BULLETIN

INTERNATIONAL CENTER FOR MATHEMATICS

June 2003 — 14

Contents

Coming Events

Thematic Term on Mathematics and Engineering

Coordinator

Isabel Maria Narra de Figueiredo (Uni. of Coimbra)

Dates

June-September 2003

The **Thematic Term** for 2003 will be dedicated to Mathematics and Engineering. The application of mathematics to engineering is crucial to knowledge and the development of science. The main objective of the thematic term for 2003 is to improve and emphasize the interdependence between the most recent and important research fields in mathematics and the most important fields of contemporary engineering: informatics engineering, chemical engineering, mechanical engineering, civil engineering and electronics engineering.

The thematic term 2003 consists of four events. The first event is devoted to mathematics and informatics engineering and focuses on soft computing and complex systems. The second event deals with modelling and simulation in chemical engineering. The third event is related to modelling and numerical simulation in continuum mechanics. The fourth event is concerned with mathematics and telecommunications.

Each one of these events is an Advanced School and Workshop, where short courses, lectures and invited talks will be given by well-known invited scientists. So it is expected that the thematic term 2003 will attract a large number of postgraduate students, mathematicians and engineers, interested in contributing to the development of mathematics and its applications to engineering.

The programme of events is the following:

23-27 June: Workshop on Soft Computing and Complex Systems

Organizers

António Dourado Correia (Univ. Coimbra), Ernesto Jorge Costa (Univ. Coimbra), José Félix Costa (I. Superior Técnico - Lisbon), Pedro Quaresma (Univ. Coimbra).

Aims

The main scientific goal of the workshop is to introduce recent developments in mathematical techniques applied to complex engineering problems. In particular, the workshop will focus on different aspects of the area called soft computing, including fuzzy and conexionist systems, evolutionary computation, artificial life and complex systems.

Harnessing complexity is an important aspect of today problem solving. Complexity may be due to the presence of uncertain information or because the regularities of a system, we are trying to understand, cannot be briefly described. We will discuss recent developments in dealing with complexity, by means of introducing the methods and their sound mathematical foundations, as well as through the work of some difficult problems.

The workshop will be held at the Mathematics Department - University of Coimbra.

Lectures

Multi-criteria Genetic Optimisation

Carlos Fonseca, University of Algarve, Portugal

Neural Computation and Applications in Time Series and Signal Processing

Georg Dorffner, Department of Medical Cybernetics and Artificial Intelligence, University of Vienna, Austria

Analog Computation

José Félix Costa, Department of Mathematics, Technical University of Lisbon, Portugal

Universal Learning Algorithms

Juergen Schmidhuber, IDSIA- Instituto Dalle Molle di Studi sull'Intelligenza Artificiale, Switzerland

Neuro-Fuzzy Modelling

Intelligent Control

Robert Babuska, Delft University of Technology, Holland.

For more information on this event, please visit the site

http://hilbert.mat.uc.pt/∼softcomplex/

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30 June - 4 July: Workshop on Modelling and Simulation in Chemical Engineering

Organizers

Alírio Egídio Rodrigues (Univ. Porto), Paula Oliveira (Univ. Coimbra), José Almiro Meneses e Castro† (Univ. Coimbra), José Augusto Mendes Ferreira (Univ. Coimbra), Maria do Carmo Coimbra (Univ. Porto).

Aims

The main objective is to bring together mathematicians and chemical engineers to improve the understanding of the problems encountered in process engineering and tools available to solve them. To reach that objective the Workhop is designed:

- To provide the basis for mathematical modelling of chemical engineering systems
- To present some numerical methods to solve model equations in particular in cases of steep moving fronts
- To stress the use of dynamic simulators
- To introduce optimization techniques

The workshop will be held at the CIM headquarters: Complexo do Observatório Astronómico - Universidade de Coimbra.

Short Courses

Modelling in Chemical Engineering

S. Sotirchos and A. Rodrigues, University Rochester, USA and LSRE-FEUP, University of Porto, Portugal

Numerical Simulations with Advection-Diffusion-Reaction Systems

W. Hundsdorfer, Center for Mathematics and Computer Science, The Netherlands

Optimization and Control of Chemical Processes

N. Oliveira, University of Coimbra, Portugal

Invited talks

Adaptive finite element solutions of dependent partial differential equations using moving grid algorithms

J. M. Baines, Department of Mathematics, University of Reading, United Kingdom

Numerical analysis of the motion of glass under external pressure

R.Mattheij, Department of Mathematics and Computer Science, Tech. University of Eindhoven, The Netherlands

Adaptive numerical methods for sensitivity analysis of differential-algebraic equations and partial differential equations

Linda Petzold, UC Santa Barbara, USA

Splitting Methods for Advection-Diffusion-Reactions Problems

J. G. Verwer, Center for Mathematics and Computer Science, CWI, Amsterdam, The Neterlands

Numerical and Computational Challenges in Environmental Modelling

Z. Zlatev, National Environmental Research Institute, Denmark

For more information on this event, please visit the site

http://www.fe.up.pt/lsre/cim2msce/workshop.html

14-18 July: Advanced School and Workshop on Modelling and Numerical Simulation in Continuum Mechanics

Organizers

Luís Filipe Menezes (Univ. Coimbra), Isabel Maria Narra de Figueiredo (Univ. Coimbra), Juha Videman (I. Superior Técnico - Lisbon).

Aims

The scientific goals of this event are the following:

- to present some of the most important recent fields of research in mathematics and its applications to civil and mechanical engineering
- to promote the interdisciplinary aspects of the field by establishing contacts between mathematicians and engineers
- to provide an opportunity for Portuguese scientists to present and discuss their research work.

This event will take place at the Department of Mechanical Engineering - University of Coimbra.

Short Courses

Numerical analysis of discrete schemes approximating grade-two fluid models. Recent results and open problems

Vivette Girault (Université Pierre et Marie Curie, France)

Shape optimization

Patrick Le Tallec (École Polytechnique, France)

Advances in the finite point method for meshless analysis of problems in solid and fluid mechanics

Eugenio Oñate (CIMNE, Universitat Politècnica de Catalunya, Spain)

Mathematics and numerics of shell problems

Juhani Pitkäranta (Helsinki University of Technology, Finland)

Computational mechanics of solid materials at large strains

Cristian Teodosiu (Université de Paris Nord, France)

Invited Plenary Lectures

Finite element simulation of sheet metal forming

Kjell Mattiasson (Volvo Car Corporation, Göteborg, Sweden)

A thermodynamic framework for dissipative processes

K.R. Rajagopal (Texas A&M University, USA)

Virtual metal forming

Karl Roll (Daimler Chrysler AG, Germany)

Analysis and simulation of non-newtonian models for blood flow microvessels

Adélia Sequeira (Instituto Superior Técnico, Lisboa, Portugal)

Numerical analysis and simulation of some contact problems in visco-elasto-plasticity

Juan Viaño (Universidade de Santiago de Compostela, Spain)

For more information on this event, please visit the site

http://www.math.ist.utl.pt/wmnscm/

8-12 September: Mathematical Techniques and Problems in Telecommunications

Organizers

Carlos Salema (I. Superior Técnico - Lisbon), Joaquim Júdice (Univ. Coimbra), Carlos Fernandes (I. Superior Técnico - Lisbon), Mário Figueiredo (I. Superior Técnico - Lisbon), Luís Merca Fernandes (I. P. Tomar).

Aims

The goals are three fold. Firstly we will try to identify and possibly provide solutions for a number of mathematical problems in the field of Telecommunications. Secondly we intend to disseminate among telecommunications engineers some mathematical techniques which are not widely known in this community even if they are being applied in modern communication techniques. Finally we would like to improve mutual understanding and recognition between mathematicians and telecommunication engineers, one of the heaviest users of mathematical techniques in the field of engineering.

This event comes in the follow-up of rather successful, even if less ambitious event, "Matemática em Telecomunicações: Que Problemas?" with similar objectives organized by IT in 1997.

This event will take place at the Instituto Politécnico de Tomar.

Invited Lectures

Combinatorial Optimization in Telecommunications

Mauricio Resende, ATT, USA

Transforms, Algorithms and Applications

Joana Soares, U. Minho, Portugal

Controllability of PDE's and its Discrete Approximations

Enrique Zuazua, U. A. Madrid, Spain

Evolutionary Computing

Eckart Zitzler, SFIT, Switzerland

Stochastic Processes in Telecommunications Traffic

Ivette Gomes, CEAUL, Portugal

For more information on this event, please visit the site

http://www.lx.it.pt/mtpt/

Third Debate on Mathematical Research in Portugal

Porto, 25 October 2003

Organizers: José Ferreira Alves (Univ. Porto), José Miguel Urbano (Univ. Coimbra).

This debate is a continuation of the two previous ones, held on December 1997 and April 2000, whose proceedings have been published by CIM.

Due to limitations of space, those interested in participating should register, by sending an e-mail to jfalves@fc.up.pt.

Themes

- **The Challenge of Excellence**
  Jacob Palis (IMPA)
  José Francisco Rodrigues (UL)
  Rui Loja Fernandes (IST)
- **Evaluation**
  Irene Fonseca (CMU)
  José Basto Gonçalves (UP)
- **Mathematical Research in Industry**
  Charles Tresser (IBM)
  Pedro Lago (UP)

This event will take place at the Pure Mathematics Department, University of Porto.

For more information on this event, please visit the site

http://www.fc.up.pt/cmup/jfalves/debate/

CIM News

CIM Events for 2004

The CIM Scientific Committee, in a meeting held in Coimbra on February 8, approved the CIM scientific program for 2004.

The **Thematic Term** for 2004 will be dedicated to Mathematics and the Environment. The Organizers-Coordinators are Juha Videman (IST, Lisbon, Portugal) and José Miguel Urbano (University of Coimbra, Portugal).

The list of events is the following:

School and Workshop on Dynamical Systems and Applications

3-8 May 2004

Organizers:

José F. Alves, Univ. Porto, Portugal

Marcelo Viana, IMPA, Rio de Janeiro, Brasil

Workshop on Forest Fires

3-5 June 2004

Organizers:

Jorge C. S. André, University of Coimbra, Portugal

José Miguel Urbano, University of Coimbra, Portugal

School on Atmospheric Sciences and Climate Dynamics

12-16 July 2004

Organizers:

Didier Bresch, CNRS/Université Blaise-Pascal, France

José Miguel Urbano, University of Coimbra, Portugal

Juha Videman, IST, Lisbon, Portugal

School and Workshop on Oceanography, Lakes and Rivers

19-25 July 2004

Organizers:

Didier Bresch, CNRS/Université Blaise-Pascal, France

José Miguel Urbano, University of Coimbra, Portugal

Juha Videman, IST, Lisbon, Portugal

Furthermore, the 2004 program will contain the following events:

Workshop on Nonstandard Mathematics

5-11 July, 2004

Organizers:

Imme van den Berg, University of Évora, Portugal

Francine Diener, Université de Nice, France

A. J. Franco de Oliveira, University of Évora, Portugal

Vítor Neves, University of Aveiro, Portugal

Keith D. Stroyan, University of Iowa, USA

João Paulo Teixeira, IST, Lisbon, Portugal

Summer School on Mathematics in Biology and Medicine

20-24 September, 2004

Organizers:

Jorge Careneiro, IGC, Oeiras, Portugal

Francisco Dionísio, IGC, Oeiras, Portugal

José Faro, IGC, Oeiras, Portugal

Gabriela Gomes, IGC, Oeiras, Portugal

Isabel Gordo, IGC, Oeiras, Portugal

Autumn School & International Conference on Stochastic Finance

20-30 September, 2004

Organizers:

Paulo Brito, ISEG, Lisbon, Portugal

Manuel L. Esquível, New University of Lisbon, Portugal

Maria do Rosário Grossinho, ISEG, Lisbon, Portugal

João Nicolau, ISEG, Lisbon, Portugal

Paulo Eduardo Oliveira, University of Coimbra, Portugal

Research in Pairs at CIM

CIM has facilities for research work in pairs and welcomes applications for their use for limited periods.

These facilities are located at Complexo do Observatório Astronómico in Coimbra and include:

- office space, computing facilities, and some secretarial support;
- access to the library of the Department of Mathematics of the University of Coimbra (30 minutes away by bus);
- lodging: a two room flat.

At least one of the researchers should be affiliated with an associate of CIM, or a participant in a CIM event.

Applicants should fill in the electronic application form

http://www.cim.pt/cim.www/cim_app/application.htm

CIM on the WWW

Complete information about CIM and its activities can be found at the site

http://www.cim.pt

This is mirrored at

http://at.yorku.ca/cim.www/

Feature Article

Profinite structures and dynamics

Jorge Almeida

Departamento de Matemática
Universidade do Porto

Surprising as it may be at first sight, there are a number of connections between the theories of finite semigroups and dynamical systems, both viewed in a broad sense. For instance in symbolic dynamics, ideas or analogies from the theory of finite automata find a natural setting for application in sofic systems [10, 24] and, even though not usually formulated in dynamical terms, the dynamical behavior of various operators on finite groups has been extensively studied. The purpose of this note is to review some further connections that have emerged recently driven mainly by work on finite semigroups and thus perhaps open the path to new investigations in this area.

The main tool underlying our approach is found in profinite constructions, be it semigroups, groups, graphs or categories. Generally speaking, profinite structures are a way of encoding, with the help of an additional topological structure, common properties of a class of finite structures of the same type. This idea can be found in various areas, from Galois theory [17] to finite semigroup theory [6, 35, 4].

Results which are given without reference are announced here for the first time and will be proved elsewhere.

A general framework for dynamics in profinite structures

We start by quickly recalling some terminology from model theory. See [27] for details.

Let $\mathcal{L}$ be a *first-order language* given by a finite set $\mathcal{F}$ of *operation symbols* and a finite set $\mathcal{R}$ of *relation symbols* together with a function α with nonnegative integer values describing the *arity* of each symbol. Let $\mathfrak{A}$ be an *$\mathcal{L}$-structure*, which is determined by a choice of a nonempty set A (the *universe*), for each operation symbol $f \in \mathcal{F}$ an operation $f^{\mathfrak{A}} : A^{\alpha(f)} \to A$, and for each relation symbol $R \in \mathcal{R}$ a relation $R^{\mathfrak{A}} \subseteq A^{\alpha(R)}$. For example, semigroups are structures in the language with one binary operation symbol and ordered semigroups are structures in the language that has an additional binary relation symbol, in both cases with the usual properties bing assumed.

A *homomorphism* of $\mathcal{L}$-structures $\mathfrak{A} \to \mathfrak{B}$ is a function $\gamma : A \to B$ between the corresponding universes such that, for every operation symbol $f \in \mathcal{F}$ with arity m, and all $a_1, \ldots, a_m \in A$,

$$\gamma\big(f^{\mathfrak{A}}(a_1, \ldots, a_m)\big) = f^{\mathfrak{B}}\big(\gamma(a_1), \ldots, \gamma(a_m)\big) \quad (1)$$

and for every relation symbol $R \in \mathcal{R}$ with arity n and all $a_1, \ldots, a_n$,

$$(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Rightarrow \big(\gamma(a_1), \ldots, \gamma(a_n)\big) \in R^{\mathfrak{B}}. \quad (2)$$

Note that the reverse implication of (2) is not assumed in our definition of homomorphism. So, for the definition of *isomorphism* we take a bijective homomorphism whose inverse is also a homomorphism.

A *substructure* of a structure $\mathfrak{A}$ is a structure $\mathfrak{B}$ such that the corresponding universes satisfy the inclusion $B \subseteq A$, and each operation $f^{\mathfrak{B}}$ and each relation $R^{\mathfrak{B}}$ is the restriction to the set B of the corresponding operation $f^{\mathfrak{A}}$ and relation $R^{\mathfrak{A}}$ on A. Given a subset X of the universe A of a structure $\mathfrak{A}$, the *substructure generated by X* is the structure $\mathfrak{B}$ with universe B the smallest subset of A that contains X and that is closed under every operation $f^{\mathfrak{A}}$ with $f \in \mathcal{F}$. Direct products of structures are defined by taking the Cartesian product of their universes and interpreting operation and relation symbols component-wise.

From this point on we will abuse notation and talk about structures rather than $\mathcal{L}$-structures and a structure $\mathfrak{A}$ with universe A will be referred simply as 'the structure A' and we will talk of an operation f and a relation R instead of $f^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$, respectively.

We say that a structure A is *finite* if the set A is finite. If the set A is endowed with a topology such that each operation f is continuous and each relation R is closed, then we say that A is a *topological structure*.

Finite structures are viewed as topological structures for the discrete topology. For a class $\mathcal{C}$ of topological structures, a topological structure A is said to be *residually in* $\mathcal{C}$ if for any two distinct points $a, b \in A$ there is a continuous homomorphism $\gamma : A \to F$ into some $F \in \mathcal{C}$ such that $\gamma(a) \neq \gamma(b)$. A compact, Hausdorff, residually in $\mathcal{C}$, structure is called a *pro-$\mathcal{C}$ structure*. In case $\mathcal{C}$ consists of all finite structures, then we talk respectively of a *residually finite* and a *profinite* structure. Note that a structure is profinite if and only if it embeds as a closed substructure in a product of finite structures.

For instance, profinite groups have been extensively studied in connection with Galois theory, number theory, and model theory [17, 29], and free profinite semigroups play a prominent role in the theory of pseudovarieties of finite semigroups [4, 6, 35], which will be introduced in the next section.

We say that a topological structure A is *finitely generated* if there is a finite subset of A such that the substructure it generates is dense in A.

We denote by $\operatorname{End} A$ the set of continuous endomorphisms of a topological structure A. Note that it is a monoid under the operation of composition. Its group of units is the group $\operatorname{Aut} A$ of continuous automorphisms of A.

For the study of a profinite structure A, it is useful to have at hand a topology on $\operatorname{End} A$ for which $\operatorname{End} A$ is a profinite monoid and the evaluation mapping

$$\begin{array}{rcl} \operatorname{End} A \times A & \to & A \\ (\gamma, a) & \mapsto & \gamma(a) \end{array} \tag{3}$$

is continuous. Two classical candidates are the point-wise convergence topology, that is the induced topology from the product topology in A^A, and the compact-open topology. These topologies do not always satisfy the above requirements but we do have the following result that extends well-known facts in the theory of profinite groups [29].

Theorem 1. *Let A be a finitely generated profinite structure. Then $\operatorname{End} A$ is a profinite monoid and $\operatorname{Aut} A$ is a profinite group under the point-wise convergence topology, which coincides with the compact-open topology, and the evaluation mapping (3) is continuous.*

Dynamics of continuous endomorphisms

A *topological dynamical system* (T, f) is a topological structure T for the language with only one operation symbol f, which is unary, and no relation symbols. Two topological dynamical systems (T, f) and (U, g) are said to be *conjugate* if they are isomorphic as topological structures; an isomorphism $\varphi : T \to U$ between them is usually called a *conjugacy*, since it is a homoeomorphism which satisfies $\varphi \circ f = g \circ \varphi$.

For example, if A is a finitely generated profinite structure then, fixing $\gamma \in \operatorname{End} A$, we have a topological dynamical system (A, γ), which just says that A is a topological space and γ is a continuous transformation of A. For the infinite iteration of γ, we use Theorem 1 to introduce an operation that is well-known in finite semigroup theory.

For an element m of a finite monoid M, the sequence $(m^{n!})_n$ becomes constant for $n \geq |M|$, therefore it converges in M, and moreover this eventual constant value is an idempotent. Since a profinite monoid embeds in the product of its finite continuous homomorphic images, if m is an element of a profinite monoid M, then the sequence $(m^{n!})_n$ also converges in M; its limit is denoted m^ω and by the above it is an idempotent. Similarly, we may define $m^{\omega+k}$ to be the limit of the sequence $(m^{n!+k})_{n \geq |k|}$ for any integer k. Note that, if G is a profinite group, then $g^{\omega+k} = g^k$ for every $g \in G$ and integer k.

Going back to our dynamical system (A, γ), we have a very special infinite iterate γ^ω of γ, which is an idempotent, namely the only idempotent in the (closed) subsemigroup of $\operatorname{End} A$ generated by γ. We proceed to examine how the dynamics of the system is determined by this particular iterate.

Recall that a point x of a topological dynamical system (X, φ) is *periodic* if there exists k such that $\varphi^k(x) = x$; the point x is *recurrent* if, for every neighborhood U of x and every k, there exists $\ell \geq k$ such that $f^\ell(x) \in U$; and x is *uniformly recurrent* if there exists m such that, for every neighborhood U of x and every k, there exists $\ell \in \{k+1, \ldots, k+m\}$ such that $f^\ell(x) \in U$. Note that periodicity implies uniform recurrence which in turn implies recurrence.

Of course, if X is finite then the above three properties are equivalent and the periodic points are the elements of the image of φ^ω (note that φ is an element of the finite monoid of all transformations of X). So in particular, if A is a finite structure and $\gamma \in \operatorname{End} A$, then the three notions are equivalent for points of the dynamical system (A, γ). For general topological dynamical systems, there are well-known examples in which no two of the three notions are equivalent. But, what about dynamical systems of the form (A, γ) with A a finitely generated profinite structure? It is easy to construct examples in which periodicity and uniform recurrence are inequivalent but it turns out that the two forms of recurrence coincide in such systems. The following result improves [2, Proposition 3.1].

Proposition 1. *Let A be a finitely generated profinite structure and let γ be a continuous endomorphism of A. Then every recurrent point of A under the action of γ is uniformly recurrent and the set of all such points is the image of γ^ω.*

Relatively free structures and implicit operations

We extend the notion of generating set X of a structure A by allowing X to be a topological space for which there is a continuous function $X \to A$ (the generating mapping) whose image generates A in the previous sense. In general we will omit reference to the generating mapping although we always consider a specific one when we talk about a generating space. Note that a generating mapping may not be injective. To avoid degenerate cases, from hereon we will consider only nonempty generating spaces.

We say that a structure A is *weakly free* with respect to a generating mapping $\iota : X \to A$ if every continuous mapping $\varphi : X \to A$ *extends* (uniquely) to a continuous endomorphism $\hat{\varphi}$ of A in the sense that $\hat{\varphi} \circ \iota = \varphi$. There is a related notion of relatively free structure that we proceed to introduce.

By a *pseudovariety* of finite structures (always of a fixed first-order language) we mean a class of such structures that is closed under taking homomorphic images, substructures and finite direct products. Note that, if $\varphi : A \to B$ is an onto homomorphism, then we call B a homomorphic image of A even though relation symbols may be interpreted in B as larger sets than the images of their interpretations in A. Pseudovarieties of finite semigroups and monoids have been extensively studied in connection with applications to automata, formal languages, circuit complexity, and temporal logic [15, 1, 30] and embody at present the most developed part of finite semigroup theory.

Let V be a pseudovariety of finite structures. We say that a pro-V structure A is V*-free* with respect to a generating mapping $\iota : X \to A$ if every continuous mapping $\varphi : X \to B$ into another pro-V structure *extends* (uniquely) to a continuous homomorphism $\hat{\varphi} : A \to B$ in the sense that $\hat{\varphi} \circ \iota = \varphi$. A profinite structure is *relatively free* with respect to a generating mapping ι if it is V-free with respect to ι for some pseudovariety V. Elements of a generating set for a relatively free structure are often called *letters*. In case $|X| = n$, we will usually presume an ordering $x_1, \ldots, x_n$ of the letters.

Proposition 2. *A profinite structure A is relatively free with respect to a generating mapping ι if and only if A is weakly free with respect to ι.*

From the definition of V-free structure A with respect to a generating mapping $\iota : X \to A$ it follows that, for a fixed space X, it is unique up to isomorphism. The existence of such a structure is established by observing that it may be constructed as the projective limit of all X-generated members of V. In general the generating mapping is understood and we talk simply about the relatively V-free structure on the space X. It will be denoted $\overline{\Omega}_X\mathsf{V}$. In case X is a (nonempty) finite set, we sketch an alternative construction of $\overline{\Omega}_X\mathsf{V}$. See [4] for details.

Let $F(X)$ denote the absolutely free structure on the set X, whose algebraic structure is that of the algebra of terms in X in the fixed first-order language $\mathcal{L}$, and where all relational symbols are interpreted as the empty set. The intersection of all kernels of homomorphisms into members of V is a congruence θ on $F(X)$. Endow the quotient $\Omega_X\mathsf{V} = F(X)/\theta$ with the structure in which, for an n-ary relational symbol R in $\mathcal{L}$, and $w_1, \ldots, w_n \in F(X)$, we set $(w_1/\theta, \ldots, w_n/\theta) \in R$ in $\Omega_X\mathsf{V}$ if and only if $(\varphi(w_1), \ldots, \varphi(w_n)) \in R$ in B for every $B \in \mathsf{V}$ and every homomorphism $\varphi : F(X) \to B$. Then, by construction, $\Omega_X\mathsf{V}$ is a minimal V-free abstract structure in the sense that, for the natural mapping $\iota : X \to \Omega_X\mathsf{V}$ and any mapping $\varphi : X \to B$ with $B \in \mathsf{V}$, there is a unique homomorphism $\hat{\varphi} : \Omega_X\mathsf{V} \to B$ such that $\hat{\varphi} \circ \iota = \varphi$ and any homomorphism of $\Omega_X\mathsf{V}$ onto a structure with the same property is an isomorphism. It is an easy exercise to show that $\Omega_X\mathsf{V}$ embeds in $\overline{\Omega}_X\mathsf{V}$ as the substructure generated by X and this partly explains the notation since this substructure is dense. The letter Ω is meant to suggest that the elements of $\Omega_X\mathsf{V}$ may be viewed as polynomial operations over V in the set X of variables. We also give below an interpretation of the elements of $\overline{\Omega}_X\mathsf{V}$ as operations.

We may define a metric structure on $\Omega_X\mathsf{V}$ by setting $d(u, v) = 2^{-r(u,v)}$, for distinct $u, v \in \Omega_X\mathsf{V}$, where $r(u, v)$ denotes the minimum cardinality of $B \in \mathsf{V}$ for which there exists a homomorphism $\varphi : \Omega_X\mathsf{V} \to B$ such that $\varphi(u) \neq \varphi(v)$, and taking $d(u, u) = 0$. Instead of proving the triangle inequality, it is more natural to establish the stronger ultra-metric inequality $d(u, w) \leq \max\{d(u, v), d(v, w)\}$. A sequence in $\Omega_X\mathsf{V}$ is a Cauchy sequence if and only if its image under any homomorphism into a member of V converges. This implies that, in $\Omega_X\mathsf{V}$, $\mathcal{L}$-operations are uniformly continuous with respect to d and that $\mathcal{L}$-relations are closed sets. Hence the completion of $\Omega_X\mathsf{V}$ with respect to the metric d is a topological structure and one can show that it is isomorphic with $\overline{\Omega}_X\mathsf{V}$.

The elements of $\overline{\Omega}_X\mathsf{V}$ may also be viewed as operations as follows. Let A be a pro-V structure. For $w \in \overline{\Omega}_X\mathsf{V}$, we define an operation $w_A : A^X \to A$ by letting, for a function $\varphi : X \to A$, $w_A(\varphi) = \hat{\varphi}(w)$ where $\hat{\varphi} : \overline{\Omega}_X\mathsf{V} \to A$ is the unique continuous homomorphism such that $\hat{\varphi} \circ \iota = \varphi$. Thus w becomes an $|X|$-ary operation with a 'natural' interpretation on every pro-V structure, and it is an easy exercise to show that this interpretation commutes with continuous homomorphisms between pro-V structures; such an operation is said to be an *implicit operation* (on the class of pro-V structures). We say w 'becomes' an operation since the fact that $\overline{\Omega}_X\mathsf{V}$ is residually in V implies that already the natural interpretations of w as an operation in the members of V completely determine w. Moreover, one can show that every implicit operation on V arises in this way. In other words, the natural interpretation determines a bijection between $\overline{\Omega}_X\mathsf{V}$ and the set of $|X|$-ary implicit operations on V and therefore we may think of the elements of $\overline{\Omega}_X\mathsf{V}$ themselves as implicit operations.

Since, up to isomorphism, $\overline{\Omega}_X\mathsf{V}$ depends only on $n = |X|$ and V, we may write $\overline{\Omega}_n\mathsf{V}$ instead of $\overline{\Omega}_X\mathsf{V}$.

Sometimes it is also useful to consider some structure on the generating set X. Usually this is done by reducing the first-order language by dropping some operation or relation symbols. The case described above in some detail corresponds to dropping all such symbols, so that structures are plain sets. Of course then, rather than considering functions from X into structures of the given language, one takes homomorphisms in the reduced language. The above may be carried out in this context, *mutatis mutandis*. A further restriction which is sometimes useful is to assume that X is a topological structure of the reduced language, in which case homomorphisms from X are also assumed to be continuous, as we already did in the definition of relatively free profinite structure.

Dynamics of implicit operators

The implicit operation point of view is particularly suited for iteration, and thus for a dynamical study. It was basically as a result of this observation that the author started getting involved with dynamical systems [3].

Let us concentrate on the case of a finite generating set $X = \{x_1, \ldots, x_n\}$. Since $\overline{\Omega}_X\mathsf{V}$ is weakly free, a continuous endomorphism γ of $\overline{\Omega}_X\mathsf{V}$ is completely determined by the n-tuple $(\gamma(x_1), \ldots, \gamma(x_n))$. Thus, giving an element of $\gamma \in \operatorname{End}\overline{\Omega}_X\mathsf{V}$ is equivalent to choosing an n-tuple $(w_1, \ldots, w_n)$ of n-ary implicit operations on V. We will abuse notation and write $\gamma = (w_1, \ldots, w_n)$. Moreover, for any pro-$\mathsf{V}$ structure A, we have an associated transformation $\gamma_A : A^n \to A^n$ defined by the natural interpretations of the w_i as follows:

$$v \in A^n \mapsto ((w_1)_A(v), \ldots, (w_n)_A(v)).$$

Such a transformation of A^n is called an *n-ary implicit operator on A*, as in [3] from where the following result can be derived.

Proposition 3. *The set of n-ary implicit operators on a profinite structure A is a profinite monoid with respect to the component-wise point-wise convergence topology and the evaluation mapping is continuous. Moreover, in case A is weakly free on n generators, this profinite monoid is isomorphic with* $\operatorname{End} A$ *via the correspondence described above.*

One may thus consider an arbitrary pro-V structure A and implicit operations $w_1, \ldots, w_n \in \overline{\Omega}_n\mathsf{V}$ and the idempotent infinite iterate $(w_1, \ldots, w_n)^\omega$ on A^n. The behavior of this operator may be closely linked with structural properties of A. Examples of this situation are explored in [2] for pseudovarieties of finite groups. We present next a few examples of this phenomenon.

Denote by S the pseudovariety of all finite semigroups. Note that the subclass G consisting of all finite groups is also a pseudovariety. Define the *commutator* of x and y to be $[x, y] = x^{\omega-1}y^{\omega-1}xy$, which determines a binary implicit operation on finite semigroups that coincides with the usual commutator on finite groups.

Example 2. Note that, on finite groups, the first component of $([x, y], y)^n$ is the usual iterated commutator $[x, {}_n y]$. Similarly, for an integer k, denote by $[x, {}_{\omega+k} y]$ the binary implicit operation defined by taking the first component of $([x, y], y)^{\omega+k}$. Then, by a theorem of Zorn [37], a finite group G is nilpotent if and only if G satisfies the operation equation $[x, {}_\omega y] = 1$.

In the preceding example, strictly speaking 1 is not an operation in our chosen language but we could take any idempotent like x^ω in its place. Or we could take, for an implicit operation w, $w = 1$ to be an abbreviation of the equations $wy = yw = w$ where y is a new variable. In general, an equation whose sides are implicit operations on V (which can always be viewed as being of the same arity) is called a *pseudoidentity*. It is said to be *valid* in a pro-V structure A if the natural interpretations in A of both sides coincide. For a set Σ of pseudoidentities, the class of all structures from V that satisfy all pseudoidentities from Σ is denoted $[\![\Sigma]\!]$. It is a pseudovariety and every pseudovariety W contained in V is of the form $\mathsf{W} = [\![\Sigma]\!]$ for some set Σ of pseudoidentities, in which case we also say that Σ is a *basis of pseudoidentities* of W or that W *is defined by* Σ. This is an extension of Reiterman's Theorem [28] that has been independently established in [26, 27].

Example 3. Let w denote the ternary operation defined by $(w, y, z) = ([[x, y], [x, z]], y, z)^\omega$. B. Plotkin has proposed a conjecture that translates into saying that the pseudovariety of all finite solvable groups is defined by the pseudoidentity $w([x, y], x, y) = 1$ [18]. In the same vein the author [2] has proposed the following alternative pseudoidentity: $u = v$ where $(u, v) = ([x, y], [x^{\omega-1}, y^{\omega-1}])^\omega$. The proof that such characterizations of solvability for finite groups hold is not likely to be very simple since one consequence of them is that a finite group is solvable if and only if all its 2-generated subgroups are solvable. This property was first established by Thompson [32] as a consequence of his monumental classification of finite simple groups whose proper subgroups are solvable, whose proof extends over 400 printed pages and which earned J. G. Thompson the Fields Medal in 1970. A much shorter yet rather involved proof of the 2-generator characterization of finite solvable groups has been given by Flavell [16].

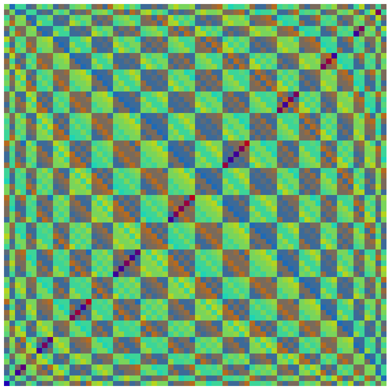

Figure 1: The Thuë-Morse operator on $\mathbb{Z}/70\mathbb{Z}$

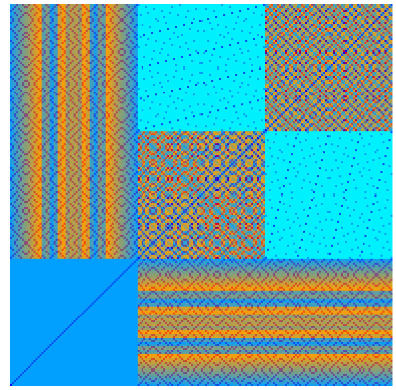

Figure 2: The Thuë-Morse operator on $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}/3\mathbb{Z}$

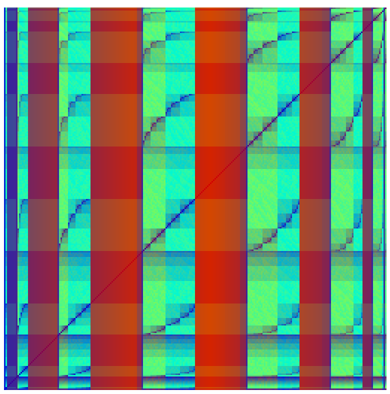

Figure 3: Action of the operator $(y^{\omega-1}xy, x)$ on D_{256}

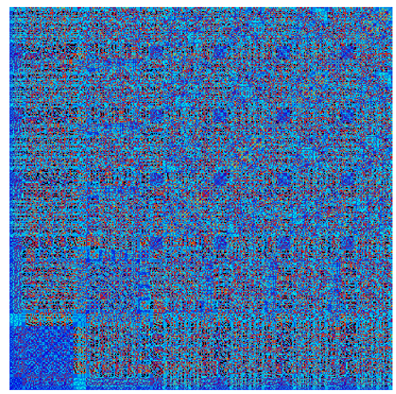

Figure 4: Action of the Thuë-Morse operator on A_6

One may try to visualize the dynamical behavior of an implicit operator on a finite structure. The examples in Figures 1 through 4 were calculated using GAP [31] for the group calculations and Mathematica [36] for converting them into a picture of the action of a binary implicit operator on a finite group G. The method used was to draw a square grid of pixels, each pixel representing a point in $G \times G$. Each pixel is colored with the three basic colors green, red and blue. The intensity of green represents the distance of the point to the cycle in its orbit so that, in particular, pixels corresponding to periodic points get no green color component. By taking a total ordering of the cycles and associating to each cycle an increasing intensity of blue and a decreasing intensity of red, according to its position in the ordering, each pixel gets the blue and red tonality determined by the cycle in its orbit. Of course, the final picture will depend on the ordering of the elements of the group G and the ordering of the cycles. We just took the ordering of the groups given by GAP and the ordering of cycles is by first appearance as the cycle in the orbit of the successive elements of G.

The picture for the *Thuë-Morse* implicit operator $(x, y) \mapsto (xy, yx)$ acting on the cyclic group $\mathbb{Z}/70\mathbb{Z}$ is shown on Figure 1, where the intensities of the basic colors have been weighted to increase the spatial visual effect. Figure 2 represents the action of the same operator on the wreath product of the Klein 4-group by the group of order 3. The fractal-like Figure 3 portrays the action of the iterated conjugation operator $(x, y) \mapsto (y^{\omega-1}xy, x)$ on the dihedral group D_{256} of order 256. Finally, in contrast, with the above examples, where one immediately recognizes patterns, the much more "chaotic" Figure 4 represents the action of the Thuë-Morse operator on the alternating group A_6.

So far these examples have only been used to experimentally explore the behavior of operators or simply for their aesthetic appeal. They may be viewed as approximations or as representing a small portion of the pictures of the action of the same operators on profinite groups, which seems to explain their fractal-like appearance.

Here is a small sample of other results concerning the action of implicit operators on finite groups.

Theorem 4 (Širšov [34]). *Consider the binary implicit operations u and v defined by $(u, v) = (xy, yx)^\omega$, the idempotent iterate of the Thuë-Morse operator. Then a finite group satisfies the pseudoidentity $u = v$ if and only if it is an extension of a nilpotent group by a 2-group.*

Theorem 5. *Let $(u,v) = (y^{\omega-1}xy, x)^{\omega}$. Then the pseudoidentity $u = 1$ defines the pseudovariety of all finite nilpotent groups.*

The following result provides partial information on finite groups for which the iterated commutator has period 1. A structure A is said to *divide* a structure B if A is a homomorphic image of a substructure of B.

Theorem 6. *Let G be a finite group satisfying the pseudoidentity $[x, {}_{\omega+1}y] = [x, {}_{\omega}y]$. Then*

a. G is supersolvable and G is a direct product of a group of order relatively prime to 6 with a group of order $2^m 3^n$ which has a normal Sylow 3-subgroup (Brandl [11]) ;

b. G is either nilpotent or divisible by the symmetric group S_3 (A. Costa [13]).

Dynamics of implicit operators on free profinite semigroups

We first introduce briefly the most basic tools in semigroup theory. Readers interested in more details might wish to consult a book in the area such as [23].

In a semigroup S, say that an element s is a *factor* of (or *lies $\mathcal{J}$-below*) another element t if t can be written as a product $t = t_1 \cdots t_r$ with $r \geq 1$ and some $t_i = s$. Two elements are *associates* if they are factors of each other. This defines an equivalence relation on S which is one of *Green's relations*, denoted $\mathcal{J}$. Similarly, one may consider left factors or *prefixes*, with corresponding equivalence relation $\mathcal{R}$, and right factors or *suffixes*, with corresponding equivalence relation $\mathcal{L}$. For a compact semigroup, the smallest equivalence relation, denoted $\mathcal{D}$, containing both $\mathcal{R}$ and $\mathcal{L}$ is precisely $\mathcal{J}$. The intersection $\mathcal{R} \cap \mathcal{L}$ provides the last of Green's relations, denoted $\mathcal{H}$. The maximal subgroups of S are precisely the $\mathcal{H}$-classes that contain idempotents and any two of them contained in the same $\mathcal{D}$-class are isomorphic.

An element s of a semigroup S is *regular* if there exists $t \in S$ such that $sts = s$. All or none of the elements in a $\mathcal{D}$-class are regular, and the former condition holds if and only if the $\mathcal{D}$-class contains an idempotent.

Free pro-V semigroups have been computed for some very special examples of pseudovarieties of semigroups, often with numerous applications as in the case of the pseudovariety

$$\mathsf{J} = [\![(xy)^{\omega} = (yx)^{\omega}, x^{\omega+1} = x^{\omega}]\!]$$

which consists of all finite semigroups in which the $\mathcal{J}$-classes are singletons. It turns out that $\overline{\Omega}_n\mathsf{J}$ is a relatively free structure in the language with a symbol added for the ω-power operation, and a finite basis of equations (that is, a finite presentation consisting of universal relations) has been given and the word problem has been solved for this structure [1].

But for instance very little is known about the free profinite semigroups $\overline{\Omega}_n\mathsf{S}$. As in the previous section, we may use infinite iteration of implicit operators to define complex implicit operations from simple ones. This has been recently used as a tool to study the semigroups $\overline{\Omega}_n\mathsf{S}$ in [8]. We proceed to review a sample of results from that paper.

The semigroup $\Omega_n\mathsf{S}$ is the free semigroup on n letters and so its elements may be viewed as words on the letters, for which an appropriate model is the sequence of letters in the unique factorization into letters. Since implicit operations are limits of sequences of finite words, we may also call them *profinite words*. So, of course, a profinite word $w \in \overline{\Omega}_n\mathsf{S}$ is said to be *finite* if it belongs to $\Omega_n\mathsf{S}$ and we will say it is *infinite* otherwise. The *length* of a finite word is the length of the sequence of letters that compose it.

An infinite profinite word w is said to be *recurrent* if every finite factor of w is also a factor of every infinite factor of w; and we say that w is *uniformly recurrent* if every finite factor of w is also a factor of every sufficiently long finite factor of w. One can easily show that these two notions are equivalent. We prefer to refer to uniformly recurrent profinite words for reasons that will be made clear in the next section.

An implicit operator $\varphi = (w_1, \ldots, w_n)$ ($w_i \in \overline{\Omega}_n\mathsf{S}$) is *finite* if its components are finite words; we say that φ is *primitive* if, for some finite exponent k, all components of φ^k admit all letters as factors; and φ is G-*invertible* if the induced operator on $(\overline{\Omega}_n\mathsf{G})^n$ is invertible. These notions are carried to continuous endomorphisms of $\overline{\Omega}_n\mathsf{S}$ via the isomorphism of Proposition 3. One can easily show that, for a primitive implicit operator $(w_1, \ldots, w_n)$, all components of the idempotent iterate $(v_1, \ldots, v_n) = (w_1, \ldots, w_n)^{\omega}$ are $\mathcal{J}$-equivalent [8]. Moreover, in case the w_i are finite, then the v_i are uniformly recurrent.

Theorem 7. *[8] Let $(w_1, \ldots, w_n)$ be a primitive, G-invertible, implicit operator all of whose components start with the same letter and end with the same letter. Let $(v_1, \ldots, v_n) = (w_1, \ldots, w_n)^{\omega}$. Then $\{v_1, \ldots, v_n\}$ freely generates a profinite subgroup of $\overline{\Omega}_n\mathsf{S}$ which is a retract of $\overline{\Omega}_n\mathsf{S}$. Moreover, every retract subgroup isomorphic with $\overline{\Omega}_n\mathsf{G}$ is obtained in this way.*

For example, the components of $(xyx, x)^{\omega}$ freely generate a profinite subgroup of $\overline{\Omega}_2\mathsf{S}$ but those of the operator $(xy, yx)^{\omega}$ do not even belong to the same $\mathcal{H}$-class although they are $\mathcal{J}$-equivalent.

The interest in finding n-tuples $(v_1, \ldots, v_n)$ of profinite words which freely generate profinite retract subgroups of $\overline{\Omega}_n\mathsf{S}$, which are called *group-generic*, stands from the fact that such n-tuples may be used to construct bases of pseudoidentities for pseudovarieties of semigroups which are derived from pseudovarieties of groups as follows. Let H be a pseudovariety of groups. Then the class $\overline{\mathsf{H}}$ of all finite semigroups whose subgroups belong to H is a pseudovariety. To obtain a basis of pseudoidentities for $\overline{\mathsf{H}}$ from a given basis for H simply transform each pseudoidentity $u(x_1, \ldots, x_n) = w(x_1, \ldots, x_n)$ into $u(v_1, \ldots, v_n) = w(v_1, \ldots, v_n)$ where $(v_1, \ldots, v_n)$ is a group-generic n-ary implicit operator. As an example, if Ab is the pseudovariety of all finite Abelian groups and $(v_1, v_2) = (xyx, x)^\omega$, then $\overline{\mathsf{Ab}} = [\![v_1v_2 = v_2v_1]\!]$. An alternative approach for the construction of group-generic n-tuples of profinite words which involves idempotents from the minimal ideal of $\overline{\Omega}_n\mathsf{S}$ is presented in [7].

The iteration φ^ω of finite implicit operators $\varphi = (w_1, \ldots, w_n)$ is of special interest because the elements of $\overline{\Omega}_n\mathsf{S}$ with which we start are particularly simple and because similar iterations take place in other areas of Mathematics, from symbolic dynamics to the theory of computation. In the case of a finite, primitive, G-invertible, implicit operator, we have the following improvement of Theorem 7.

Theorem 8. *Let φ be a finite, primitive, n-ary, implicit operator and let J be the $\mathcal{J}$-class of $\overline{\Omega}_n\mathsf{S}$ containing the $\varphi^\omega(x_i)$ $(i = 1, \ldots, n)$. If φ is G-invertible then there is at least one maximal subgroup H of $\overline{\Omega}_n\mathsf{S}$ contained in J which satisfies $H = \varphi^\omega(H)$. Moreover, H is a free profinite group on n generators of the form $\varphi^\omega(u)$ with $u \in \Omega_n\mathsf{S}$.*

For example, taking $\varphi = (xy, zx, yzx)$, with a little additional calculation one can show that the profinite words $\varphi^\omega(x), \varphi^{\omega+1}(x), \varphi^{\omega+2}(x)$ freely generate a maximal subgroup of $\overline{\Omega}_3\mathsf{S}$. We do not know if this subgroup is a retract of $\overline{\Omega}_3\mathsf{S}$ although we conjecture it is not.

In general one cannot expect the retract subgroups of $\overline{\Omega}_n\mathsf{S}$ isomorphic with $\overline{\Omega}_n\mathsf{G}$ to be maximal subgroups. Indeed, by [7, 8] one can find such subgroups in the minimal ideal and there one can show that maximal subgroups are not n-generated for $n > 1$.

To show more generally that, for $n > 1$, the minimal ideal of $\overline{\Omega}_n\mathsf{S}$ cannot be reached through iteration of finite n-ary implicit operators, we introduce some numerical parameters. We first consider the *factor complexity* of a profinite word $w \in \overline{\Omega}_n\mathsf{S}$ which is given by a function q_w that associates to a positive integer k the number of factors of w of length k. One can easily show that the limit

$$h(w) = \lim_{k\to\infty} \frac{1}{k} \log_n q_w(k)$$

exists for every infinite $w \in \overline{\Omega}_n\mathsf{S}$ with $n > 1$ and we call it the *entropy* of w. Note that $\mathcal{J}$-equivalent infinite elements of $\overline{\Omega}_n\mathsf{S}$ have the same complexity and entropy.

Theorem 9. *[8] Entropy does not increase by applying an implicit operation nor by iteration. More precisely:*

a. if $u \in \overline{\Omega}_m\mathsf{S}$ and $v_1, \ldots, v_m \in \overline{\Omega}_n\mathsf{S}$, then

$$\begin{aligned} &h(u(v_1, \ldots, v_m)) \\ &\quad \le \max\{h(u)\log_n m, h(v_1), ..., h(v_m)\}; \end{aligned}$$

b. if $w_1, \ldots, w_n \in \overline{\Omega}_n\mathsf{S}$ and $z_1, \ldots, z_n$ are the components of the iterate $(w_1, \ldots, w_n)^\omega$, then

$$\max_{1\le i\le n} h(z_i) \le \max_{1\le i\le n} h(w_i).$$

We say that a subset X of $\overline{\Omega}_n\mathsf{S}$ is *closed under iteration* if, whenever $w_1, \ldots, w_n \in X$, the components of $(w_1, \ldots, w_n)^\omega$ also belong to X.

Consider the minimal ideal I of $\overline{\Omega}_n\mathsf{S}$. It is a $\mathcal{J}$-class and every element of I admits every element of $\overline{\Omega}_n\mathsf{S}$ as a factor. Hence elements of I have entropy 1 and, conversely, one can show that every profinite word of entropy 1 belongs to I. We thus obtain the following corollary of Theorem 9 which in particular states that the minimal ideal is inaccessible by iteration for $n > 1$.

Corollary 1. *[8] For $n > 1$, the complement of the minimal ideal I of $\overline{\Omega}_n\mathsf{S}$ is closed under iteration and under the application of implicit operations $w \in \overline{\Omega}_m\mathsf{S}$ with $h(w) < \frac{1}{\log_n m}$.*

Symbolic dynamics

We proceed to relate more closely free profinite semigroups with symbolic dynamics. Consider the pseudovarieties defined by the following pseudoidentities:

$$\begin{aligned} \mathsf{K} &= [\![x^\omega y = x^\omega]\!] \\ \mathsf{D} &= [\![yx^\omega = x^\omega]\!] \\ \mathsf{LI} &= [\![x^\omega y x^\omega = x^\omega]\!] \end{aligned}$$

In words: K consists of all finite semigroups in which idempotents are left zeros; D is the left-right dual of K; LI consists of all *locally trivial* finite semigroups in which every submonoid is trivial and it is the smallest pseudovariety containing both K and D.

The free pro-K semigroup $\overline{\Omega}_n\mathsf{K}$ on n letters is the completion of $\Omega_n\mathsf{K} = \Omega_n\mathsf{S}$ with respect to the metric d defined by $d(u, v) = 2^{-p(u,v)}$ where $p(u, v)$ is the length of the longest common prefix of u and v. A sequence of words which is not eventually constant is a Cauchy sequence if and only if prefixes of any given length stabilize for sufficiently large indices and the limit is completely determined by these successive prefixes, or in other words it may be identified with a right infinite word $x_{i_0}x_{i_1}\ldots x_{i_r}\ldots$. Such infinite words are one of the objects studied in symbolic dynamics, precisely under the metric resulting from d. Multiplication in $\overline{\Omega}_n\mathsf{K}$

is by concatenation of words except that right infinite words are declared to be left zeros.

Dually, $\overline{\Omega}_n\mathsf{D}$ is a compactification of $\Omega_n\mathsf{D} = \Omega_n\mathsf{S}$ by adding all left infinite words $\ldots x_{i_r} \ldots x_{j_2} x_{j_1}$ and declaring words (finite or infinite) to be close if they have a long common suffix. As for $\overline{\Omega}_n\mathsf{LI}$, it embeds naturally in the product $\overline{\Omega}_n\mathsf{D} \times \overline{\Omega}_n\mathsf{K}$ as follows: add to $\Omega_n\mathsf{LI} = \Omega_n\mathsf{S}$ points at infinity consisting of pairs $(\ldots x_{j_2}x_{j_1}, x_{i_0}x_{i_1}\ldots)$ of a left infinite and a right infinite word, which may also be identified with a doubly infinite word $\ldots x_{j_2}x_{j_1}\underline{x_{i_0}}x_{i_1}\ldots$ with a marked origin, that is a function $w \in X^{\mathbb{Z}}$ defined on the integers with values in $X = \{x_1, \ldots, x_n\}$. Two doubly infinite words with marked origins are close if they coincide in a large factor centered at the origin, which induces the product topology on $X^{\mathbb{Z}}$.

The *shift* transformation sending $w \in X^{\mathbb{Z}}$ to the function $\sigma(w) : n \mapsto w(n+1)$ corresponds to a letter conjugation $v \mapsto a^{-1}va$ in $\overline{\Omega}_n\mathsf{LI}$ where $a = w(0)$. The shift defines the natural action of the cyclic group $\mathbb{Z}$ on $X^{\mathbb{Z}}$. A *symbolic dynamical system* or *subshift*, is a closed subset $\mathcal{S}$ of $X^{\mathbb{Z}}$ which is stable under the group action. It is easy to see that a subshift $\mathcal{S}$ is completely determined by the language $L(\mathcal{S}) \subseteq \Omega_n\mathsf{S}$ of its finite factors and that the languages that arise in this way are precisely the subsets L of $\Omega_n\mathsf{S}$ which are *factorial*, that is they are closed under taking factors, and *extendable*, that is for any $w \in L$ there are $a, b \in X$ such that $aw, wb \in L$. A subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ is viewed as a topological dynamical system $(\mathcal{S}, \sigma|\mathcal{S})$.

A subshift whose factors are the factors of the powers of a finite word is said to be *periodic*. The subshift $\mathcal{S}$ is said to be *sofic* if the language $L = L(\mathcal{S})$ can be recognized by a homomorphism $\varphi : \Omega_n\mathsf{S} \to S$ into a finite semigroup S in the sense that $L = \varphi^{-1}\varphi(L)$. If, moreover, $S \in \mathsf{LI}$ then L is said to be *locally testable* and $\mathcal{S}$ is called a subshift *of finite type*. Equivalently, a subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ is of finite type if and only if there is a finite set W of words such that $L(\mathcal{S})$ consists of the finite words over X which do not admit any word from W as a factor. A subshift $\mathcal{S}$ is *irreducible* if, for all $u, v \in L(\mathcal{S})$, there exists $w \in \Omega_n\mathsf{S}$ such that $uwv \in L(\mathcal{S})$. A *minimal subshift* is a nonempty subshift which does not properly contain any other nonempty subshift. It is well known that a subshift is minimal if and only if its language consists of all finite factors of a uniformly recurrent doubly infinite word.

A major open problem in symbolic dynamics is whether conjugacy is decidable for sofic subshifts, or even just for subshifts of finite type. There is a coarser equivalence relation, the *eventual conjugacy* or *shift-equivalence*, for which complete invariants are given by dimension groups [24]. These are ordered Abelian groups which are effectively computable and so eventual conjugacy is decidable. To define eventual conjugacy, one considers first the power $\mathcal{S}^n$ of a subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ whose alphabet is the set X^n of all length n words over X. Elements of $\mathcal{S}$ are considered as words over X^n by scanning the successive non-overlapping factors of length n that compose them. The so-called *eventual conjugacy* of subshifts $\mathcal{S}$ and $\mathcal{T}$ means that their powers $\mathcal{S}^n$ and $\mathcal{T}^n$ are conjugate for all sufficiently large n. Eventual conjugacy is known to be strictly coarser than conjugacy even for irreducible subshifts of finite type [21, 22].

Given a subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$, we may consider the closure $\overline{L(\mathcal{S})}$ of its language of finite factors in $\overline{\Omega}_n\mathsf{S}$. The set $\overline{L(\mathcal{S})}$ completely determines $\mathcal{S}$ since the language of its finite factors is precisely $L(\mathcal{S})$. This suggests doing symbolic dynamics in $\overline{\Omega}_n\mathsf{S}$, an object that has a much richer structure than $X^{\mathbb{Z}}$. The question that immediately comes to mind is what transformation of $\overline{\Omega}_n\mathsf{S}$ should we consider. The shift transformation corresponds to the conjugation $\chi : w \mapsto a^{-1}wa$, where a is the first letter of w, which means sending $w = av$ to va. However, a finite iterate of this transformation conjugates by a finite factor and coinciding in finite factors corresponds to the completion $\overline{\Omega}_n\mathsf{LI}$ of the free semigroup $\Omega_n\mathsf{S}$ rather than the much richer structure $\overline{\Omega}_n\mathsf{S}$ which really interests us here. We do not know of any single transformation which plays for $\overline{\Omega}_n\mathsf{S}$ the role the shift plays in the case of $\overline{\Omega}_n\mathsf{LI}$. Our connection between $\overline{\Omega}_n\mathsf{S}$ and subshifts proceeds in a different direction.

By Zorn's Lemma and compactness, the closed set $\overline{L(\mathcal{S})}$ must contain elements which are $\mathcal{J}$-equivalent to all other elements of $\overline{L(\mathcal{S})}$ of which they are factors. This suggests studying the $\mathcal{J}$-classes of such elements, which we will call the *minimal $\mathcal{J}$-classes* of $\mathcal{S}$. The following results provide the basis for this study.

Proposition 4. *Let $\mathcal{S} \subseteq X^{\mathbb{Z}}$ be a subshift and let w be a regular element of $\overline{\Omega}_n\mathsf{S}$. Then the following conditions are equivalent:*

a. *$w \in \overline{L(\mathcal{S})}$;*

b. *w is $\mathcal{J}$-equivalent to some element of $\overline{L(\mathcal{S})}$;*

c. *all finite factors of w belong to $L(\mathcal{S})$.*

Theorem 10. *Let $\mathcal{S} \subseteq X^{\mathbb{Z}}$ be a subshift.*

a. *If $\mathcal{S}$ is sofic, then there are only finitely minimal $\mathcal{J}$-classes of $\mathcal{S}$ and $\overline{L(\mathcal{S})}$ is a union of $\mathcal{J}$-classes.*

b. *The subshift $\mathcal{S}$ is irreducible if and only if $\mathcal{S}$ has only one minimal $\mathcal{J}$-class and it is regular. The regular $\mathcal{J}$-classes that appear in this way are those that contain profinite words which are limits of sequences of finite factors.*

c. *The subshift $\mathcal{S}$ is minimal if and only if $\mathcal{S}$ has only one minimal $\mathcal{J}$-class J and J contains all its regular factors. The $\mathcal{J}$-classes that appear in this way are those that contain uniformly recurrent profinite words or, equivalently the $\mathcal{J}$-classes which contain infinite profinite words and all their regular factors.*

In terms of the factor ($\mathcal{J}$-)ordering, minimal subshifts are thus in bijective correspondence with $\mathcal{J}$-maximal regular $\mathcal{J}$-classes. One might expect such $\mathcal{J}$-classes to have low entropy since they are far from the minimal ideal, provided the alphabet has more than one letter. However, it has been recently shown that there are uniformly recurrent doubly infinite words with arbitrarily large entropy $h < 1$ [14].

Corollary 2. *For $n \geq 2$, there are $\mathcal{J}$-maximal regular $\mathcal{J}$-classes in $\overline{\Omega}_n\mathsf{S}$ of arbitrarily large entropy $h < 1$.*

At the other end, we already know that there are $\mathcal{J}$-maximal regular $\mathcal{J}$-classes of $\overline{\Omega}_n\mathsf{S}$ with zero entropy, such as the $\mathcal{J}$-class containing the $\varphi^\omega(x_i)$ for any finite primitive continuous endomorphism φ of $\overline{\Omega}_n\mathsf{S}$.

The study of sofic subshifts and of minimal subshifts correspond to major subareas of symbolic dynamics. In general, the dynamics of a sofic subshift is determined by that of certain irreducible sofic subshifts associated with it. Since minimal subshifts are irreducible (but not sofic, unless they are periodic), irreducibility is usually assumed and it is therefore not a serious restriction, which we will assume from hereon.

In semigroup theory, when a semigroup has a nontrivial minimal ideal, a lot of its structural properties are reflected in the minimal ideal and in the action of the semigroup on this ideal. Although $\overline{L(\mathcal{S})}$ is not in general a subsemigroup of $\overline{\Omega}_n\mathsf{S}$, it does have a minimal $\mathcal{J}$-class J, which is regular, and so one may view it as a partial semigroup, for which J plays the role of the minimal ideal. One way to formalize this idea is to consider a profinite category associated with $\mathcal{S}$ as follows.

By the transition graph $\Gamma(\mathcal{S})$ of a subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ we mean the (directed) graph with vertex set $\mathcal{S}$ and an edge $v \to \sigma(v)$ for each vertex v. As a purely combinatorial graph, this is a rather uninteresting graph in which every vertex has in-degree and out-degree 1 and, for instance, all (nonempty) subshifts without periodic points over finite alphabets have isomorphic graphs. But both the sets of vertices $\mathcal{S}$ and edges $\{(v, \sigma(v)) : v \in \mathcal{S}\} \subseteq \mathcal{S} \times \mathcal{S}$ have a topological structure induced from $X^{\mathbb{Z}}$ and the partial operations of taking the beginning and end vertices of an edge are continuous.

This suggests coming back to the general framework of structures of first-order languages at the beginning of the paper. However, the treatment of partial operations, which has important applications for instance in computer science, is much more delicate and apparently has only be done in special cases in the sense of obtaining Birkhoff/Reiterman-type theorems characterizing certain classes of structures by means of equations [12]. One of the difficulties lies in the definition of a suitable notion of substructure and homomorphic image. For (small) categories, this has been done by Tilson [33] with the profinite approach added in [20, 9].

For our present purposes we do not need Birkhoff/Reiterman-type theorems, but rather just free profinite constructions. This does carry through from the discussion in earlier sections of this paper with a few minor adjustments. For substructures we take subsets such that whenever an operation is defined on elements of the subset then the resulting value is also in the subset. For a homomorphism, whenever an operation is defined on elements of the domain, the corresponding operation should also be defined on their images and the usual relation (1) should hold. We assume further that there are unary relations in the language which are interpreted in structures so as to form partitions of their universes (into *sorts* in the language of computer science) and so that all operations take their arguments in one sort and all their values are also of a single sort. Note that this is a nontrivial restriction. It allows us to define products of structures as subsets of the Cartesian product consisting of elements in which all components have the same sort, and then define operations and relations component-wise. Profinite structures are defined as in the case of fully-defined operations and free profinite structures may be constructed by taking projective limits, which in turn are realized as appropriate substructures of products of finite structures.

In our case, we may view (small) categories as structures of a suitable first-order language, namely the language with unary relation symbols V and E, unary operation symbols α, ω and I, and binary operation symbol π. Their interpretation in a category C is the following: V is the set (sort) of vertices (or objects); E is the set (sort) of edges (or morphisms); α is the partial operation defined on edges where $\alpha(e)$ is the vertex where the edge starts; ω is the partial operation defined on edges where $\omega(e)$ is the vertex where the edge ends; I is the partial operation defined on vertices where $I(v)$ is the identity at v; π is the partial associative operation defined on edges e, f such that $\omega(e) = \alpha(f)$ and the edge $\pi(e, f)$ starts at $\alpha(e)$ and ends at $\omega(f)$.

Graphs may be viewed as structures of the reduced language in which the symbols I and π are dropped. *Semigroupoids* are structures of the language with the symbol I dropped. The general framework gives us the right notions of graph homomorphism, category homomorphism (or functor), topological graph, profinite category, and so on.

Back to subshifts, with the above topology, not only $\Gamma(\mathcal{S})$ is a topological graph but, more precisely, we have the following expected result.

Proposition 5. *The graph $\Gamma(\mathcal{S})$ is profinite.*

Recall that a homomorphism (or functor) $\varphi : C \to D$ between two categories is *faithful* if its restriction to every set of edges of C with fixed beginning and end is injective. We say that a graph is *strongly connected* if, for all vertices v and w, there is an edge $v \to w$. *Groupoids* are strongly connected categories in which all morphisms are isomorphisms.

Note that the class Cat of all finite categories is a pseudovariety. The free structure $\Omega_\Gamma\mathsf{Cat}$ on a graph Γ is then the free category on Γ, whose edges are the finite paths in Γ. In case Γ is a profinite graph, $\overline{\Omega}_\Gamma\mathsf{Cat}$ may be constructed as in an earlier section as the completion of $\Omega_\Gamma\mathsf{Cat}$ with respect to a suitable metric. We call the edges of $\overline{\Omega}_\Gamma\mathsf{Cat}$ *profinite edges* and we say they are *infinite* if they do not lie in $\Omega_\Gamma\mathsf{Cat}$.

Note that from the free profinite category $\overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat}$ one can reconstruct the subshift $\mathcal{S}$ as a topological dynamical system: the space $\mathcal{S}$ is the closed subspace V of vertices and the shift transformation $v \to \sigma(v)$ is characterized by the edges which are not local identities and which cannot be factorized nontrivially. In particular, two subshifts are conjugate if and only if their associated profinite graphs (respectively categories) are isomorphic.

A subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ further determines a labeling of its associated profinite graph $\Gamma(\mathcal{S})$: label the edge $v \to \sigma(v)$ with the letter $v(0)$ across which the shift moves the origin of the doubly infinite word v. This labeling extends uniquely to a continuous homomorphism $\lambda : \overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat} \to \overline{\Omega}_X\mathsf{M}$ to the free profinite monoid on X, which is obtained from $\overline{\Omega}_X\mathsf{S}$ by adding an identity as an isolated point, where monoids are seen as one (virtual) vertex categories.

Proposition 6. *The mapping λ is faithful.*

We thus have another strong, "geometrical", connection between subshifts and free profinite semigroups. The next result summarizes some relationships between the profinite constructions associated with a subshift.

Theorem 11. *Let $\mathcal{S} \subseteq X^{\mathbb{Z}}$ be a subshift.*

a. The subshift $\mathcal{S}$ is irreducible if and only if the category $\overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat}$ is strongly connected. In this case, the labeling λ embeds the minimal ideal of each local monoid of $\overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat}$ in the minimal $\mathcal{J}$-class of $\mathcal{S}$ as a union of maximal subgroups of $\overline{\Omega}_n\mathsf{S}$.

b. The subshift $\mathcal{S}$ is minimal if and only if the category $\overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat}$ is strongly connected and its subsemigroupoid whose edges are the infinite profinite paths of $\Gamma(\mathcal{S})$ is a groupoid.

In particular, for an irreducible subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$, the maximal subgroups of the minimal ideals of local monoids of the profinite category $\overline{\Omega}_{\Gamma(\mathcal{S})}\mathsf{Cat}$ are mutually isomorphic and they are isomorphic to the maximal subgroups of the minimal $\mathcal{J}$-class of $\mathcal{S}$. This gives a geometrical meaning to the groups computed in the preceding section. We also obtain the following result.

For shortness, let us denote $G(\mathcal{S})$ any of the maximal subgroups of the minimal $\mathcal{J}$-class of an irreducible subshift $\mathcal{S}$.

Corollary 3. *The group $G(\mathcal{S})$ is a conjugacy invariant of $\mathcal{S}$.*

A subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ is said to be *generated* by a finite primitive endomorphism φ of $\overline{\Omega}_n\mathsf{S}$ if $L(\mathcal{S})$ is the set of factors of the words of the form $\varphi^n(x_i)$ or, equivalently, the finite factors of the profinite words $\varphi^\omega(x_i)$. Since for such φ, $\varphi^\omega(x_i)$ is uniformly recurrent, we do always generate a subshift in this way. The subshifts thus obtained are also called *substitution subshifts*.

As a consequence of Theorem 8 we should note that $G(\mathcal{S})$ is a very rough conjugacy invariant. However, it is easy to see that the action of the alphabet on the minimal $\mathcal{J}$-class of an irreducible subshift $\mathcal{S}$ is sufficient to allow us to recover $\mathcal{S}$. Hence, one should be able to extract from this action enough information to characterize the conjugacy class of $\mathcal{S}$. At present it remains an open problem how to do it and whether that may lead to a solution of the conjugacy problem for subshifts of finite type or even for sofic subshifts.

We end this section with a partial extension of Theorem 8 to non-substitution subshifts. A subshift $\mathcal{S}$ is said to be *Sturmian* if $L(\mathcal{S})$ has exactly $n+1$ elements of length n for every $n \geq 1$. It is well known that this is the minimum possible value for a non-periodic subshift and that Sturmian subshifts are minimal [19]. Taking $n = 1$, we see that a Sturmian subshift involves only two letters and so it may be considered as a subshift over a two-letter alphabet.

The following result has also been announced in [5].

Theorem 12. *Let $\mathcal{S}$ be a Sturmian subshift. Then the group $G(\mathcal{S})$ is a free profinite group on two generators.*

For example, the continuous endomorphism of $\overline{\Omega}_2\mathsf{S}$ defined by $\varphi = (xy, x)$ generates the so-called *Fibonacci subshift*, which has many remarkable properties [25]. The name is justified since the number of occurrences of y in $\varphi^n(x)$ is the nth term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ The associated group is a free profinite group on two generators by Theorem 12.

Sturmian substitution subshifts have been characterized as those subshifts on two-letter alphabets which are generated by finite primitive G-invertible continuous endomorphisms of $\overline{\Omega}_2\mathsf{S}$ [25, Chapter 2]. Hence Theorem 12 is indeed an extension of Theorem 8 for two-letter alphabets. The following partial extension to larger alphabets has also been announced in [5].

We say that a word w is *right special* for a subshift $\mathcal{S}$, if there are at least two letters a, b such that $wa, wb \in L(\mathcal{S})$. In this case, the number of such letters is called the *right-degree* of w. The left analogues of this notion are defined dually. A subshift $\mathcal{S} \subseteq X^{\mathbb{Z}}$ is said to be an *Arnoux-Rauzy subshift* if, for every positive integer n, there is exactly one right special word of length n, which is of right-degree $|X|$, and one left special word of length n, which is of left-degree $|X|$. One can easily show that an Arnoux-Rauzy subshift is minimal.

Theorem 13. *Let $\mathcal{S}$ be an Arnoux-Rauzy subshift over an alphabet with m letters. Then the group $G(\mathcal{S})$ is a free profinite group on m generators.*

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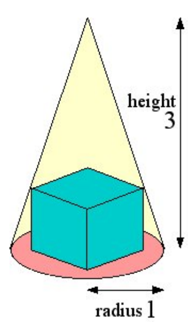
What's New in Mathematics

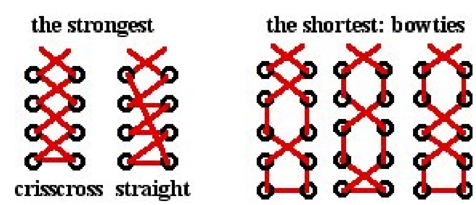
The Putnam in *Time*. "Crunching the Numbers" is the title of a piece by Lev Grossman, in the December 23 2002 *Time* magazine, about the William Lowell Putnam Mathematical Competition. "Every year," it begins, "on the first Saturday in December, 2,500 of the most brilliant college students in North America take what may be the hardest math test in the world." Grossman gives a quick survey of the history of the exam, a summary of the daunting statistics ("the median score on last year's test was 1 point. Out of a possible 120.") and a *Time*-like glimpse of its mystique ("think of it as a coming-out party for the next generation of beautiful minds"). He interviews Leonard Klosinski (Santa Clara; the competition director), Richard Stanley (coach of the MIT team) and Kevin Lacker, one of last year's winners, who remarks: "Doing well on the Putnam and doing good math research are two different tasks that take two different kinds of intelligence."

The piece includes a sample problem, labeled "An Easy One." "A right circular cone has a base of radius 1 and a height of 3. A cube is inscribed on the cone so that one face of the cube is contained in the base of the cone. What is the length of an edge of the cube?" Check *Time* for the answer.



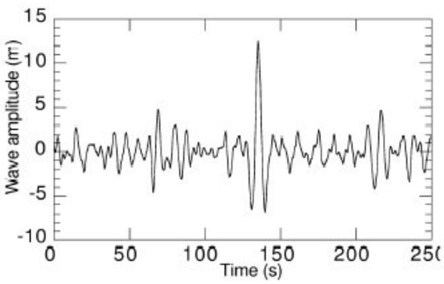
Too much pi? Under the title "How to Slice the Pi Very, Very Thin," the December 7, 2002 *New York Times* ran an AP dispatch from Tokyo reporting on the calculation of π to 1.24 trillion places, "six times the number of places recognized now." A ten-person team led by Yasumasa Kanada broke the trillion-place barrier with the help of a Hitachi supercomputer at the Information Technology Center of Tokyo University. The report quotes David Bailey (Lawrence Berkeley Lab): "It's an enormous feat of computing, not only for the sheer volume, but it's an advance in the technique he's using. All known techniques would exceed the capacity of the computer he's using." Which is, we are told, two trillion calculations a second. Note that light travels .15 mm in one two-trillionth of a second. This must be a very small or very parallel computer.

The best ways to lace your shoes has been worked out by Burkard Polster, a mathematician at Monash University (Victoria, Australia). His report, in the December 5 2002 *Nature*, was picked up in the December 10 *Boston Globe* (via Reuters) and in *Time* magazine for December 23.



The best way to lace depends on your criteria, but in all allowable lacings each eyelet is connected to at least one eyelet on the opposite side. The strongest lacings with n pairs of eyelets are the "crisscross" (when the ratio h of vertical eyelet spacing to horizontal is below a certain value h_n) and the "straight" (when h is greater than h_n). The shortest lacings are the "bowties". There is only one minimal bowtie lacing when n is even, but there are $(n+1)/2$ when n is odd. The shortest "dense" lacing (no vertical segments) is the crisscross.

Freak waves. BBC Two, on November 14, 2002, aired a program on this phenomenon and its recent mathematical analysis. Freak waves, also "rogue waves," "monster waves," are extraordinarily tall and steep waves that appear sporadically and wreck havoc with shipping. One is suspected to have washed away the German cargo München which went down with all hands in the midst of a routine voyage in 1978. More recently, the cruise ship Caledonian Star was struck by a 30m wave on March 2, 2001. The standard analysis of ocean waves predicts a Gaussian-like distribution of heights; extreme heights, although possible, should be very rare - a 30m wave is expected once in ten thousand years, according to the BBC. But these waves occur much more frequently than predicted. The program focused on new methods of analysis, and on the work of the mathematician A. R. Osborne (*Fisica Generale*, Torino). Osborne has applied the inverse scattering transform, which he describes as "nonlinear Fourier analysis," to the time series analysis of wave data. He conducted simulations using the nonlinear Schrödinger equation and found near agreement with the standard analysis, except that "every once in a while a large rogue wave rises up out of the random background noise." His paper, available online, gives an example of such a simulation:



TIME SERIES OF A RANDOM WAVE TRAIN SHOWING THE APPEARANCE OF A LARGE ROGUE WAVE WITH HEIGHT 20M OCCURRING AT 140 SECONDS.

Mathematical oncology. "Clinical oncologists and tumor biologists posess virtually no comprehensive model to serve as a framework for understanding, organizing and applying their data." This statement is featured in a box at the top of Robert A. Gatenby and Philip K. Maini's "Concepts" piece in the January 23 2003 *Nature*. They point out that despite the glut of publication (over 21000 articles on cancer in 2001) oncology has not been pursuing "quantitative methods to consolidate its vast body of data and integrate the rapidly accumulating new information." The explanations they suggest are mostly cultural. For example: "... medical schools have generally eliminated mathematics from admission prerequisites ..." They also blame "those of us who apply quantitative methods to cancer" for not having "clearly demonstrated to our biologist friends a dominant theme of modern applied mathematics: that simple underlying mechanisms may yield highly complex observable behaviors." An illustration from Wolframscience.com drives home the point. They end with an apology for mathematical modeling, showing how a verbal schema may be be enriched and strengthened by incorporation into a mechanistic and quantitative model which can handle, through computation, properties such as stochasticity and nonlinearity which cannot be handled by verbal reasoning alone. "As in physics, understanding the complex, nonlinear systems in cancer biology will require ongoing, interdisciplinary, interactive research in which mathematical models, informed by extant data and continually revised by new information, guide experimental design and interpretation."

4 log 3 - a new cosmic constant? John Baez (UC Riverside) has a "news and views" piece in the February 13 2003 *Nature* entitled "The Quantum of Area?". We start by asking whether black holes have a discrete spectrum of energy levels. According to Baez, a complete answer would require an understanding of "how quantum mechanics and general relativity fit together – one of the great unsolved problems in physics." But two completely different ways of guessing have recently come to the same answer: the spectrum of discrete energy levels is related to the surface area of the black hole, and the quantum of surface area is exactly 4 times the natural logarithm of 3 times the Planck area (which itself is about 10^{-70} m^2). The "surface" is actually the event horizon - "the closest distance an object can approach a black hole before being sucked in," so it is an imaginary boundary, but nevertheless acts in many ways "like a flexible membrane," and has a geometry of its own: it is flat except at points where it is punctured by one of the "threads" postulated by loop quantum gravity theory. Recent work by Shahar Hod (Hebrew University), Olaf Dreyer (Penn State; available online at http://arxiv.org/list/gr-qc/0211) and Lubos Motl (Harvard; available online at http://arxiv.org/list/gr-qc/0212) relates to earlier research by Hawking, Ashketar and Baez himself.

The Poincaré Conjecture. The *New York Times*, in their Science section for April 15, 2003, ran a piece by Sara Robinson entitled "Celebrated Math Problem Solved, Russian Reports." The problem is the 100-year-old Poincaré Conjecture; the Russian is Grigory Perelman of the Steklov Institute in St. Petersburg. As Robinson describes it, Perelman is claiming even more: a proof of a conjecture due to William Thurston, that "three-dimensional manifolds are composed of ... homogeneous pieces that can be put together only in prescribed ways." The Poincaré Conjecture, about the possible topology of a three-dimensional manifold in which every loop can be shrunk to a point, follows because now it would be known what possible geometric structure such a manifold could have. Robinson comments briefly on the method of proof. There is a natural way for the geometry of a manifold to evolve in time: this is the Ricci flow, "an averaging process used to smooth out the bumps of a manifold and make it look more uniform." Its application to Thurston's geometrization conjecture was pioneered by Richard Hamilton (now at Columbia) and carried out in full, we hope, by Perelman. Robinson remarks on the interesting parallels between Perelman's odyssey and that of Andrew Wiles (who recently proved Fermat's Theorem) and also on Perelman's eligibility, if his proof sustains scrutiny, for one of the Clay Mathematical Institute's million-dollar prizes. The *Times* picked up the story again in the "Week in Review" section on Sunday, April 20: "A Mathematician's World of Doughnuts and Spheres," by

George Johnson. "Poincaré proof adds up to potential payday" is the tack *Nature* chose to follow in a News in Brief item (April 24, 2003). The math got mangled: "Closed two-dimensional surfaces without holes can be transformed onto the surface of a sphere, and Henri Poincaré suggested that similar surfaces with higher dimensions should also transform back to spheres." But they did give a link to one of Perelman's preprints.

"The Superformula". *Nature* Science Update ran a piece on April 3, 2002 by John Whitfield: "Maths gets into shape." Whitfield was reporting on an article by Johan Gielis (Nijmegen) in the March 2003 American Journal of Botany in which Gielis proposes his superformula ("A generic geometric transformation that unifies a wide range of natural and abstract shapes"). The superformula, in slightly different notation, is the following polar equation:

$$r(\varphi) = f(\varphi)(|A\,cosM|^p + |B\,sinM|^q)^{-1/n}, \quad (4)$$

which, for various values of the parameters A, B, M, p, q, n and various choices of the function $f(\varphi)$ does in fact give a wide variety of interesting shapes. Whether this mathematical unity is of any botanical significance is harder to see. Whitfield quotes Ian Stewart (Warwick): "I'm not convinced ... , but it might turn out to be profound if it could be related to how things grow" as is the case, for example, with D'Arcy Thompson's explanation of the logarithmic spiral in mollusk shells. Gielis' position, as quoted by Whitfield: "Description always precedes ideas about the real connection between maths and nature." A botanical Kepler awaiting his Newton. Meanwhile, Gielis has applied for a patent on his discovery: Methods and devices for synthesizing and analyzing patterns using a novel mathematical operator, USPTO patent application No. 60/133,279 (1999).

Math in *Nature*. The May 15 2003 issue of *Nature* has at least three articles with interesting mathematical aspects.

* Astronomy. "Chaos-assisted capture of irregular moons" is a comparative study of the irregular moon systems of the gas giants Jupiter and Saturn. Irregular moons have highly inclined orbits (but never more than 55 degrees) with respect to the planet's equatorial plane. Their motion may be prograde, counterclockwise when viewed from above, like our Moon and Jupiter's Galilean moons, or retrograde. In fact in the Jupiter system, the retrogrades outnumber the progrades 26 to 6. The authors study the 3-dimensional circular restricted three-body problem, focussing on the Sun-Jupiter-moon system. They use a Monte Carlo simulation to show how, in phase space, "the chaotic layer selects for the sense of the angular momentum of incoming and outgoing particles," i.e. sorts them into prograde and retrograde. (Authors: S. A. Astakhov, A. D. Burbanks, S. Wiggins, D. Farrelly)

* Econophysics. "A theory of power-law distributions in financial market fluctuations" sets up a model to explain the empirical probabilities:

$$\begin{aligned} P(|r_t| > x) &\sim x^{-3} \\ P(V > x) &\sim x^{-1.5} \\ P(N > x) &\sim x^{-3.4} \end{aligned}$$

where r_t is the change of the logarithm of stock price in a given time interval Δt (for a given stock), V is trading volume and N is the number of trades. The model "is based on the hypothesis that large movements in stock market activity arise from the trades of large participants." (Authors: X. Gabaix, P. Gopihrishnan, V. Plerou, H. E. Stanley).

* Neurophysiology. In "Attractor dynamics of network UP states in the neocortex" the authors report that in analyzing the dynamics of spontaneous activity of neurons in the mouse visual cortex, they detected "synchronized UP state transitions" occurring in "spatially organized ensembles involving small numbers of neurons." (UP is short for the membrane potential depolarized state). They argue that the these synchronized transitions, or 'cortical flashes,' are dynamical attractors, and that "a principal function of the highly recurrent neocortical networks is to generate persistent activity that might represent mental states." (Authors: R. Cossart, D. Aronov, R. Yuste)

The Poincaré Conjecture (cont.) The recent developments were also covered by *Science*, in an April 18 2003 piece by Dana Mackenzie whose title, "Mathematics World Abuzz Over Possible Poincaré Proof," correctly suggests his Variety-style approach to the subject. "Furthermore, what was to keep the surgeries, like plastic surgeries on a Hollywood star, from going on endlessly?" Nevertheless Mackenzie gives the best layman's guide so far to the history of the problem and to Perelman's innovations. An excellent presentation, ending in a lovely quote from Bennett Chow (UCSD): "It's like climbing a mountain, except in the real world we know how high the mountain is. What Hamilton did was climb incredibly high, far beyond what anyone expected. Perelman started where Hamilton left off and got even higher yet - but we still don't know how high the mountain is." *Nature* came back to the story, after last month's "News in Brief" item, with a more elaborate, and mathematically substantial, report by Ian Stewart (May 8, 2003). This account, also excellent, is complementary to Mackenzie's: they emphasize different aspects of the problem and of the putative solution.

An Interview with with M. J. D. Powell

I am sure that our readers would like to know a bit about your academic education and professional career first. Why did you choose to go to the Atomic Energy Establishment (Harwell) right after college in 1959?

When I studied mathematics at school, nearly all of my efforts were applied to solving problems in text books, instead of reading the texts. Then my teachers marked and discussed my solutions instead of instructing me in a formal way. I enjoyed this kind of work greatly, especially when I was able to find answers to difficult questions myself. Thus I gained a good understanding of some fields of mathematics, but I became unwilling to learn about new subjects at a general introductory level, because I do not have a good memory, and to me it was without fun. I also disliked the breadth of the range of courses that one had to take at Cambridge University as an undergraduate in mathematics. Fortunately, I was able to complete that work adequately in two years, which allowed me to study for the Diploma in Numerical Analysis and Computation during my third year. It was a relief to be able to solve problems again most of the time, and the availability of the Edsac 2 computer was a bonus. I welcomed the use of analysis and the satisfaction of obtaining answers. I wished to continue this kind of work after graduating, but the possibility of remaining in Cambridge for a higher degree was not suggested to me. Contributing to academic research and publishing papers in journals were not suggested either, although I developed a successful algorithm for adaptive quadrature in a third year project. Therefore in 1959 I applied for three jobs at government research establishments, where I would assist scientists with numerical computer calculations. I liked the location of Harwell and the people who interviewed me there, so it was easy for me to accept their offer of employment.

You obtained your doctor of science only in 1979, twenty years after your bachelors degree and three years after being hired as a professor in Cambridge. Why was that the case?

After graduating from Cambridge in 1959 with a BA degree, I had no intention of obtaining a doctorate. All honours graduates from Cambridge are eligible for an MA degree after about 3 further years, without taking any more courses or examinations, but from my point of view that opportunity was not advantageous, partly because one had to pay a fee. When I became the Professor of Applied Numerical Analysis at Cambridge in 1976, I was granted all the privileges of an MA automatically, and my official degree became BA with MA status. Two years later, I was fortunate to be elected as a Professorial Fellow at Pembroke College, and the Master of Pembroke suggested that I should follow the procedure for becoming a Master of Arts. Rather than expressing my reservations about it, I offered to seek an ScD degree instead, which required an examination of much of my published work. Thus I became an academic doctor in 1979.

M. J. D. Powell

Was it hard to adapt to the academic life after so many years in Harwell?

After about five years at Harwell, most of my time was spent on research, which included the development of Fortran software for general computer calculations, the theoretical analysis of algorithms, and of course the publication of papers. The purpose of the administrative staff there was to make it easier for scientists to carry out their work. On the other hand, I found at Cambridge that one had to create one's own opportunities for research, which required some stubbornness and lack of cooperation, because of the demands of teaching, examining and admitting students, and also because administrative duties at universities can consume the time that remains, especially during terms. This change was particularly unwelcome, and is very different from the view that most of my relatives and friends have of university life. Indeed, when I was at Harwell they did not doubt that I had a full time job, but they assume that at Cambridge the vacations provide a life of leisure.

In your work in optimization we find several interesting and meaningful examples and counter-examples. Where did you get this training (assuming that not all is natural talent)? From your exposure to approximation theory? From the hand calculations of the old computing times?

The construction of examples and counter-examples is a natural part of my strong interest in problem solving, and of the development of software that I have mentioned. Specifically, numerical results during the testing of an algorithm often suggest the convergence and accuracy properties that are achieved, so conjectures arise that may be true or false. Answers to such questions are either proofs or counter-examples, and often I have tried to discover which of these alternatives applies. Perhaps my training started with my enjoyment of geometry at school, but then the solutions were available. I am pleased that you mention hand calculations, because I still find occasionally that they are very useful.

Was exemplification a relevant tool for you when you taught numerical analysis classes? Did your years as a staff member at Harwell influence your teaching?

My main aim when teaching numerical analysis to students at Cambridge was to try to convey some of the delightful theory that exists in the subject, especially in the approximation of functions. Only 36 lectures are available for numerical analysis during the three undergraduate years, however, except that there are also courses on computer projects in the second and third years, where attention is given to the use of software packages and to the numerical results that they provide. Moreover, in most years I also presented a graduate course of 24 lectures, in order to attract research students. The main contribution to my teaching from my years at Harwell was that I became familiar with much of the relevant theory there, because it was developed after I graduated in 1959, but I hardly ever mentioned numerical examples in my lectures, because of the existence of the Cambridge computer projects, and because the mathematical analysis was more important to my teaching objectives. Therefore my classes were small. Fortunately, some of the strongest mathematicians who attended them became my research students. I am delighted by their achievements.

Could you tell us how computing resources evolved at Harwell in the sixties and seventies and how that impacted on the numerical calculations of those times?

Beginning in 1958, I have always found that the speed of computers and the amount of storage are excellent, because of the huge advances that occur about every three years. On the other hand, the turnaround time for the running of computer programs did not improve steadily while I was at Harwell. Indeed, for about four years after I started to use Fortran in 1962, those programs were run on the IBM Stretch computer at Aldermaston, the punched cards being transported by car. Therefore one could run each numerical calculation only once or twice in 24 hours. Of course it was annoying to have to wait so long to be told that one had written DIMESNION instead of DIMENSION, but ever since I have been grateful for the careful attention to detail that one had to learn in that environment. Moreover, it was easier then to develop new algorithms that extend the range of calculations that can be solved. Conveying such advances to Harwell scientists was not straightforward, however, mainly because they wrote their own computer programs, using techniques that were familiar to them. The Harwell Subroutine Library, which I started, was intended to help them, and to reduce duplication in Fortran software. Often it was highly successful, but many computer users, both then and now, prefer not to learn new tricks, because they are satisfied by the huge gains they receive from increases in the power of computers.

You once wrote: "Usually I produced a Fortran program for the Harwell subroutine library whenever I proposed a new algorithm,..."[1]. In fact, writing numerical software has always been a concern of yours. Could you have been the same numerical analyst without your numerical experience?

My principal duty at Harwell was to produce Fortran programs that were useful for general calculations, which justified my salary. My work on the theoretical side of numerical analysis was also encouraged greatly, and its purpose was always to advance the understanding of practical computation. Indeed, without numerical experience, I would be cut off from my main source of ideas. It is unusual for me to make progress in research by studying papers that other people have written. Instead I seek fields that may benefit from a new algorithm that I have in mind. I also try to explain and to take advantage of the information that is provided by both good and bad features of numerical results.

Roger Fletcher wrote once that "your style of programming is not what one might call structured". Some people think that a piece of software should be well structured and documented. Others that it should be primarily efficient and reliable. What are your views on this?

I never study the details of software that is written by other people, and I do not expect them to look at my

[1] *A View of Nonlinear Optimization*, History of Mathematical Programming: A Collection of Personal Reminiscences (J.K. Lenstra, A.H.G. Rinnooy Kan, and A. Schrijver eds), North-Holland (Amsterdam), 119-125 (1991).

computer programs. My writing of software always depends on the discipline of subroutines in Fortran, where the lines of code inside a subroutine can be treated as a black box, provided that the function of each subroutine is specified clearly. Finding bugs in programs becomes very painful, however, if there are any doubts about the correctness of the routines that are used. Therefore I believe that the reliability and accuracy of individual subroutines is of prime importance. If one fulfils this aim, then in my opinion there is no need for programs to be structured in a formal way, and conventional structures are disadvantageous if they do not suit the style of the programmer who must avoid mistakes. Those people who write reliable software usually achieve good efficiency too. Of course it is necessary for the documentation to state what the programs can do, but otherwise I do not favour the inclusion of lots of internal comments.

And by the way, how do you regard the recent advances in software packages for nonlinear optimization?

Most of my knowledge of recent advances in software packages has been gained from talks at conferences. I am a strong supporter of such activities, as they make advances in numerical analysis available for applications. My enthusiasm diminishes, however, when a speaker claims that his or her software has solved successfully about 90% of the test problems that have been tried, because I could not tolerate a failure rate of 10%. Another reservation, which applies to my programs too, is that many computer users prefer software that has not been developed by numerical analysts. I have in mind the popularity of simulated annealing and genetic algorithms for optimization calculations, although they are very extravagant in their use of function evaluations.

Many people working in numerical mathematics undervalue the paramount importance of numerical linear algebra (matrix calculations). Would you like to comment on this issue? How often was research in numerical linear algebra essential to your work in approximation and optimization?

An optimization algorithm is no good if its matrix calculations do not provide enough accuracy, but, whenever I try to invent a new method, I assume initially that the computer arithmetic is exact. This point of view is reasonable for the minimization of general smooth functions, because techniques that prevent serious damage from nonlinear and nonquadratic terms in exact arithmetic can usually cope with the effects of computer rounding errors, as in both cases one has to restrict the effects of perturbations. Therefore I expect my algorithms to include stability properties that allow the details of the matrix operations to be addressed after the principal features of the algorithm have been chosen. Further, I prefer to find ways of performing the matrix calculations myself, instead of studying relevant research by other people.

I read in one of your articles that "a referee suggested rejection because he did not like the bracket notation". What is your view about the importance of refereeing? How do you classify yourself as a referee?...

The story about the bracket notation is remarkable, because the paper that was nearly rejected is the one by Roger Fletcher and myself on the Davidon–Fletcher–Powell (DFP) algorithm. As a referee, I ask whether submitted work makes a substantial contribution to its subject, whether it is correct, and whether the amount of detail is about right. I believe strongly that we can rely on the accuracy of published papers only if someone, different from the author(s), checks every line that is written, and in my opinion that task is the responsibility of referees. When it is done carefully, then refereeing becomes highly important. I try to act in this way myself, but, because my general knowledge of achievements in my fields is not comprehensive, I often consider submissions in isolation, although I should relate them to published work.

Actually, in my previous question I had in mind the difficulty that others might face to meet your high standards. This brings me to your activity as a Ph.D adviser. What difficulties and what rewards do you encounter when advising Ph.D. students?

Of course I take the view that my requirements for the quality of the work of my PhD students are reasonable. I require their mathematics to be correct, I require relevance to numerical computation, and I require some careful investigations of new ideas, instead of a review of a subject with some superficial originality. Further, I prefer my students to work on topics that are not receiving much attention from other researchers, in order that they can become leading experts in their fields. Some of them have succeeded in this way, which is a great reward, but two of them switched to less demanding supervisors, and another one switched to a well paid job instead of completing his studies. I also had a student that I never saw after his first four terms. Eventually he submitted a miserable thesis, that was revised after his first oral examination, and then the new version was passed by the examiners, but the outcome would have been different if university regulations had allowed me to influence the result. On the other hand, all my other students have done excellent work and have thoroughly deserved their PhDs. One difficulty has occurred in several cases, namely that, because each student has to gain experience and to make advances independently, one may have to allow his or her rate of progress to be much slower than one could achieve oneself. Another difficulty is that my knowledge of pure mathematics has been inadequate for easy communication between myself and most of my students who have studied approximation theory. Usually they were very tolerant about my ignorance of distributions and properties of Fourier transforms, for example, but my heart sinks when I am asked to referee papers that depend on these subjects.

Most of your publications are single-authored. Why do you prefer to work on your own?

I believe I have explained already why I enjoy working on my own. Therefore, when I begin some new research, I do not seek a co-author. Moreover, as indicated in the last paragraph, I prefer my students to make their own discoveries, so usually I am not a co-author of their papers.

I have been trying to avoid technical questions but there is one I would like to ask. What is your view on interior-point methods (a topic where you made only a couple — but as always relevant and significant — contributions)?

My view of interior point methods for optimization calculations with linear constraints is that it seems silly to introduce nonlinearities and iterative procedures for following central paths, because these complications are not present in the original problem. On the other hand, when the number of constraints is huge, then algorithms that treat constraints individually are also unattractive, especially if the attention to detail causes the number of iterations to be about the number of constraints. It is possible, however, to retain linear constraints explicitly, and to take advantage of the situation where the boundary of the feasible region has so many linear facets that it seems to be smooth. This is done by the TOLMIN software that I developed in 1989, for example, but the number of variables is restricted to a few hundred, because quadratic models with full second derivative matrices are employed. Therefore eventually I expect interior point methods to be best only if the number of variables is large. Another reservation about this field is that it seems to be taking far more than its share of research activity.

You published a book in approximation theory. Have you ever thought about writing a book in nonlinear optimization?

My book on Approximation Theory and Methods was published in 1981. Two years later, my son died in an accident, and then I wished to write a book on Nonlinear Optimization that I would dedicate to him. I have not given up this idea, but other duties, especially the preparation of work for conferences and their proceedings, have caused me to postpone the plan. Of course, because of the circumstances, I would try particularly hard to produce a book of high quality.

Let me end this interview with the very same questions I asked T.R. Rockafellar (who, by the way, shared with you the first Dantzig Prize in 1982). Have you ever felt that a result of yours was unfairly neglected? Which? Why? What would you like to prove or see proven that is still open (both in approximation theory and in nonlinear optimization)? What was the most gratifying paper you ever wrote? Why?

I was taught the FFT (Fast Fourier Transform) method by J.C.P. Miller in 1959, and then it made Cooley and Tukey famous a few years later. Moreover, my 1963 paper with Roger Fletcher on the DFP method is mainly a description of work by Davidon in 1959, and it has helped my career greatly. Therefore, by comparison, none of my results has been unfairly neglected. My main theoretical interest at present is trying to establish the orders of convergence that occur at edges, when values of a smooth function are interpolated by the radial basis function method on a regular grid, which is frustrating, because the orders are shown clearly by numerical experiments. In nonlinear optimization, however, most of my attention is being given to the development of algorithms. Since you ask me to mention a gratifying paper, let me pick "A method for nonlinear constraints in minimization problems", because it is regarded as one of the sources of the "augmented Lagrangian method", which is now of fundamental importance in mathematical programming. I have been very fortunate to have played a part in discoveries of this kind.

Interview by Luís Nunes Vicente (Uni. of Coimbra)

M.J.D. Powell completed his undergraduate studies at the University of Cambridge in 1959. From 1959 to 1976 he worked at the Atomic Energy Establishment, Harwell, where he was Head of the Numerical Analysis Group from 1970. He has been John Humphrey Plummer Professor of Applied Numerical Analysis, University of Cambridge since 1976 and a Fellow of Pembroke College, Cambridge since 1978.

He made many seminal contributions in approximation theory, nonlinear optimization, and other topics in numerical analysis. He has written a book in approximation theory and more than one hundred and fifty papers.

José Tiago de Oliveira

José Tiago da Fonseca Oliveira was an eminent statistician and university professor. His name is already registered in the history of the 20th century Statistics due to his important contributions to the development of the theory of extreme values. As a Portuguese scientist his name will remain forever associated to the recognition of Statistics as a science in Portugal.

Tiago de Oliveira was born in Lourenço Marques, Mozambique, on the 22th of December 1928. A very interesting account of his times in Mozambique, where he lived until 1945, is given by Eugénio Lisboa, one of his friends from childhood, in *Tiago de Oliveira, O Homem e a Obra*, 1993, eds. Colibri.

Tiago de Oliveira finished his high school education in 1945. Because of his outstanding performance he was awarded, that year, the prize for the best student of Liceu Lourenço Marques. He also received a grant from Caixa Económica Postal which helped him to leave Mozambique and pursue his studies in Porto. His intention was to study Naval Engineering at the University of Porto. However, during his trip back to the Continent he stopped in Lobito, Angola. A visit to a local bookshop led him to buy a book on Statistics, written in Spanish. It was then, according to his son José Carlos Tiago de Oliveira (in *Tiago de Oliveira, O Homem e a Obra*, 1993, eds. Colibri), that he found is vocation. Instead of Naval Engineering he studied Mathematics and finished his degree in 1949. In 1950 he got a degree in Geographic Engineering, and in 1951 he received the Rotary Club Prize for the best student of the Faculty of Sciences.

Tiago de Oliveira's political views against Salazar's regime were well known. As a consequence it was not easy for him to get a job despite his achievements as a student. Twice he was invited for the place of assistant at the Faculty of Sciences in Porto, but twice he saw his appointment denied for political reasons. He moved to Lisbon in 1951 and got a job at the Institute of Marine Biology as a research assistant in biometry and biostatistics. By the time he left the Institute in 1953 to become an assistant lecturer at the Faculty of Sciences at the University of Lisbon, he had already published seven papers in Statistics. This was only the beginning of an extraordinary career in the area of probability and statistics.

Tiago de Oliveira

He entered the Faculty of Sciences as an assistant, thanks to the influence of Prof. António Almeida e Costa, a true scientist and a person with vision, who knew how to separate science from politics. Tiago de Oliveira studied under his supervision and in 1957 he finished his doctoral thesis in the area of Algebra with a dissertation entitled "Residuais de Sistemas e Radicais de Anéis". However, his interest in Statistics had not died out and it was with a thesis on "Estatística de Densidades; resultados Assintóticos" that he applied in 1965 for the position of Professor Extraordinário. He studied probability and statistics as an autodidact. His "bible", as he used to call it, was the work of Kendall and Stuart. In 1967, when he became a full professor, he had already 63 publications, some of them in well-known periodicals such as *Annals of Mathematical Statistics* and *Bulletin of the International Statistical Institute*, among others.

It is not clear how Tiago de Oliveira got interested in the theory of extreme values, his main area of research. His first publication in this area, "Extremal Distributions", dates back to 1959. In 1960 he went for the first time to Columbia University, as Senior Research Assistant, and there he had the opportunity to work with the most prominent scientist in the area, E. J. Gumbel. This collaboration marked the beginning of a very fruitful research career for Tiago de Oliveira. In 1961 he published some extensions of Gumbel's results in the theory of univariate extremes, to the bivariate and multivariate cases. His pioneer work was followed by many other important contributions and new developments in the area of multivariate extremes. He also developed several methods for the estimation of the parameters of Gumbel, Fréchet and Weibull models and for the estimation of high quantiles. Together with S. B. Littauer he worked on prediction of extremal models. He also had important contributions in statistical decision problems related to the Weibull distribution, and in the study of univariate extremes in dependent sequences. Another pioneer work of Tiago de Oliveira was on the statistical choice of univariate extremal models. In a paper published in *Statistical Distributions in Scientific Work*, vol. 6, in 1981, he developed locally most powerful (LMP) tests for discrimination between extremal models. This problem was approached from a computational point of view in a joint paper with A. Frasen, and with M. I. Gomes he studied exact and asymptotic behaviour of alternative statistical tests to the same problem.

Although Tiago de Oliveira is well known due to his work in Extreme Value Theory, his research went well beyond this particular area. He had important contributions in many other themes such as Demography, Quality Control, Outliers, Mixtures, Non-parametric Statistics, Risk Theory, Actuarial Mathematics, just to mention a few.

Tiago de Oliveira was also a man of broad interests, both scientific and cultural. He had a deep understanding of history and Portuguese political culture. He wrote several historical, philosophical and didactical articles. Particularly interesting are his views on the development of mathematics in Portugal from the XVI to the XIX centuries (in *Collected Works of J. Tiago de Oliveira*, vol. II). Overall, he published around 160 scientific papers, 9 books, 22 historical and philosophical papers, 18 didactic and expository articles, and 21 other papers on miscellaneous subjects. At the time of his death, on the 23th of June 1992, he had six papers and four more books in preparation. His book *Statistical Analysis of Extremes* was posthumously published due to the efforts of his son, José Carlos Tiago de Oliveira, who also compiled all his works in a six-volume series entitled *Collected Works of J. Tiago de Oliveira* and published by Pendor.

Tiago de Oliveira was not just a great scientist. He was a man with strong views and strong convictions who would fight for his own ideals. He fought for the autonomy of the area of Applied Mathematics in the Faculty of Sciences at the University of Lisbon, and later for the autonomy of Statistics and Operations Research, founding in 1981 a Department of Statistics, Operations Research and Computation, today the Department of Statistics and Operations Research of the FCUL. He was also a founder of the Center of Statistics and Applications of the University of Lisbon and the Portuguese Statistical Society. Due to his trust in the younger generations and constant encouragement he brought, in the late seventies and early eighties, many people to the areas of Statistics, Operations Research and Computation. The "Portuguese Statistical School of Extremes", which today is internationally respected, owes its existence to him. Later, when in 1987 he left the University of Lisbon and went to the Faculty of Sciences and Technology of the New University of Lisbon, he again put all his efforts in bringing up a new group of people working in his areas of choice. In that Faculty he founded the Laboratory of Statistics and Actuarial Mathematics. He also served the scientific community as Secretary of State for scientific research from 1976 to 1978.

Tiago de Oliveira had been a Fellow of the Royal Statistical Society since 1952. However, in 1987, in recognition of his merit and important contributions to the area of Statistics, he was awarded the title of Honorary Fellow of the Royal Statistical Society. He was also a member of the International Statistical Institute, a member of the Bernoulli Society for Mathematical Statistics, a Fellow of the Institute of Mathematical Statistics, a full member of the Academia das Ciências de Lisboa, a corresponding member of the Real Academia de las Ciencias Exactas, Físicas y Naturales de Madrid, among many other scientific associations.

During his life he was awarded three prizes in recognition of his outstanding scientific work. The A. Malheiros Prize for Mathematical Sciences of the Academy of Sciences of Lisbon, in 1969; the Calouste Gulbenkian Foundation Prize for Sciences and Technology in 1984; the Science Prize of the Oriente Foundation in 1992.

The sphere of activity of Tiago de Oliveira was not limited to the academic level. He was deeply interested and involved in the problems of society in general and of the Portuguese society in particular. As such he was a founding member of the Socialist party, a member of the Union of Teachers of Greater Lisbon (Sindicato dos Professores da Grande Lisboa), a member of the Association of Statisticians for Human Rights, a member of the Portuguese Association of Human Rights, and a member of the Portuguese Section of the International Amnesty.

For the outstanding scientific legacy Tiago de Oliveira left behind, he deserves a very special place among the

Great Portuguese Mathematicians of the 20th Century.

[To write this short sketch I based myself on the following documents:

- *J. Tiago de Oliveira: O Homem e a Obra*, edições Colibri, 1993 - a book organized by José Carlos Tiago de Oliveira, and published to commemorate the first aniversary of Tiago de Oliveira´s death.
- Special edition of the *Boletim Informativo da Sociedade Portuguesa de Estatística* in honour of Tiago de Oliveira, 22 December 1998. This edition was specially organized to commemorate the day of his 70th anniversary. It contains testimonies of his children (José Carlos and Luisa), many of his friends, colleagues and former students.
- The text "José Tiago de Oliveira - Um estatístico eminente" by Margarida Mendes Leal contained in the book *Memórias de Professores Cientistas*, published in 2001 to commemorate the 90th anniversary of the Faculty of Sciences of Lisbon.
- *Collected Works of J. Tiago de Oliveira*, vol II, 1995; compiled by José Carlos Tiago de Oliveira, edições Pendor.]

Maria Antónia Amaral Turkman

Errata

Na versão impressa do boletim 13 de Dezembro de 2002, o artigo *Warp Drive with Zero Expansion* de José Natário, continha erros em várias expressões. A razão desses erros é técnica (de transferência de ficheiros) da responsabilidade dos editores do boletim. A versão electrónica (disponível em http://www.cim.pt/cim.www/cimE/boletim.html) encontra-se corrigida. Pelo facto pedimos desculpa ao autor e aos leitores.

Os editores

Editors: Jorge Buescu (jbuescu@math.ist.utl.pt)
F. Miguel Dionísio (fmd@math.ist.utl.pt)
João Filipe Queiró (jfqueiro@mat.uc.pt).

Address: Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal.

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