrightarrow{f_c} \mathcal{S}_{m-1}$ be an (m,n)-set cycle for f_c. The *basin of attraction* of $\mathcal{C}_{m,n}$, denoted by $\mathcal{B}(\mathcal{C}_{m,n})$, is given by

$$\mathcal{B}(\mathcal{C}_{m,n}) = \bigcup_{\ell=0}^{m-1} \mathcal{B}(\mathcal{S}_\ell),$$

where

$$\mathcal{B}(\mathcal{S}_\ell) := \{q \in \mathbb{H}_{coq} : \lim_k d(f_c^{km}(q), \mathcal{S}_\ell) = 0\}$$

and d is a distance function.

Naturally, when a set cycle reduces to a cycle of isolated points, we recover the usual definition of basin of attraction of that cycle.

As an illustrative example, we consider now two different cycles for the map f_c: the 2-cycle of isolated complex points

$$\mathcal{C}_2 : q_1 \xrightarrow{f_c} q_2,$$

where

$$q_{1,2} = \frac{1}{2}(1 \pm \sqrt{-3-4c}),$$

and the (1,2)-set cycle $\mathcal{C}_{1,2}$ defined by (5) with $\mathcal{P}$ the set given by (4), for a particular choice of the parameter c, the complex number $c = -0.95 + 0.2i$.

In Fig. 2 we present plots of the basins of attraction of these two cycles. The representations are two-dimensional plots obtained by assuming $q_3 = 0$ and considering different values for q_2, i.e., all the pictures correspond to plots in planes parallel to the complex plane. In the plots, the points in the basin of attraction of the cycle $\mathcal{C}_2$ are colored in red and the points in the basin of attraction of the cycle $\mathcal{C}_{1,2}$ are colored in blue.

The plot on the top-left of Fig. 2 corresponds to $q_2 = 0$, i.e., is a plot in the complex plane, and we immediately recognize the picture associated with the dynamics of the quadratic complex map f. As the value of q_2 increases, the two coquaternionic basins of attraction appear, showing an interesting intertwined structure.

4 Conclusions

As it is well-known, to study the dynamics of complex quadratic maps we only have to consider the particular family of maps of the form $f_c(x) = x^2 + c$, since any quadratic map may be converted, by conjugacy, to a member of this family. In the coquaternionic case, the situation is totally different.

Due to the non-commutativity of the product of coquaternions, the sum of two mth degree monomials $a_0 x a_1 x \cdots a_{m-1} x a_m$ and $a'_0 x a'_1 x \cdots a'_{m-1} x a'_m$ can not be written simply in the form $A_0 x A_1 x \cdots A_{m-1} x A_m$ and hence, the general expression of a quadratic coquaternionic polynomial is

$$\sum_{j=1}^{n} a_0^j x a_1^j x a_2^j + \sum_{j=1}^{k} b_0^j x b_1^j + c, \quad n, k \in \mathbb{N},$$

with a_i^j, b_i^j and c coquaternions. Not surprisingly, contrary to what happens in the commutative case, no conjugacy equivalence of a quadratic coquaternionic polynomial to a simple form is available.

The important differences from the complex setting already observed for the simple coquaternionic quadratic family $f_c(q) = q^2 + c$ and the interesting results obtained for the zeros of unilateral coquaternionic polynomials lead us to believe that coquaternions — in particular the study of more general coquaternionic quadratic maps and of more general polynomials — are an area worth exploring.

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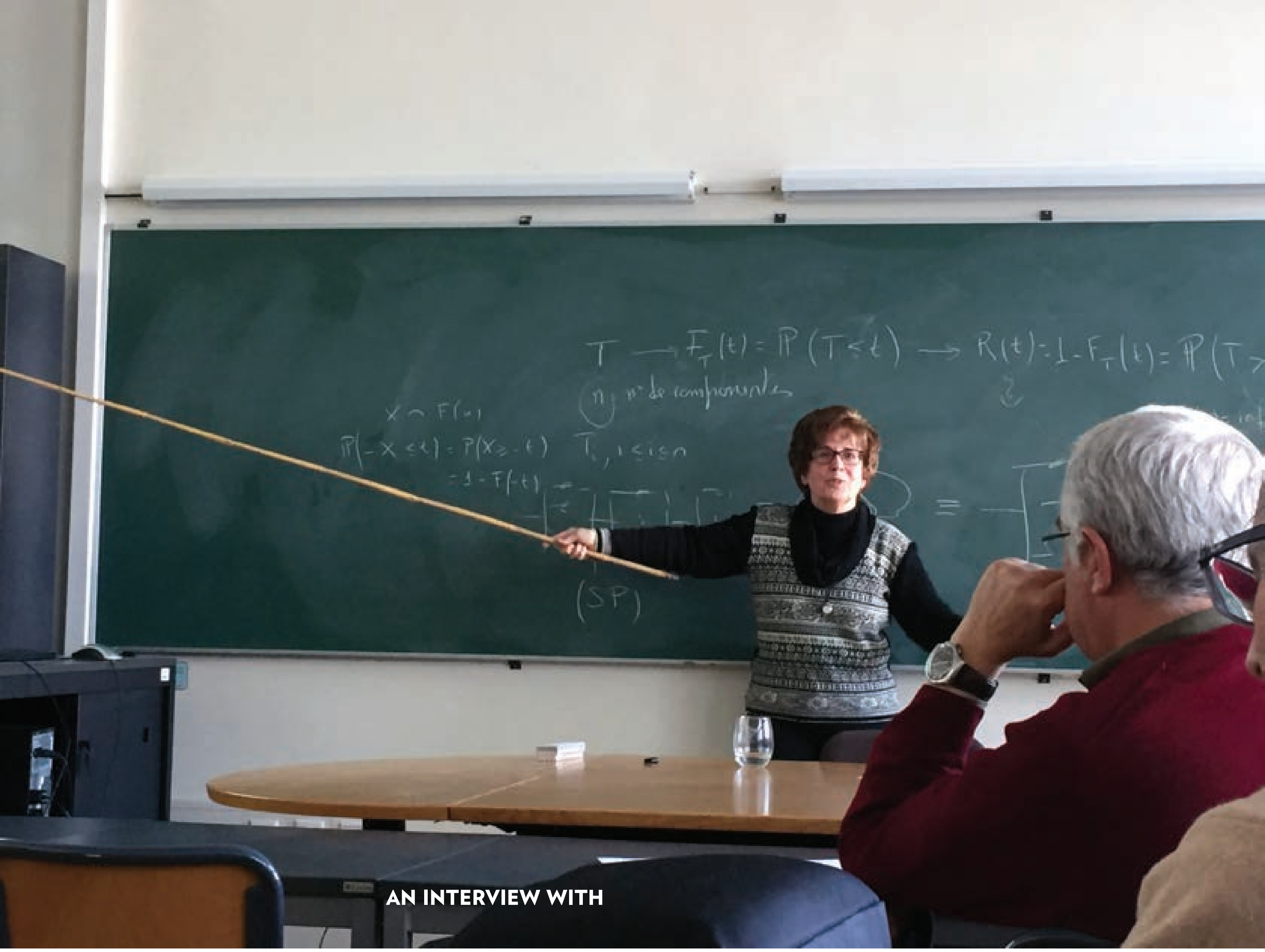

AN INTERVIEW WITH

Ivette Gomes

by **Ana Cristina Moreira Freitas*** and **Jorge Milhases Freitas****

Maria Ivette Gomes (born 21 July 1948) is an Emeritus Professor of the University of Lisbon, editor-in-chief of REVSTAT, member of the editorial board of Extremes and researcher of the research centre of statistics and its applications of the University of Lisbon (CEAUL). She obtained a PhD in Statistics in 1978 from the University of Sheffield. She became a renown influential specialist in the field of Extremes. Together with Tiago de Oliveira, she is responsible for dynamizing a School of Extremes in Portugal, which, nowadays, counts with many researchers. She was one of the founders of SPE (the Portuguese society of statistics), which she presided from 1990 to 1993.

She was a member of the first Scientific Committee of CIM, in 1996, and a vice-president of the direction board from 2004 to 2008.

She was principal investigator of several national and international projects, she authored 5 books and published more than 100 publications in peer reviewed journals, including several of the most reputed journals in Probability and Statistics.

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When and how did you decide to become a mathematician?

Indeed, I think I decided to become a mathematician in my fifth year in the secondary school and the main reason was because I could not go to Architecture. I would like to go to Architecture because I liked very much geometry and drawing, but I did not have very good marks in History. To pursue a degree in Architecture, at the time, we needed to have very good marks in that subject and so, I decided to go to Sciences and my second choice was Mathematics, for sure.

So, we lost an architect but won a mathematician

Yes...

How does a graduate in pure Mathematics with a passion for Algebra get interested in Statistics and, in particular, in Extreme Value Theory?

I went to Mathematics, which was a five years degree. After the first three years we had to choose between three different topics: Pure Mathematics, Applied Mathematics and Mechanics. Indeed, I liked Algebra very much. I had the first course in Probability and Statistics taught by Tiago de Oliveira in the third year of our course in Mathematics. At the time I liked Probability and Statistics a lot but my decision was already taken and I decided to study Algebra and to go to Pure Mathematics. Afterwards, I had to choose again, in the fifth year, two different subjects. I decided to choose two courses, one in Probability Theory and the other one in Mathematical Statistics. In the course of Probability Theory I got to know the book by Gnedenko, I read the book and I liked it very much. So I think maybe that was the first push I had towards the field of Probability. Afterwards, I also got interested in the field of Statistics due to the wonderful course on Mathematical Statistics given by Tiago.

So, Gnedenko is the guy to blame.

Yes . . . He is the guy to blame . . . So, this was in my fifth year of Pure Mathematics, after a lot of very nice courses in Algebra and Analysis. I had courses given by Sebastião e Silva . . .

Ah, you were one of the lucky ones . . .

Yes, I was lucky to still have Sebastião e Silva as a professor. He gave me a course of Distribution Theory and there I could also see some connections to Probability Theory. He also gave me a very nice course about the history of mathematical thinking. But indeed, I always found that there was some kind of magic in the random and in Probability. Algebra was too much deterministic, you see ... Life is not so deterministic. It is more random. And so, I thought that maybe I should take the degree in Applied Mathematics, which meant two extra years. But then, Tiago de Oliveira learned about my intentions and he decided to offer me a position as assistant. So I was directly hired as an assistant of the department of Mathematics to work with Tiago in the Section of Applied Mathematics, which also meant to work with computers and I had never worked with computers . . .

Isabel (Fraga Alves) told me once that you know how to program quite well. So, is that how you learned?

Yes . . . At the time I learned by myself the Basic language and afterwards I had to learn Fortran.

So you worked with Tiago and that is what lead you to go and obtain a PhD in the field of Statistics . . .

Yes . . .

And how does the Extreme Value Theory come into the picture? Was Gnedenko, again?

Maybe not entirely Gnedenko, this time. Indeed, when I went to Sheffield, in 1975 to earn a PhD I was supposed to choose a topic different from the Extreme Value Theory. The idea was to go possibly to Nonparametric Statistics, which was a topic that I enjoyed and there was nobody here working in that field. But it happened that, at the time, Clive Anderson, who earned a PhD at the Imperial College and in the field of Extreme Value Theory, got to know that I was coming from Portugal and came to me and asked whether I did not want to work in the field of Extremes because he would like to supervise my PhD in that topic. I said I would think about it. I talked with Tiago and he said that it would be my decision. And then I decided to accept to be supervised by Clive. Hence, one can say that Clive was the responsible person for my dedication to Extreme Value Theory.

Do you have a favourite Mathematician that has particularly inspired you?

Gnedenko is obviously one of the names. But I would add two other names: Fisher and Tippett.

Some people say that Statistics is not Mathematics. What do you think?

It is a difficult question . . . I think that when we go to real applied statistics and applied data analysis, sometimes they are far away from Mathematics. But when we speak about it through Probability Theory, I think we can include it in Mathematics, obviously. Moreover, when we consider theoretical statistics and mathematical statistics, we definitely cannot say that they are not mathematics. But in my opinion we should keep some individuality for Statistics, because some people who are working in applied statistics are a bit away from Mathematics.

But they still need a mathematician . . . Or not?

They need . . . Yes, they need a mathematician, obviously.

You were a founder of SPE (Sociedade Portuguesa de Estatística — the Portuguese Society of Statistics) and its second president, after Tiago de Oliveira. Do you think we lived up to the expectations you had for the Portuguese community of statisticians when you helped to found SPE, back in 1980?

You know that before SPE we had SPEIO (Sociedade Portuguesa de Estatística e Investigação Operacional — Portuguese Society of Statistics and Operational Research). Afterwards they thought that Operational Research should have an Association and so there was no reason to go on with the joint society SPEIO. Then, we decided to found the Portuguese Statistical Society. And I think that SPE was quite important for the development of Statistics in Portugal. At the beginning, it was a surprise for me because at the time we were a very small group, then we

Figure 1.—Conference on Extremwerttheorie, Oberwolfach, 1987

had the first meeting of SPE in Vimeiro (1993), and after that the number of people in the Portuguese Statistical Society increased suddenly. We had a lot of activities at the time and we managed to have annual meetings of SPE, from 1993 to 2013, which gave some enthusiasm to people and allowed researchers from different places in Portugal to be together. Unfortunately, recently we lost something with the change from annual meetings to bi-annual meetings.

So, you think that if Tiago de Oliveira could now see what he started he would be happy with what he would see today, except for the change to the bi-annual meetings . . .

Yes, I think he would be happy to see how things have grown . . .

You were elected for the first Scientific Committee of CIM, in 1996, which was created to develop and promote research in Mathematics in Portugal. When you look back to that time and compare with today, how much do you feel we have evolved?

I get the impression that we are doing quite well. I have not followed how CIM has been working in the last years. I am a bit out of that, but I think that the research in Mathematics, in general, has evolved quite well, in terms of numbers and in terms of quality. I would say that things are going in the right direction.

How would you describe this evolution? What were the key aspects or events that you witnessed in first hand? Do you see anything that was crucial for this development?

I think that both FCT and Calouste Gulbenkian Foundation have supported many mathematicians and Mathematics in general. I do not know how CIM is now, but for instance I remember that when I was in that first Scientific Committee of CIM and even later, when I was in the board of directors, CIM had a strong support from both FCT and Calouste Gulbenkian Foundation. But I do not know how CIM is now . . .

Now, we depend essentially on the money coming from our associates. So, we have a very small budget and we try to do the best we can with it.

I think it would be important to find support from FCT. I definitely think that CIM should have the support from FCT.

Because, at the time, with that support, we were able to organize conferences in Portugal and that obviously boosted a lot the development of the field. I remember that at the time we were able to support several different conferences, organised around the country. We even had those SPM/CIM meetings or SPE/CIM meetings, which were also quite interesting and, usually, that was possible due to the support from FCT and also from Calouste Gulbenkian Foundation.

We now have some support of Calouste Gulbenkian but for very specific initiatives, like for example Pedro Nunes Lectures. But we do have to ask every time we organize it. It's not a regular thing. So that makes a bit the difference, I guess.

You have been the editor in chief of REVSTAT, which, essentially, in a 10 year period went from a national journal to an international respected journal classified at the Web of Science, with a five year impact factor of 1.4. Would you like to share with us how you did it?

A lot of work . . .

That I would suspect . . .

A lot of work, but no tricky way of getting any kind of impact factor. Indeed, as you used to say, I am from the *stone age* and at that time there were no impact factors, and for me is still a bit strange to work with these numbers. They don't tell me too much. Anyway, at the beginning it was a bit difficult . . . I had to invite a few people to submit papers to the journal and we had a very small number of papers submitted. But afterwards we had the organisation of the International Statistical Institute (ISI) World Statistical Congress (WSC), in 2007, and that was a big help. The first issue of the journal came out in 2003 and in 2007 we were already with a bulk of papers. Indeed, in the beginning, I was advised by the president of the National Statistical Institute, who told me that it would be a better strategy to begin with two issues and then, possibly, after some time, increase to three issues. Nowadays, we are already with four issues and a lot of papers in the list of forthcoming papers. We have around 200 submissions per year. As I said, the ISI WSC in 2007 was a big push because, at the time, we had different sections in the field of Extremes and so a few people offered to have a special issue related to papers at the ISI 2007. Then, with the boost of interest, I managed to write to the Web of Science and give them information about the journal, which was ultimately the main reason why they decided that, in 2010, we would be referenced in the Science Citation Index.

We have made a big effort to build a reputation, which was achieved also by the prestige of some associate editors who agreed to be with us from the very beginning. The idea to found the journal was not actually mine. The idea was of the people in charge of the *Revista de Estatística* from INE (the National Institute of Statistics) who contacted me, during the European Meeting of Statisticians, in 2001, in Funchal, to be editor in chief of the journal. Then, we thought it would be a good idea to seize the opportunity and invite some important people participating in the meeting to assemble a strong editorial board. Among them, were sir David Cox, Jef Teugels, . . . There were a few people that from the very beginning were very enthusiastic about the launching of REVSTAT, who helped a lot. Some of them were from Portugal like Antónia and Dinis. So, I was not alone, fortunately.

A controversial topic in the order of the day refers to the difficulties women go through in order to thrive in their careers. Worldwide there are much more male mathematicians than female and only in 2014 a woman was awarded a Fields medal. We are interviewing a very successful Portuguese mathematician who happens to be woman. Did you ever feel any difficulty because of that?

Never. Fortunately, I always felt that I was treated as everyone else. Easy question.

One of the most impressive aspects that we find in your career is the fact that you are responsible for the creation of strong community of statisticians dedicated to Extremes, in Portugal. Leadbetter even kindly described it as Ivette and her chicks.

The gang, the portuguese gang . . .

Do you also consider this as one of your best accomplishments in your career?

Yes, I think so. Indeed, I never had that specific objective. But the truth is that when I look now at what is happening in Portugal, I think I helped a lot with the obvious help of Tiago de Oliveira. Tiago was the pioneer and, together with Tiago, I helped a lot in the building of what we can nowadays call a school of Extremes. And so, ok, I'm proud of it.

So, you are proud of this legacy.

I'm proud. I'm proud of this legacy. And I'm proud of people who have really been involved in the field of Extremes. And you are two of them, obviously. Ana Cristina began with Margarida Brito, another pioneer in the field, who came from Paris, and worked under the supervision of Paul Deheuvels . . . And many more. Fortunately, I'm not alone . . .

Sure. But responsible for the . . .

Responsible maybe for some dynamism.

Is it a coincidence that most of your *descendants* are women?

In our faculty, for instance, there are more women than men working in Statistics. I mean, there are more women getting degrees in Statistics.

It's not an extreme thing, then . . .

I do not think so . . . But, in fact, among around twenty PhD students, I only had two male students.

You were president of SPE, a member of CIM's scientific committee, editor-in-chief of REVSTAT, member of the editorial board of Extremes, a full time full professor, advisor of many students and postdocs, a wife, a mother and, more recently, a dedicated grand-mother. How did you manage to cope it all? Did you ever feel you had too many responsibilities?

Not really . . . Nowadays, things are different, obviously. Nowadays I feel a bit tired, a bit old, I have not time for doing many things. But when I was doing those jobs, let's say, I was doing them enthusiastically. But I also had a lot of help, help from my mother, which I'll never forget, and also the help of my husband. The help of my mother and of my husband were crucial. And afterwards I was able, with enthusiasm and maybe with some dynamism, to work almost around 100%, or maybe more than 100%.

I would say that I was able to manage on both duties, reasonably well.

We have always known you as a very active and committed person. In fact, we have seen you being invited in several occasions either to participate, speak or simply borrow your experience and reputation to the events and projects, in some of which we were also involved. Not only you always say yes as you always engage enthusiastically and with your characteristic good mood in all initiatives. Have you ever had to say no?

It's very difficult for me to say no. Sometimes, even when I say no, I keep reflecting on it and afterwards the *no* becomes a *yes*.

But nowadays I am already able to say no several times. But sometimes it's difficult. For example, one month ago I had an invitation to go to Kiev. I would like to go there and I had to say no. Sometimes it's just not possible . . .

How did an average guy and an extreme girl come together to work on an extreme paper published in the Journal of the American Statistical Association?

You know that my husband, Dinis Pestana, also always liked the field of Extremes. And there is a big link between Extreme Value Theory and some of Dinis' works related with stable laws. So, we have worked jointly a long time ago, essentially in the topic of Penultimate Approximations and Rates of Convergence. Afterwards, he also became interested in Risk and that was maybe the main reason why he was involved in that paper in JASA, which is related to the estimation of the Value-at-Risk in the reduced-bias framework. Dinis has some nice ideas and indeed he was the guy who gave me the idea of reducing the bias, first in the extreme value index and next in the Value-at-Risk estimation. More recently, he gave me another nice idea related to the mean-of-order-p. He just asked me and Isabel why we were always studying the Hill estimator. And instead of working with the Hill estimator, which just considers the logarithm of the mean of order zero, he suggested us to work with the logarithm of the mean of order p, for any real p . . .

How was to work with your husband? Did you ever fight because of that?

Sometimes we had some discussions but not very strong discussions . . .

And do you have any future project with him?

In a certain sense . . . I think we can continue to work on these generalised means topic. But we are working less and less. It's true that research is a vice, but we need to be more and more calm.

Besides that particular paper, among your many results, is there one that you are particularly proud of?

Good question, but I go again back to the Penultimate Approximations and Rates of Convergence. I like the topic, it's more Mathematics than Applied Statistics but as you know, recently, but not with Dinis, we have been applying it to Reliability. We think that there is still something that can be done in that field. But I also like the generalized means . . .

Scientifically speaking, do you have any particular unfulfilled goal that you still would like to accomplish?

Indeed, I think I would like to go into a spacial framework . . . But it's too much for me at the moment . . . I have it in my head, but it's not going to be fulfilled, for sure.

Are you sure?

Almost sure . . .

Since your retirement in 2013, there have been a few conferences and events in your honour, such as the EVT2013, in Vimeiro, or the 7th International Conference on Risk Analysis, ICRA2017, in Chicago, or the award of the title of *Professor Emérito* of the University of Lisbon. Yet, who knows you well, knows you are a very humble and simple person, who we have already seen running after our six-year-old boy in the streets of Sevilla, a year ago. Do you ever feel a bit embarrassed with all such praising during those events? Or you just take it naturally?

Sometimes I feel a bit embarrassed. I felt particularly a bit embarrassed when I got the degree of *Professor Emérito*, the title of *Emeritus*. Indeed, at the time, I just mentioned that I felt that there were many people in the faculty who also deserved such a kind of distinction. I think it was the unique time where I indeed felt a bit bad. The other times, no. EVT in Vimeiro . . ., it was among friends . . . Well, I cannot say that I was not among friends when I got the distinction of Emeritus. But the truth is that I think that there were people who possibly deserved more . . .

For sure you deserved it . . .

I don't know . . . Ok, I am not humble enough to say that I didn't deserve, but I think that there are other people who also deserved such a kind of distinction, at least as much. They are very strict with the number . . . Only four people in the entire Faculty of Sciences got it and I think it's a very small number when compared with the number of people who indeed have contributed a lot for the image of the faculty.

In light of the recent developments and events in the Portuguese scientific policy, do you have any comment/suggestion/advice for the people in charge of the ministry of science, technology and higher education and the national science foundation?

The last known FCT evaluation of Research Centres in Portugal, associated with the period 2008–2013, was a nightmare and introduced a strong retroaction factor in the development of Science in Portugal, when such a development had been quite positive in the most diverse areas, including Statistics. Among others, I can mention two of the big problems related to the evaluation: 1) The specificity of Statistics in the area of Mathematics is recognised internationally, and it is crucial to have in the evaluation panel at least a recognised scientist in the area. Indeed, this happened in prior panels, with the integration of researchers like David Cox, Anthony Davison and Richard Smith, among others. But it did not happen for the aforementioned period; 2) Also, FCT was unable to ask in time to questions placed by researchers. I merely mention, among others, what happened with our research centre (CEAUL), with an excellent production in the above mentioned period. No answer was given by FCT to our rebutal for more than two years, and this led to a null funding in 2015. We could survive only due to to some projects CEAUL had in hands, but the process was exhausting and will surely have a strong negative reflex in our research centre and in the development of Science in Portugal.

Given your extensive experience, do you have any advice for these young researchers who are struggling to build a career in Mathematics?

Be enthusiastic, work hard and never give up.

REPORT

Workshop on Graphs Spectra and Combinatorics

by **Paula Carvalho***, **Sofia Pinheiro****, and **Tatiana Tchemisova*****

WGSCO 2018, Workshop on Graph Spectra, Combinatorics and Optimization was organized by a group of docents of the Mathematics Department and members of the Research Unit CIDMA —- Center for Research and Development in Mathematics and Applications of the University of Aveiro, Portugal, on occasion of the 65th birthday of Prof. DomingosM. Cardoso, prominent professor and researcher of the University of Aveiro.

The topics of Workshop reflected the diversity of the scientific interests of Prof. Domingos M. Cardoso and the main lines of research of the Group on Graph Theory, Optimization and Combinatorics, which is been coordinating by him during many years within the Research Unit CIDMA of the Mathematics Department of the University of Aveiro: Algebraic Combinatorics, Algebraic Graph Theory, Algorithms and Computing Techniques, Combinatorial Optimization, Communications and Control Theory, Enumerative Combinatorics, Extremal Combinatorics, Graph Theory, Optimization in Graphs, Graph Spectra and applications, Linear Optimization, Networks, Nonlinear Optimization and others.

The organizational duties were distributed between an International Scientific Committee, the local Organizing Committee and three co-chairs: Paula Carvalho, Sofia Pinheiro, and Tatiana Tchemisova.

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The Scientific Program included 9 plenary talks: *Scale-free graphs* by Jerzy Szymański (University of Adam Mickiewicz, Poznan, Poland), *On the scheduling of periodic events* by Jorge Orestes Cerdeira (New University of Lisbon, Portugal), *On the various concepts of central vertex sets of a graph* by Nair Abreu (Federal University of Rio de Janeiro, Brazil), *Eigenvalue Multiplicity in Regular Graphs* by Peter Rowlinson (University of Stirling, Scotland, UK), *An Introduction to Combinatorial Matrix Theory* mod *k* by Richard A. Brualdi (University of Wisconsin, Madison, Wisconsin, USA), *Salem Graphs*" by Slobodan Simić (Serbian Academy of Sciences and Arts, Belgrade, Serbia), *Prison Guards Dilemma: Optimal Inmate Assignment by Multiobjective Mixed Integer Linear Optimization* by Tamás Terlaky (Lehigh University, Bethlehem, Pennsylvania, USA), *From structure to algorithms: breaking the boundaries* by Vadim Lozin (University of Warwick, UK), and *Spectral characterizations of graphs* by Willem H. Haemers (University of Tilburg, Netherlands). Unfortunately, the talk of Prof. Slobodan Simić was cancelled by the health problems of the speaker. The workshop was attended by 115 registered participants coming from 27 countries.

During the Opening Session in the morning of January 25, the participants were welcomed by the Rector of the University of Aveiro, Prof. Manuel Assunção, by the Director of the Department of Mathematics, Prof. João Santos, by the Scientific Coordinator of the CIDMA Research Unit, Prof. Luís Castro, and by the co-chairs of the Workshop.

The scientific program of the Workshop was composed by 32 sessions with about 80 talks divided into 4 main streams: Graph Theory, Graph Spectra, Combinatorics, and Optimization. There was 8 Invited Talks and 4 Special Sessions, as follows: *Spectral Graph Theory and its Applications*, organized by Irene Sciriha, *Matrices and Applications I* organized by Enide Andrade and Rute Lemos, *Matrices and Applications II*, organized by Raquel Pinto and Diego Napp, and *Bruhat Order*, organized by Rosário Fernandes and Henrique Cruz.

Among the participants, there were about 20 young researchers and PhD Students. All submitted abstracts were refereed by the members of the International Scientific Committee and published in the Book of Abstracts.

The participants of the Workshop are invited to submit their papers to two Special Issues of international journals: Optimization by Francis & Taylor and DAM, Discrete and Applied Mathematics by Elsevier.

The success of the Workshop owes much to the support of different national and international organizations: Mathematics Department of the University of Aveiro, CIDMA — Center for Research and Development in Mathematics and Applications of the University of Aveiro, FCT — Portuguese Foundation for Science and Technology, APDIO — Portuguese Association of the Operations Research, FLAD — Luso-American Development Foundation, ILAS — International Linear Algebra Society, EURO Working Group on Continuous Optimization EUROPT.

The Organizing Committee from the Mathematics Department of the University of Aveiro has made the best to offer to the participants an interesting Social Program, get them acquainted with the University of Aveiro and the beautiful region of Portugal where it is situated.

The social program of the conference included a Welcome Reception on the first day of the Workshop and a guided visit to the Vista Alegre Porcelain Factory, Museum and Chapel, on the last day. These two events were included into program of the event and were free of charge for all participants. On occasion of 65th birthday of Professor Domingos M. Cardoso, a small festive ceremony was organized during the Conference Dinner in the Hotel Melia, Ria, of Aveiro. During the dinner, took place a short presentation of the life and work done until this moment of Prof. Domingos M. Cardoso, many nice friendly words were said, and participants had opportunity to listen some music, including Fado, try typical Portuguese gastronomy, and communicate in a friendly and informal atmosphere.

More information about the program of the Workshop, participants, Invited Speakers, Special and Contributes sessions, photo gallery and other, can be found on the conference webpage http://wgsco2018.web.ua.pt/

Mathematics and Wild Fires

by **J. A. Ferreira***; **I. N. Figueiredo****; and **J. F. Rodrigues*****

In 2017 the center of Portugal, where the city of Coimbra is localised, was devastated by severe wild fires. Taking into account that Mathematics can play a significant role in predicting fire behaviour and spread, as well as, in defining suitable strategies to prevent and combat fires, the CIM (Centro Internacional de Matemática), in cooperation with CMUC (Centre for Mathematics of the University of Coimbra), CMAFcIO (Center for Mathematics, Fundamental Applications and Operations Research of the University of Lisbon) and APMTAC (Portuguese Association of Theoretical, Applied and Computational Mechanics) organized a small workshop, with about forty registered participants, that took place the 8-9 November 2018 at the Mathematics Department of the University of Coimbra.

Alberto Bressan (Penn State University, USA) opened the meeting with a description of a new class of variational problems, seeking optimal strategies for the confinement of forest fires. Assuming the spreading of the wild fire can be controlled by constructing barriers, he addressed the existence (or non-existence) of a strategy that completely blocks the spreading of the fire, within a bounded domain by reducing it to an optimization problem, where one seeks to minimize the total value of the burned region, plus the cost of building the barrier. Deriving necessary conditions for optimality, nearly optimal strategies can be numerically constructed. Various open problems were also presented.

A generalization of the Richards' equations for the parametric spread of ellipse-borne wildfires, was present-

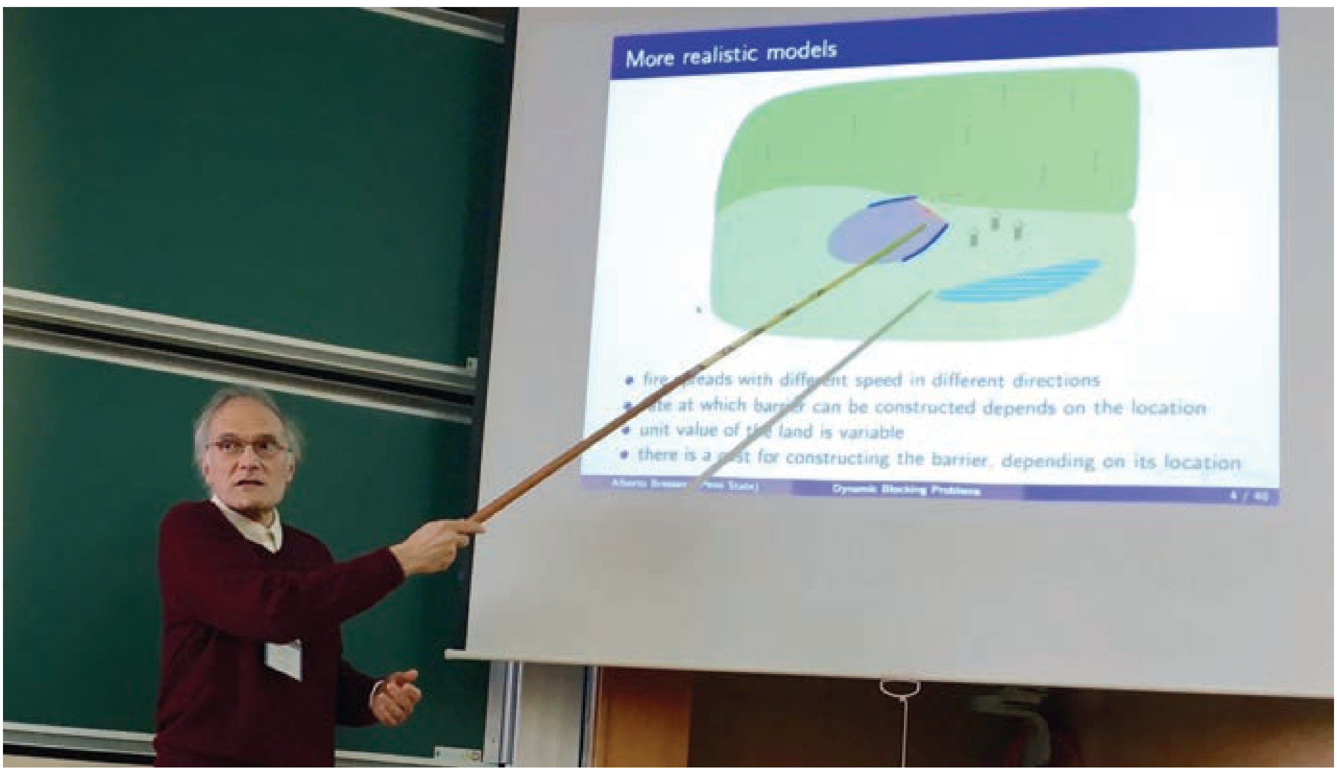


Figure 1.—A. Bressan — Dynamic blocking problems for a model of fire confinement

* APMTAC & UCoimbra; ** CMUC & UCoimbra; *** CMAFcIO & ULisboa

ed as a Finsler geometric paradigm for wildfire spread modelling by Steen Markvorsen (Technical University of Denmark). He aimed to cover any type of (possibly time-dependent) ovaloid-borne wildfires in dimensions 2 and 3. Considering a given ovaloid at a given point in space-time as the local indicatrix, i.e. the local firelet, that is obtained by short (unit)time spread of a model fire from that point, ignited at that given time, and under the assumption of homogeneous (linearized) measures of fuel, wind, and topography, he considered a Finsler geometric framework. In this setting the generalized Richards' equations can be formulated as a time-dependent eikonal type Finsler-Hamilton-Jacobi equation, which can be solved – and thence the corresponding wildfire spread problem – using results from the differential geometry of geodesic sprays and/or the control geometry of differential inclusions. Finally he also addressed the important problem of including the curvature of the front by modifying the Finsler eikonal equation into a second order equation, which (by construction) produces the observed initial increase in fire particle speed from point ignitions.

Gianni Pagnini (Basque Center for Applied Mathematics, Spain) talked on a versatile probabilistic approach for modelling fire-spotting. He based his approach on the weighted superposition of random fronts whose fluctuations of the position are distributed according to proper densities functions including the random effects of turbulent heat transfer and fire-spotting. In particular, a Gaussian density was considered for turbulence and a lognormal for the landing distance of the firebrands and a global sensitivity analysis for the proposed fire-spotting model has been performed. The proposed fire-spotting model has been implemented in WRF-Sfire, a full-fledged coupled fire-atmosphere model (http://www.openwfm.org/wiki/WRF-SFIRE), using the Weather Research and Forecasting model (WFR). A test case has been performed and was discussed at the workshop.

The need of efficient fire prediction and fighting systems was emphasized by Wenceslao González Manteiga (Universidad de Santiago de Compostela, Spain), by using and developing statistics and optimization techniques. Reporting his research work with members of the MODESTYA group at his university, he described applied nonparametric inference techniques for spatial and spatio-temporal point processes to understand wildfire behavior, and operations research techniques for an optimal allocation of limited resources, such as aircrafts, in fire extinction. Working with a large number of data (more than 100.000 wildfires between 1999 and 2014) in Galicia (NW Spain), and a large number of covariates that may be in-

Figure 3.— M. A. Turkman — Statistical methods towards the construction of decision tools to assist wildfire management.

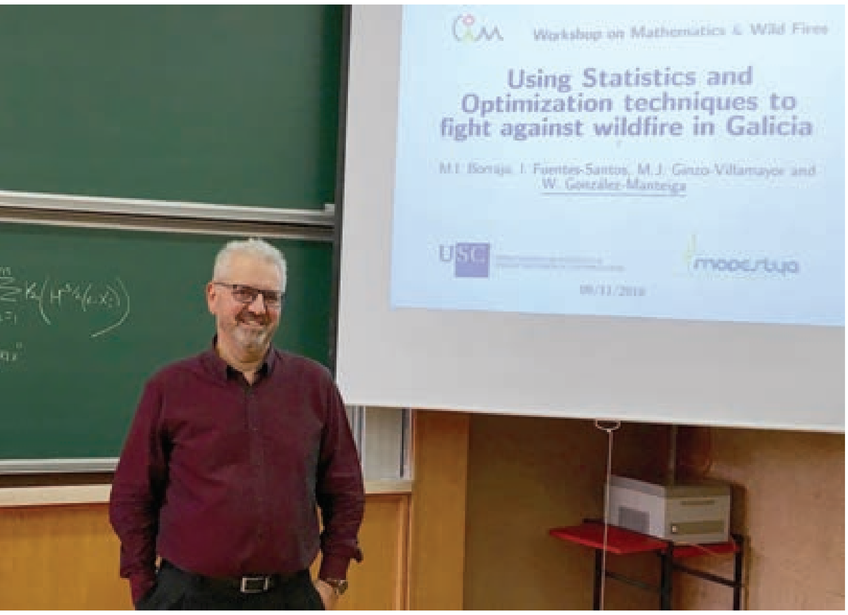


Figure 2.

Left: G. Pagnini — Fire-spotting modelling for regional-scale wildfire simulators: a case study with WRF-Sfire.

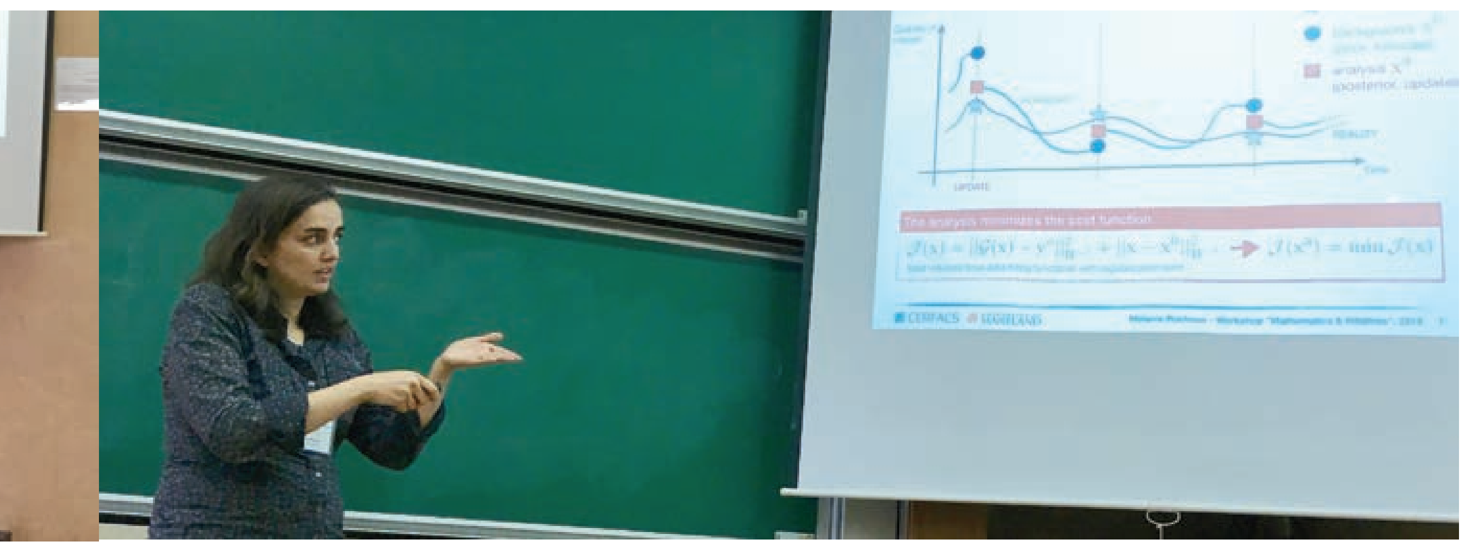
Right: W. G. Manteiga — Using statistics and optimization techniques to fight against wildfires in Galicia.

volved in wildfire risk, which increase the computational demand and complexity of a Big Data scenario, his talk outlined the challenges found in the analysis of wildfires.

Maria Antónia Amaral Turkman (Universidade de Lisboa, Portugal) presented statistical methods towards the construction of decision tools to assist wildfire management, which were exemplified with the fire risk maps for 2018, based on satellite data of fire ignition and burned area in Portugal from 1988 to 2017. These maps were obtained in May 2018 and, for instance, spotted the Monchique area, in the South of Portugal, as a potential danger place, where in fact the greatest fires of 2018 in Portugal took place, and, in the North, the area of Caminha were prevention measures were effective. They are considered useful tools for decision makers to allocate firefighting capacities and to support fire/forest management decisions in space, according to the risks involved. Using data sources coming from satellite images and ground sources, and exploiting a Markovian structure for the fire incidence data, her team constructed a model able to capture, as much as possible, the strong spatio-temporal dependence structures in the fire incidence data, allowing at the same time for the introduction of any type of dynamic explanatory variables in the model. This was achieved through Bayesian hierarchical modeling techniques and simulation-based inference.

Dominique Morvan (Université Aix-Marseille, France) spoke on wildfires physics and modelling. Observation that the behavior of wildfires is governed by various physical mechanisms, at different scales in space (and time), ranged between less than 1mm (the flame) to larger than 100km (the plume), he stressed the numerical simulation

Figure 4.—M. Rochoux — Overview and challenges of data-driven wildland fire spread modeling.



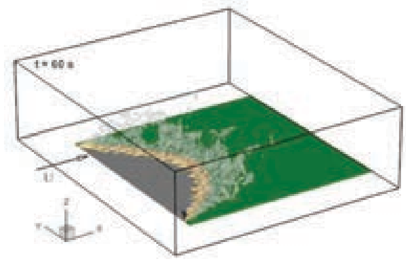


Figure 5.—Example of numerical simulation of a surface fire propagation in a grassland from Morvan's talk (left) and a view of the Mathematics and Wild Fires Workshop attendance (right).

of wildfires is a high challenging multiscale problem. Despite these difficulties, the resolution of some problems in fire safety engineering such as the propagation of a fire front through a wildland urban interface (WUI), needs to describe a fire at a relative local scale (few hundred meters), with a relatively high level of details. It is in this context, that a new class of fire models, referred in the literature as "fully physical models", has been proposed at the end of 90's. This approach, which is often referred in the literature as a multiphase formulation, consists of formulating the problem with the balance equations (mass, momentum, energy, etc.) of the coupled system formed by the vegetation and the surrounding atmosphere. A short presentation of what is a fully physical wildfires model and

Figure 6.—The table of the final session on challenges, open problems and scientific interaction (from left to right): W. G. Manteiga, M. A. Turckman, R. R. Linn, M. Rochoux, S. Markvorsen and J. F. Rodrigues.

some results obtained with this kind of approach was also given.

Although data-driven wildfire spread models are still at an early stage of development, Mélanie Rochoux (CECI, CNRS-CERFACS, France) provided an excellent overview of the current strategies and she highlighted some of the challenges and opportunities this new data assimilation approach offers. This most important aspect of wild fires modeling is a new approach that couples existing models and real-time observations of the wildland fire dynamics, with the objective of reducing the uncertainties in both model fidelity and input data. This approach, called "data-driven modeling" (or "data assimilation"), takes full advantage of the recent advances in remote sensing technology to improve forecasts of the wildland fire evolution.

Finally, all the speakers have participated at the Final Session on Challenges, Open Problems, Scientific Interaction, which was coordinated by José Francisco Rodrigues (Universidade de Lisboa) and had the special participation of Rodman Ray Linn, a well-known expert in computational models for wildfire behavior at the Los Alamos National Laboratory, USA. Rodman Linn made a short presentation plenty of mathematical opportunities in fire modeling and simulation, including the importance of the curvature in the fire front, the need to understand mathematically the breaking of the "wall of flames" and the influence of insects and of other ecological and environmental conditions in the variation of fire in forests.

W. Manteiga referred the interest of fire models for the insurance companies and shared his experience with the building of a COST proposal. J. F. Rodrigues asked for a model to describe the perturbation and breaking of telecommunications in the fire areas, a serious accident that occurred in the 2017 fires in Portugal. M. A. Turkman reported her experience in a current multidisciplinary national project on fire and M. Rochoux spoke of several open issues in fire modeling involving data sets, deep learning from images, data reduction and visualization.

Figure 7.—From left to right: A. Bressan, D. Morvan, M. Rochoux, W. G. Manteiga, M. A. Turckman, J. F. Rodrigues, S. Markvorsen, G. Pagnini, I. N. Figueiredo, J. A. Ferreira.

Figure 8.— Mathematization of Knowledge is a main theme of one of the 1969 Almada Negreiros' frescoes of the Department of Mathematics of the University of Coimbra.

Several participants also raised questions, emphasized the need of communication across disciplines and sharing data. S. Markvorsen also highlighted on the green board the significant phrase "Mathematization of Knowledge" that is written in a beautiful fresco by Almada Negreiros at the entrance of the Department of Mathematics of the University of Coimbra.

Mathematics and Literature III in Folio 2018

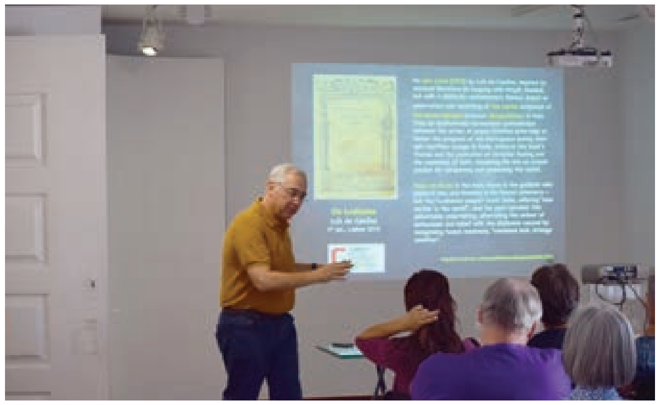
An International Workshop held in Óbidos, Portugal 29th–30th September

by **António Machiavelo***

The *International Center for Mathematics* (www.cim.pt), in partnership with the *Science Museum of the University of Coimbra* (www.museudaciencia.org) and the *Portuguese Mathematical Society* (www.spm.pt), organized the *Mathematics and Literature III*, a two-day international workshop on Literature and Mathematics, with the support of the *Center for Mathematics, Fundamental Applications and Operational Research of the University of Lisbon* (cmafcio.campus.ciencias.ulisboa.pt), and the *FCT Doctoral Program in Materialities of Literature of the Faculty of Arts and Humanities of the University of Coimbra* (www.apps.uc.pt/courses/en/course/2341). This initiative took place at the Municipal Museum, on the occasion of the FOLIO 2018 — *Óbidos Literary International Festival* (www.foliofestival.com), organised and hosted by the medieval town of Óbidos (www.obidos.pt).

* Centro de Matemática da Universidade do Porto, Dep. de Matemática da Faculdade de Ciências do Porto

António Machiavelo



José Francisco Rodrigues

Robin Wilson

Angelo Guerraggio

Krzysztof Ciesielski

The workshop consisted of four presentations in english, on the first day, and four talks in portuguese and a bilingual one in french and portuguese, on the second day. The talks covered significant aspects of some interconnections between Mathematics and Literature.

The first talk, a joint presentation by António Machiavelo (University of Porto) and José Francisco Rodrigues (University of Lisbon), brought up *some unexpected interactions between Mathematics and Literature*, ranging from Luís de Camões and Pedro Nunes, Sylvester and his *Laws of Verse or Principles of Versification*, Joaquim Namorado and António Aniceto Monteiro, to Bourbaki and Oulipo, and ending with a challenge to the audience to create long words in portuguese, or any other language, with no repeated letters.

Robin Wilson (Open University, UK) presented *the fantastical mathematical logical life of Lewis Carrol in Numberland*, describing the mathematical and literary lifes of Charles Dodgson in the context of Victorian Oxford, his writings for children and adults, his passion for photography and Euclid, his witty word play. Angelo Guerraggio (Università Bocconi, Milano) talked about the fascinating Leonardo Sinisgalli (1908–1981), a poet with a mathematical background, and a key figure at the *crossroads* of literature, mathematics and industrial culture in Italy in the middle 20th century. Krzysztof Ciesielski (Jagiellonian University, Krakow, Poland) offered several glimpses of *elements of mathematics in some science-fiction novels and stories* of Stanisław Lem (1921–2006), Januz A. Zajdel (1938–1985), as well as John Boyd, Robert Heinlein, Rudy Rucker and Alexander Zytinski.

Nuno Camarneiro

Joana Gomes

Flávio Coelho

Cristina Cordeiro and Carlota Simões

Manuel Portela and Ana Silva

On the second day, Nuno Camarneiro (Leya prize recipient in 2012, presently at the Catholic University of Porto) opened the session with some thoughts on *the fiction of time*, providing some literary and some scientific views of the elusive concept of time. Joana Gomes (University of Porto), in *The Waltz of exactness: the Vienna Circle and the Concert of Mathematics, Philosophy, and Literature in the Interwar Period*, described the principles defended by the Vienna Circle, their interaction with the political and social backgrounds of this period, and their influence in the arts and literature, pointing out the influence of some mathematicians and of Mathematics in that circle. Flávio Ulhoa Coelho (University of São Paulo, Brazil) offered a glimpse of *two faces of the same coin: Literature and Mathematics* from the perspective of his dual life, as a writer and as a mathematician.

Cristina Robalo Cordeiro and Carlota Simões, both from the University of Coimbra, presented, in a humorous and delighful duet, the *Mémoire concernant le calcul numérique de Dieu par des méthodes simples & fausses*, a text of Boris Vian, full of wit and satire, mixing mathematics, literature and religion. Finally, Ana Marques da Silva and Manuel Portela, both from the University of Coimbra, acted to offer the participants a performance from some of their experiences mixing Literature with Cybernetics.

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Symbolic Dynamics and Semigroup Theory

by **Alfredo Costa***

A major motivation for the development of semigroup theory was, and still is, its applications to the study of formal languages. Therefore, it is not surprising that the correspondence $\mathcal{X} \mapsto B(\mathcal{X})$, associating to each symbolic dynamical system $\mathcal{X}$ the formal language $B(\mathcal{X})$ of its blocks, entails a connection between symbolic dynamics and semigroup theory. In this article we survey some developments on this connection, since when it was noticed in an article by Almeida, published in the CIM bulletin, in 2003 [2].

1 Symbolic dynamics

A *topological dynamical system* is a pair (X, T) consisting of a topological space X and a continuous self-map $T: X \to X$. It is useful to think of X as representing a sort of *space*, where each point x is moved to $T(x)$ when a unit of time has passed. A morphism between two topological dynamical systems (X_1, T_1) and (X_2, T_2) is a continuous map $\varphi: X_1 \to X_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$. In this way, topological dynamical systems form a category, if we take the identity on X as the local identity at (X, T). In this category, an isomorphism is called a *conjugacy*, and isomorphic objects are said to be *conjugate*.

We focus on a special class of topological dynamical systems, the symbolic ones. Their applications in the study of general topological dynamical systems frequently stem from the following procedure: use symbols to mark a finite number of regions of the underlying space, and register, with a string of those symbols, the regions visited by a orbit. In the next paragraph we give a brief formal introduction to symbolic systems. For a more developed introduction, we refer to the book [26]. Also, the book review [33] is an excellent short introduction to the field and its ramifications.

Consider a finite nonempty set A, whose elements are called *symbols*, and the set $A^{\mathbb{Z}}$ of sequences $(x_i)_{i\in\mathbb{Z}}$ of symbols of A indexed by $\mathbb{Z}$. One should think of an element $x = (x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ as a biinfinite string $\ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2x_3\ldots$, with the dot marking the reference position. A *block* of x is a finite string appearing in x: a finite sequence of the form $x_kx_{k+1}\ldots x_{k+\ell}$, with $k \in \mathbb{Z}$ and $\ell \geq 0$, also denoted $x_{[k,k+\ell]}$. Of special relevance are the central blocks $x_{[-k,k]}$, as one endows $A^{\mathbb{Z}}$ with the topology induced by the metric $d(x, y) = 2^{-r(x,y)}$ such that $r(x, y)$ is the minimum $k \geq 0$ for which $x_{[-k,k]} \neq y_{[-k,k]}$. Hence, two elements of $A^{\mathbb{Z}}$ are *close* if they have a *long* common central block. The *shift mapping* $\sigma: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$, shifts the dot to the right. A *symbolic dynamical system*, also called *subshift*, is a pair $(\mathcal{X}, \sigma_{\mathcal{X}})$ formed by a nonempty closed subspace $\mathcal{X}$ of $A^{\mathbb{Z}}$, for some A, such that $\sigma(\mathcal{X}) = \mathcal{X}$, and by the restriction $\sigma_{\mathcal{X}}$ of σ to $\mathcal{X}$. As it is clear what self-map is considered, $(\mathcal{X}, \sigma_{\mathcal{X}})$ is identified with $\mathcal{X}$. When $\mathcal{X} = A^{\mathbb{Z}}$, the system is a *full shift*. The *sliding block code* from the subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ to the subshift $\mathcal{Y} \subseteq B^{\mathbb{Z}}$, with *block map* $\Phi: A^{m+n+1} \to B$, *memory* m and *anticipation* n, is the map $\varphi: \mathcal{X} \to \mathcal{Y}$ defined by $\varphi(x) = (\Phi(x_{[i-m,i+n]}))_{i\in\mathbb{Z}}$. It follows from the definition of the metric on a full shift that the morphisms between subshifts are precisely the sliding block codes.

2 Formal languages

Given a set A of symbols, the set of finite nonempty strings of elements of A is denoted by A^+. In the jargon of formal languages, A is said to be an *alphabet*, the elements of A and those of A^+ are respectively called *letters* and *words*, and the subsets of A^+ are *languages*. Moreover, A^+ is viewed as a semigroup for the operation *concatenation of words*. For example, in $\{a, b\}^+$, the product of aba and bab is $ababab$. In fact, A^+ is the *free semigroup generated by* A, since, for every semigroup S, every mapping $A \to S$ extends uniquely to a homomorphism $A^+ \to S$. Concerning semigroups, for-

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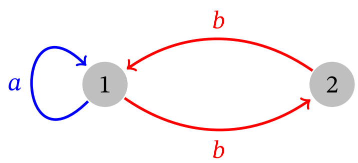
mal languages, and their interplay, we give [4] as a source of detailed references and as a very convenient guide, since, in this sort of introductory text, connections with symbolic dynamics are also highlighted.

If $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$, we let $B(\mathcal{X})$ be the language over the alphabet A such that $u \in B(\mathcal{X})$ if and only if u is a block of some element x of $\mathcal{X}$. As a concrete example, consider the subshift $\mathcal{E}$, known as the *even shift*, formed by the biinfinite sequences of a's and b's that have no odd number of b's between two consecutive a's, that is, the biinfinite paths in the following labeled graph:



A language L is *factorial* if, for each $u \in L$, every factor of u belongs to L. A factorial language over A is *prolongable* if $u \in L$ implies $aub \in L$ for some $a, b \in A$. It is easy to see that the languages of the form $B(\mathcal{X})$, with $\mathcal{X}$ a subshift of $A^{\mathbb{Z}}$, are precisely the factorial prolongable languages over A. Moreover, the correspondence $\mathcal{X} \mapsto B(\mathcal{X})$ is a bijection between subshifts and factorial prolongable languages. Moreover, one has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $B(\mathcal{X}) \subseteq B(\mathcal{Y})$. In view of this bijection, symbolic dynamics may be regarded as a subject of formal language theory.

Semigroups appear in the study of formal languages via the concept of *recognition*. In the labeled graph of the figure above, letters a and b may be seen as the binary relations $a = \{(1, 1)\}$ and $b = \{(1, 2), (2, 1)\}$. Let $S(\mathcal{E})$ be the semigroup of binary relations, on the vertices 1 and 2, generated by a and b. For example, ab is the binary relation $\{(1, 2)\}$. The words in $B(\mathcal{E})$ are precisely the words that in $S(\mathcal{E})$ are not the empty relation $\varnothing$. Formally, given a semigroup homomorphism $\varphi\colon A^+ \to S$, a language $L \subseteq A^+$ is *recognized* by φ if $L = \varphi^{-1}(P)$ for some subset P of S. Note that $B(\mathcal{E})$ is recognized by the homomorphism $\varphi\colon \{a, b\}^+ \to S(\mathcal{E})$ such that $\varphi(a) = \{(1, 1)\}$ and $\varphi(b) = \{(1, 2), (2, 1)\}$, since $B(\mathcal{E}) = \varphi^{-1}(S(\mathcal{E}) \setminus \{\varnothing\})$.

A language over A is *recognized by the semigroup S* when recognized by a homomorphism from A^+ into S. It is said to be *recognizable* if it is recognized by a finite semigroup. Recognizable languages constitute one of the main classes of languages, as they describe *finite-like* properties of words, captured by finite devices. Frequently the devices are finite automata, which are labeled graphs with a distinguished set of initial vertices and final vertices. These devices recognize the words labeling the paths from the initial to the final vertices. Recognition by a finite automaton is the same as recognition by a finite semigroup, because in fact an automaton may be seen as a semigroup with generators acting on its vertices.

Another reason why recognizable languages matter is Kleene's theorem (1956) [22], stating that the recognizable languages of A^+, with A finite, are precisely the *rational* languages of A^+, that is, the languages which can be obtained from subsets of A by applying finitely many times the Boolean operations, concatenation of languages, and the operation that associates to each nonempty language L the subsemigroup L^+ of A^+ generated by L. The rational languages obtainable using only the first two of these three sets of operations, the *plus-free* languages,[1] are precisely the languages recognized by finite aperiodic semigroups [31]. This characterization, due to Schützenberger and dated from 1965, is one of the first important applications of semigroups to languages (for the reader unfamiliar with the concept: a semigroup is aperiodic if all its subgroups, i.e., subsemigroups that have a group structure, are trivial). Eilenberg, later on (1976), provided the framework for several results in the spirit of that of Schützenberger on aperiodic semigroups, by establishing a natural correspondence between *pseudovarieties of semigroups* (classes of finite semigroups closed under taking homomorphic images, subsemigroups and finitary products) and the types of classes of languages recognized by their semigroups, called *varietes of languages* [17].

3 Classification of subshifts

The correspondence $\mathcal{X} \mapsto B(\mathcal{X})$ provides ways of classifying subshifts in special classes with *static* definitions in terms of $B(\mathcal{X})$ that, from a semigroup theorist viewpoint, may be more convenient than the alternative definitions of a more *dynamical* flavor.

As a first example, consider the *irreducible* subshifts: these are the subshifts $\mathcal{X}$ such that, for every $u, v \in B(\mathcal{X})$, one has $uwv \in B(\mathcal{X})$ for some word w. The dynamical characterization is that a subshift is irreducible when it has a dense forward orbit.

In the same spirit, a subshift $\mathcal{X}$ is *minimal* (for the inclusion) if and only if $B(\mathcal{X})$ is *uniformly recurrent*, the latter meaning that for every $u \in B(\mathcal{X})$, there is a natural number N_u such that u is a factor of every word of $B(\mathcal{X})$ of length N_u. Note that uniform recurrence implies irreducibility. A procedure for building minimal subshifts, with a semigroup-theoretic flavor that was useful for getting results mentioned in the final section, is as follows. Consider a *primitive substitution* $\varphi\colon A^+ \to A^+$, i.e., a semi-

[1] Actually, Schützenberger's result is usually formulated in terms of finite aperiodic monoids and languages admitting the empty word, with *star-free* languages in place of *plus-free* languages.

group endomorphism φ of A^+ such that every letter of A appears in $\varphi^n(a)$, for all $a \in A$ and all sufficiently large n: if φ is not the identity in an one-letter alphabet, then the language of factors of words of the form $\varphi^k(a)$, with $k \geq 1$ and $a \in A$, is factorial and prolongable, thus defining a subshift $\mathcal{X}_\varphi$, and in fact this subshift is minimal.

A subshift $\mathcal{X}$ is *sofic* when $B(\mathcal{X})$ is recognizable. Hence, the even subshift is sofic. Sofic and minimal subshifts are arguably the most important big realms of subshifts, with only periodic subshifts in the intersection. Every subshift $\mathcal{X}$ of $A^\mathbb{Z}$ is characterized by a set F of *forbidden blocks*, a language $F \subseteq A^+$ such that $x \in \mathcal{X}$ if and only if no element of F is a block of x. We write $\mathcal{X} = \mathcal{X}_F$ for such a set F. It turns out that $\mathcal{X}$ is sofic if and only if F can be chosen to be rational. A subshift $\mathcal{X}$ is of *finite type* if there is a finite set of forbidden blocks F such that $\mathcal{X} = \mathcal{X}_F$. The class of finite type subshifts is closed under conjugacy and is contained in the class of sofic subshifts. The inclusion is strict: the even subshift is not a finite type subshift.

The most important open problem in symbolic dynamics consists in classifying (irreducible) finite type subshifts up to conjugacy. A related problem is the classification of (irreducible) sofic subshifts up to *flow equivalence*. In few words, two subshifts are flow equivalent when they have equivalent mapping tori, a description that is somewhat technical, when made precise. Next is an alternative characterization (from [29]), more prone to a semigroup theoretical approach. Take $\alpha \in A$ and a letter $\diamond$ not in A. Consider the homomorphism $E_\alpha : A^+ \to (A \cup \{\diamond\})^+$ that replaces α by $\alpha\diamond$ and leaves the remaining letters of A unchanged. The symbol expansion of a subshift $\mathcal{X} \subseteq A^\mathbb{Z}$ with respect to α is the subshift whose blocks are factors of words in $E_\alpha(B(\mathcal{X}))$. Flow equivalence is the least equivalence relation between subshifts that contains the conjugacy relation and the symbol expansions. A symbol expansion on α represents a time dilation when reading α in a biinfinite string, thus flow equivalence preserves *shapes* of orbits, but not in a *rigid* way. Finite type subshifts have been completely classified up to flow equivalence [18]. The strictly sofic case remains open. In [10] one finds recent developments.

4 The Karoubi envelope of a subshift

Let L be a language over A. Two words u and v of A^+ are *syntactically equivalent* in L if they share the contexts in which they appear in words of L. Formally, the *syntactic congruence* $\equiv_L$ is defined by $u \equiv_L v$ if and only if the equivalence $xuv \in L \Leftrightarrow xvy \in L$ holds, for all (possibly empty) words x, y over A. The quotient $S(L) = A^+/{\equiv_L}$ is the *syntactic semigroup* of L. The quotient homomorphism $\eta_L : A^+ \to A^+/{\equiv_L}$ is minimal among the onto homomorphisms recognizing L: if the onto homomorphism $\varphi : A^+ \to S$ recognizes L, then there is a unique onto homomorphism $\theta : S \to S(L)$ such that the diagram

$$\begin{tikzcd} A^+ \arrow[r, "\varphi"] \arrow[dr, "\eta_L"'] & S \arrow[d, "\theta"] \\ & S(L) \end{tikzcd}$$

commutes. In particular, L is recognizable if and only if $S(L)$ is finite. More generally, L is recognized by a semigroup of a pseudovariety V if and only if $S(L)$ belongs to V. For example, a language is plus-free if and only if $S(L)$ is an aperiodic semigroup, in view of Schützenberger's characterization of plus-free languages. Since $S(L)$ is computable if L is adequately described (e.g., by an automaton), this gives an algorithm to decide if a rational language is plus-free. This example illustrates why syntactic semigroups and pseudovarieties are important for studying rational languages.

Let S be a semigroup, and denote by $E(S)$ the set of idempotents of S. The *Karoubi envelope* of S is the small category $\mathsf{Kar}(S)$ such that

- the set of objects is $E(S)$;
- an arrow from e to f is a triple (e, s, f) such $s \in S$ and $s = esf$;
- composition of consecutive arrows is given by $(e, s, f)(f, t, g) = (e, st, g)$ (we compose on the opposite direction adopted by category theorists);
- the unit at vertex e is (e, e, e).

This construction found an application in finite semigroup theory in the *Delay Theorem* [32]. Avoiding details, this result concerns a certain correspondence $\mathsf{V} \mapsto \mathsf{V}'$ between semigroup pseudovarieties, with one of the formulations of the Delay Theorem stating that a finite semigroup S belongs to V' if and only if $\mathsf{Kar}(S)$ is the quotient of a category admitting a faithful functor into a monoid in V. Interestingly, the variety of languages corresponding in Eilenberg's sense to V' is, roughly speaking, determined by the inverse images of languages recognized by semigroups of V via block maps of sliding block codes. Hence, it is natural to relate the Karoubi envelope with subshifts. This was done in the paper [15], of which we highlight some results in the next paragraphs.

The *syntactic semigroup $S(\mathcal{X})$ of a subshift $\mathcal{X}$* is the syntactic semigroup of $B(\mathcal{X})$. One finds this object in some papers [20, 21, 8, 9, 12, 13, 11], namely for (strictly) sofic subshifts. Several invariants encoded in $S(\mathcal{X})$ were deduced. The *Karoubi envelope of $\mathcal{X}$*, denoted $\mathsf{Kar}(\mathcal{X})$, is the Karoubi envelope of $S(\mathcal{X})$. Conjugate subshifts do not need to have isomorphic syntactic semigroups, but the Karoubi envelope of a subshift is invariant in the sense of the following result from [15].

Theorem 1.— If $\mathcal{X}$ and $\mathcal{Y}$ are flow equivalent subshifts, then the categories $\mathsf{Kar}(\mathcal{X})$ and $\mathsf{Kar}(\mathcal{Y})$ are equivalent.

For some classes of subshifts, the Karoubi envelope is of no use. For example, all irreducible finite type subshifts have equivalent Karoubi envelopes. But in the strictly sofic case, the Karoubi envelope does bring meaningful information, as testified by several examples given in [15]. We already mentioned the previous existence in the literature of several (flow equivalence) invariants encoded in $S(\mathcal{X})$. It turns out that the Karoubi envelope is the best possible syntactic invariant for flow equivalence of sofic subshifts: indeed, the main result in [15], which we formulate precisely below, states that all syntactic invariants of flow equivalence of sofic subshifts are encoded in the Karoubi envelope. First, it is convenient to formalize what a syntactic flow invariant is. An equivalence relation ϑ on the class of sofic subshifts is: an *invariant of flow equivalence* if $\mathcal{X}\ \vartheta\ \mathcal{Y}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are flow equivalent; a *syntactic invariant* if $\mathcal{X}\ \vartheta\ \mathcal{Y}$ whenever $S(\mathcal{X})$ and $S(\mathcal{Y})$ are isomorphic; a *syntactic invariant of flow equivalence* if it satisfies the two former properties.

Theorem 2.— If ϑ is a syntactic invariant of flow equivalence of sofic subshifts and $\mathcal{X}$ and $\mathcal{Y}$ are sofic shifts such that $\mathsf{Kar}(\mathcal{X})$ is equivalent to $\mathsf{Kar}(\mathcal{Y})$, then $\mathcal{X}\ \vartheta\ \mathcal{Y}$.

Outside the sofic realm, the Karoubi envelope was successfully applied in [15] to what is arguably an almost complete classification of the *Markov-Dyck subshifts*, a class of subshifts introduced by Krieger [23]. Loosely speaking, a Markov-Dyck subshift D_G is formed by biinfinite strings of several types of parentheses, subject to the usual parenthetic rules, and to additional restrictions defined by a graph G. The edges of G are the opening parentheses, and consecutive opening parentheses appearing in an element of D_G correspond to consecutive edges, with a natural symmetric rule for closing parentheses also holding. Flow invariance of $\mathsf{Kar}(D_G)$, together with a characterization of $S(D_G)$, implicit in [19], gives the following result (a different and independent proof appears in [24]).

Theorem 3.— Let G and H be finite graphs. If each vertex of G or of H has out-degree not equal to one and in-degree at least one, then D_G and D_H are flow equivalent if and only if G and H are isomorphic.

5 Free profinite semigroups

We already looked at the importance of (pseudovarieties of) finite semigroups in the study of (varieties of) rational languages. It is well known that free algebras (e.g., free groups, free Abelian groups, free semigroups, etc.) are crucial for the study of varieties of algebras, but for pseudovarieties, a difficulty arises: there is no universal object within the category of *finite* semigroups. To cope with this difficulty, an approach successfully followed by semigroup theorists, since the 1980's, was to enlarge the class of finite semigroups, by considering profinite semigroups. We pause to define the latter, giving [4] as a supporting reference.

A *profinite semigroup* is a compact semigroup (i.e., one with a compact Hausdorff topology for which the semigroup operation is continuous) that is *residually finite*, in the sense that every pair of distinct elements s,t of S admits a continuous homomorphism φ from S onto a finite semigroup F such that $\varphi(s)\neq\varphi(t)$, where finite semigroups get the discrete topology.

Assuming A is finite, consider in A^+ the metric $d(u,v)=2^{-r(u,v)}$ such that $r(u,v)$ is the least possible size of the image of a homomorphism $\psi\colon A^+\to S$ satisfying $\psi(u)\neq\psi(v)$. The completion $\widehat{A^+}$ of A^+ with respect to d is a profinite semigroup. Moreover, each map $\varphi\colon A\to S$ from A into a profinite semigroup S has a unique extension to a continuous homomorphism $\hat{\varphi}\colon\widehat{A^+}\to S$. Hence, $\widehat{A^+}$ is the *free profinite semigroup generated by* A. The next theorem gives a glimpse of why free profinite semigroups matter [1]. This theorem identifies the free profinite semigroup as the Stone dual of the Boolean algebra of recognizable languages.

Theorem 4.— The recognizable languages of A^+ are the traces in A^+ of the clopen subsets of $\widehat{A^+}$: if $L\subseteq A^+$ is recognizable, then $\overline{L}$ is clopen in $\widehat{A^+}$, and, conversely, if K is clopen in $\widehat{A^+}$, then $K\cap A^+$ is recognizable.

The elements of $\widehat{A^+}$ constitute a sort of generalization of the words in A^+, and for that reason they are often named *pseudowords*. The elements in $\widehat{A^+}\setminus A^+$ are the *infinite* pseudowords over A. While the algebraic-topological structure of A^+ is poor, that of $\widehat{A^+}$ is very rich: for example, A^+ has no subgroups, while $\widehat{A^+}$ contains all finitely generated free profinite groups when $|A|\geq 2$, and actually many more groups [30]. The structure of $\widehat{A^+}$ is nowadays less mysterious than it was fifteen years ago, symbolic dynamics having been very useful for achieving that. Our goal until the end of the text is to give examples of such utility.

Most connections between symbolic dynamics and free profinite semigroups developed over Almeida's idea of considering, for each subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, the topological closure $\overline{B(\mathcal{X})}$ of $B(\mathcal{X})$ in $\widehat{A^+}$ [2, 4].

In a semigroup S, the quasi-order $\leq_{\mathcal{J}}$ is defined by $s\leq_{\mathcal{J}}t$ if and only if t is a factor of s. The equivalence relation on S induced by $\leq_{\mathcal{J}}$ is denoted by $\mathcal{J}$. By standard compactness arguments, when $\mathcal{X}$ is an irreducible subshift there is a $\leq_{\mathcal{J}}$-minimum $\mathcal{J}$-class of $\widehat{A^+}$ among the $\mathcal{J}$-classes contained in

$\overline{B(\mathcal{X})}$ (equivalently, intersecting $\overline{B(\mathcal{X})}$), as explained in [14]. This $\mathcal{J}$-class is denoted $J(\mathcal{X})$. The proof of the existence of $J(\mathcal{X})$ also entails that it is a *regular* $\mathcal{J}$-class, that is, one that contains idempotents. One has $J(\mathcal{X}) = J(\mathcal{Y})$ if and only if $\mathcal{X} = \mathcal{Y}$, and so $J(\mathcal{X})$ contains all information about $\mathcal{X}$. Something more holds: one has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $J(\mathcal{Y}) \leq_{\mathcal{J}} J(\mathcal{X})$. For the next statement, have in mind that an infinite pseudoword u of $\widehat{A^+}$ is a *$\leq_{\mathcal{J}}$-maximal infinite pseudoword* if every factor of u either belongs to A^+ or is $\mathcal{J}$-equivalent to u.

THEOREM 5 ([3]).— An element u of $\widehat{A^+}$ is a $\mathcal{J}$-maximal infinite pseudoword if and only if $u \in J(\mathcal{X})$ for some minimal subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$.

The next theorem states that, in a natural sense, $\widehat{A^+}$ is very *large* and very *high* (a weaker version appears in [16], with a harder proof). Its proof is a good example of the potential of symbolic dynamics in the study of free profinite semigroups. A *regular* pseudoword is one that is $\mathcal{J}$-equivalent to an idempotent.

THEOREM 6.— Let A be an alphabet with at least two letters. For the relation $<_{\mathcal{J}}$ in $\widehat{A^+}$, there are both chains and anti-chains with $2^{\aleph_0}$ regular elements.

PROOF.— On the one hand, $A^{\mathbb{Z}}$ contains $2^{\aleph_0}$ minimal subshifts (cf. [27, Chapter 2]), and minimal subshifts clearly form an anti-chain for the inclusion. On the other hand, $A^{\mathbb{Z}}$ contains a chain of $2^{\aleph_0}$ irreducible subshifts [34, Section 7.3]. Hence, the theorem follows immediately from the equivalence $\mathcal{X} \subseteq \mathcal{Y} \Leftrightarrow J(\mathcal{Y}) \leq_{\mathcal{J}} J(\mathcal{X})$ for irreducible subshifts. ∎

Since $J(\mathcal{X})$ is regular, it contains a maximal subgroup, which is a profinite group for the induced topology. Because all maximal subgroups in a regular $\mathcal{J}$-class are isomorphic, we may consider the abstract profinite maximal subgroup $G(\mathcal{X})$ of $J(\mathcal{X})$. The group $G(\mathcal{X})$ was called in [5] the *Schützenberger group of $\mathcal{X}$*. This group is a conjugacy invariant (see [12] for a proof). We collect other facts about $G(\mathcal{X})$.

- In [3] it was shown that $G(\mathcal{X})$ is a free profinite group of rank k if $\mathcal{X}$ is a subshift over a k-letter alphabet that belongs to an extensively studied class of minimal subshifts, called Arnoux-Rauzy subshifts. On the other hand, also in [3], it was shown that the substitution φ defined by $\varphi(a) = ab$ and $\varphi(b) = a^3b$ is such that $G(\mathcal{X}_\varphi)$ is not a free profinite group. This was the first example of a non-free maximal subgroup of a free profinite semigroup. More generally, profinite presentations for $G(\mathcal{X}_\psi)$ were obtained in [5], for all primitive substitutions ψ.
- If $\mathcal{X}$ is a nonperiodic irreducible sofic subshift, then $G(\mathcal{X})$ is a free profinite group of rank $\aleph_0$ [14].
- A sort of geometrical interpretation for $G(\mathcal{X})$ was obtained in [6], when $\mathcal{X}$ is minimal. It was shown that $G(\mathcal{X})$ is an inverse limit of the profinite completions of the fundamental groups of a certain sequence of finite graphs. The n-th graph in this sequence captures information about the blocks of $\mathcal{X}$ with length $2n+1$.

While free profinite semigroups are interesting *per se*, it is worthy mentioning that some of the achievements on the Schützenberger group of a minimal subshift were used in the technical report [25] to obtain results on code theory, whose statement may appear to have nothing to do with profinite semigroups. These results were incorporated and further developed in [7].

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PEDRO NUNES LECTURES

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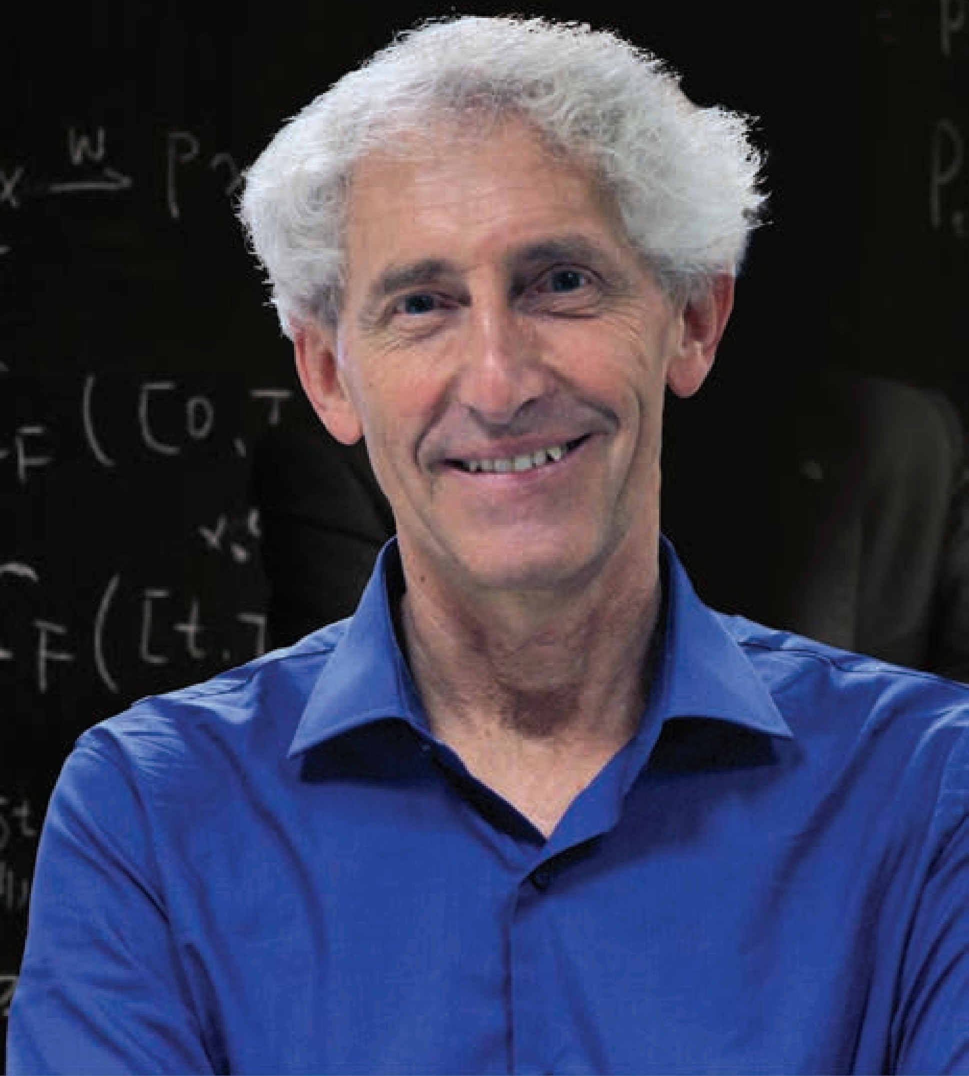

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