Continuous optimization polynomial-time upper bounds on the stability number of graphs

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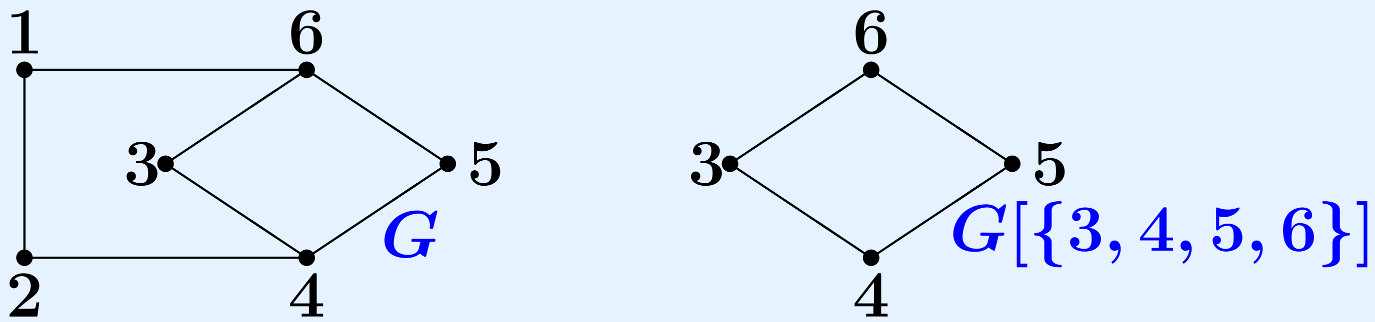
Outline

1. Basic definitions and notation
2. Continuous formulations for the stability number
3. Polynomial-time upper bounds
4. Graphs whose stability number is easily determined
5. Recognition of Q-graphs
6. References

1. Basic definitions and notation

- **We consider simple graphs G with at least one edge;**
- **$E(G)$ and $V(G)$ will denote the edge set and the vertex set of G, respectively;**
- **Given a vertex subset $S \subseteq V(G)$ a subgraph induced by S, $G' = G[S]$ is such that $V(G') = S$ and $E(G')$ are the edges of G connecting vertices of S;**



1. Basic definitions and notation (cont.)

- **The neighborhood of $v \in V(G)$, denoted by $N_G(v)$, is the subset of vertices adjacent to v, and the degree of v is**

$$d_G(v) = |N_G(v)|$$

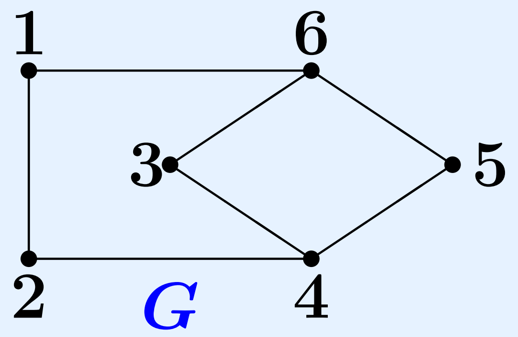


Figure 2: $N_G(6) = \{1, 3, 5\}$ and $d_G(6) = 3$.

- **If $d_G(v) = p \ \ \forall v \in V(G)$ then we say that G is p-regular.**

1. Basic definitions and notation (cont.)

- A **stable set** (**clique**) is a vertex subset inducing a null (complete) subgraph. The cardinality of a maximum size stable set (clique) of a graph G is called stability (clique) number of G and it is denoted by $\alpha(G)$ ($\omega(G)$);

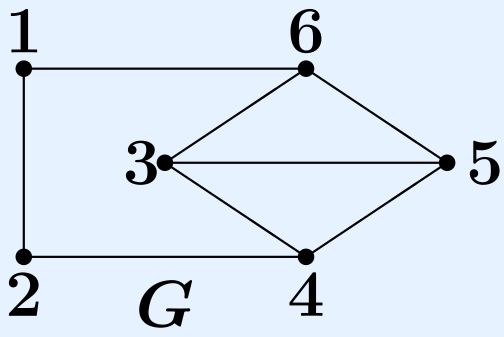


Figure 3: A stable set $S = \{2, 4, 6\}$ and a clique $K = \{3, 4, 5\}$.

1. Basic definitions and notation (cont.)

- **The complement of a graph G, denoted by $\bar{G}$, is such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{ij : i, j \in V(G) \wedge ij \notin E(G)\}$.**

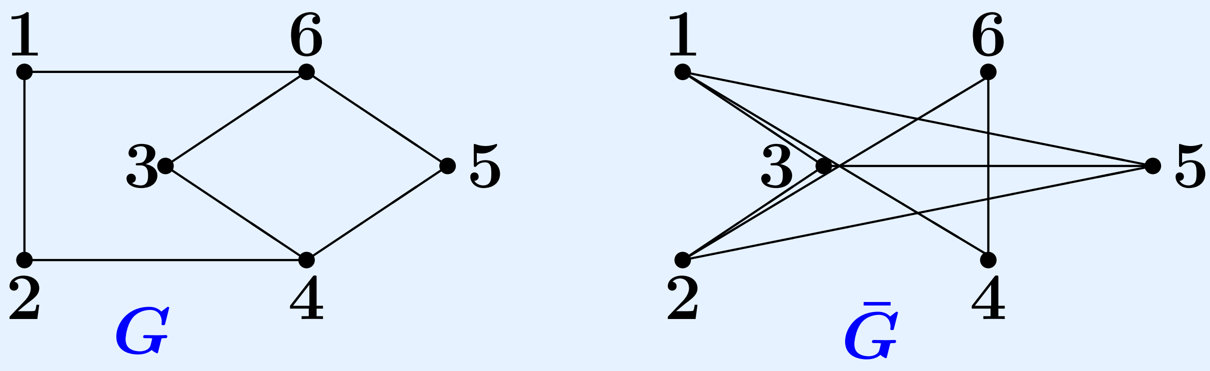


Figure 4: A graph G and its complement $\bar{G}$.

- **Then $\alpha(G) = \omega(\bar{G})$ and determine the stability number is equivalent to determine the clique number.**

1. Basic definitions and notation (cont.)

- Given a nonnegative integer k, to determine if a graph G has a stable set of size k is NP-complete [Karp, 1972].
- A matching in a graph G is a subset of edges $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is designated maximum matching.
- If for each vertex $v \in V(G)$ there is one edge $e \in M$ such that v is incident with e, then M is called a perfect matching.
- The *line graph* $L(G)$ of a graph G has the edges of G as its vertices, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G have a vertex in common.

1. Basic definitions and notation (cont.)

- Then a matching in G corresponds to a stable set in $L(G)$.

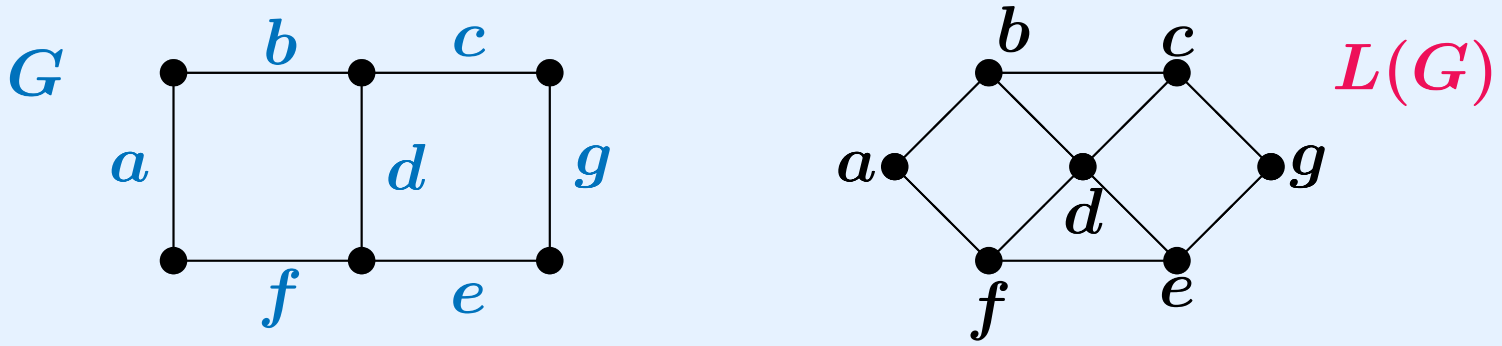


Figure 5: A graph G and its line graph $L(G)$.

- The graph G has the perfect matching $\{a, d, g\}$
- and then $L(G)$ has the maximum stable set $\{a, d, g\}$.

1. Basic definitions and notation (cont.)

- The *adjacency matrix* of a graph G, denoted by A_G, is such that $A_G = (a_{ij})_{n \times n}$, with $n > 1$, and

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

- Thus A_G is symmetric and then it has n real eigenvalues

$$\lambda_{max}(A_G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A_G).$$

- Furthermore, since G has at least one edge, the minimum eigenvalue of A_G, $\lambda_{min}(A_G)$, is not greater than -1.

2. Continuous formulations for the stability number

- **The first continuous formulation for $\omega(G)$ was obtained by Motzkin and Straus (1965):**

$$\max_{x\in\Delta} \frac{1}{2}x^T A_G x = \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right),$$

where $\Delta = \{x : \hat{e}^T x = 1, x \geq 0\}$ and $\hat{e}$ is the all ones vector.

- **Then (assuming that $|V(G)| = n$) it follows that**

$$\alpha(G) = \max_{0\neq x\in[0,1]^n} \frac{1}{x^T(A_G + I)x} \qquad (1)$$

2. Continuous formulations for the stability number (cont.)

Other continuous formulations for the stability number of a graph G

(Shor, 1990):

$$\begin{aligned} \alpha(G) = \quad & \max \sum_{v \in V(G)} x_v \\ & s.t. \quad x_i x_j = 0, \forall ij \in E(G) \qquad (2) \\ & \qquad\quad x_i^2 - x_i = 0, \forall i \in V(G) \end{aligned}$$

2. Continuous formulations for the stability number (cont.)

(Harant, 2000) and (Harant et al, 1999):

$$\alpha(G) = \max_{0 \leq x \leq \hat{e}} \Big(\sum_{i \in V(G)} (1 - x_i) \prod_{j \in N_G(i)} x_j \Big) \qquad (3)$$

$$\alpha(G) = \max_{0 \leq x \leq \hat{e}} \Big(\sum_{v \in V(G)} x_v - \sum_{ij \in E(G)} x_i x_j \Big) \qquad (4)$$

(Balasundaram and Butenko, 2005):

$$\alpha(G) = \max_{0 \leq x \leq \hat{e}} \sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N_G(i)} x_j} \qquad (5)$$

2. Continuous formulations for the stability number (cont.)

- Consider the family of quadratic programming problems

depending on a parameter $\tau > 0$ [C, 2003]:

$$v_G(\tau) = \max\{2\hat{e}^T x - x^T(\frac{A_G}{\tau} + I_n)x : x \geq 0\}, \qquad (6)$$

where I is the identity matrix.

Then, for each $\tau > 0$, we may conclude that

- $\alpha(G) \leq v_G(\tau)$;
- $1 \leq v_G(\tau) \leq n$, $v_G(\tau) = 1$ if G is a clique, and $v_G(\tau) = n$ if G has no edges;
- Furthermore, (6) is a convex program for $\tau \geq -\lambda_{min}(A_G)$.

2. Continuous formulations for the stability number (cont.)

• [C, 2003] The function $\upsilon_G :]0, +\infty[\mapsto [1, n]$ verifies:

$$0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2).$$

The following statements are equivalent:

- $\exists \bar{\tau} \in]0, \tau^*[$ such that $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$;
- $\upsilon_G(\tau^*) = \alpha(G)$;
- $\forall \tau \in]0, \tau^*]\ \upsilon_G(\tau) = \alpha(G)$.

$\forall U \subset V(G)\ \ \forall \tau > 0\ \ \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

For $\tau = 1$, (6) is equivalent to the Motzkin-Straus program [Motzkin and Straus, 1965] and $\upsilon_G(1) = \alpha(G)$.

2. Continuous formulations for the stability number (cont.)

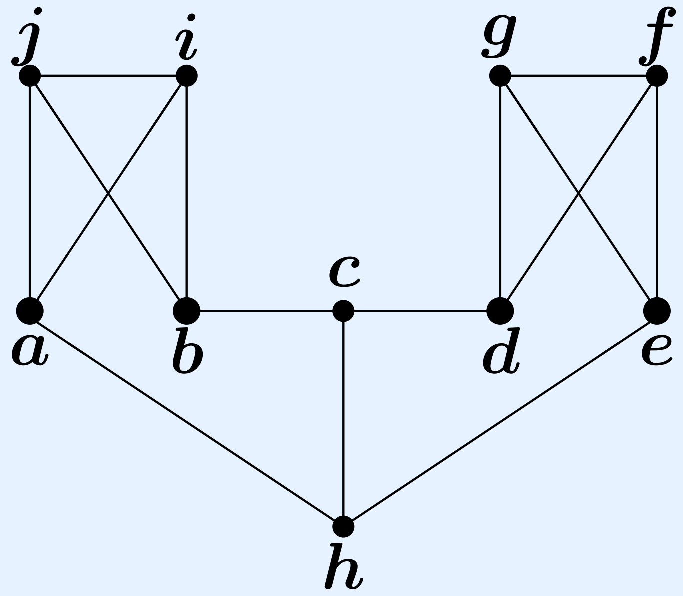


Figure 6: A cubic graph G such that $\upsilon_G(2) = 4 = \alpha(G)$.

3.Polynomial-time upper bounds

- From the above family, setting $\tau = -\lambda_{min}(A_G)$, we obtain the convex programming problem introduced in [Luz, 1995].

$$v(G) = \max_{x \geq 0} \; 2\hat{e}^T x - x^T \left(\frac{A_G}{-\lambda_{min}(A_G)} + I\right)x. \qquad (7)$$

Then $\alpha(G) \leq v(G)$ and, according to [Luz, 1995], $\alpha(G) = v(G)$ if and only if for a maximum stable set S (and then for all)

$$-\lambda_{min}(A_G) \leq \min\{|N_G(v) \cap S| : v \notin S\}. \qquad (8)$$

3.Polynomial-time upper bounds(cont.)

- Actually, $\alpha(G) = \upsilon(G)$ if and only if there exists a stable set S for which (8) holds [C and Cvetković, 2006].
- [Luz and C, 1998] If $\tilde{x}$ and $\bar{x}$ are distinct optimal solutions for (7), then the vector $\tilde{x} - \bar{x}$ belongs to the $\lambda_{min}(A_G)$-eigensubspace.

Assuming that G is regular, we may conclude that

- According to [Luz, 1995], $\upsilon(G) = |V(G)| \frac{-\lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}$ (the Hoffman bound).
- $\upsilon(G) = \alpha(G)$ if and only if there exists a stable set S for which (8) holds as equality.

3.Polynomial-time upper bounds(cont.)

• **The Lovász ϑ-number, [Lovász, 1979], is the most popular polynomial-time upper bound on the stability number.**

$$\begin{aligned} \vartheta(G) &= \max tr(JX) \\ & X_{ij} = 0, \forall ij \in E(G) \\ & tr(X) = 1 \\ & X \in \mathcal{S}_n^+, \end{aligned}$$

where $tr(A)$ is the trace of a square matrix A, J is the all ones $n \times n$ square matrix and
$\mathcal{S}_n^+ = \{X \in \mathbb{R}^{n \times n} : X = X^T, z^T X z \geq 0 \ \forall z \in \mathbb{R}^n\}$.

3. Polynomial-time upper bounds (cont.)

- Recently, in [Luz and Schrijver, 2005], the Lovász ϑ-number was redefined as follows:

$$\vartheta(G) = \min_{C} \upsilon(G, C) \qquad (9),$$

where $\upsilon(G,C) = \max\{2\hat{e}^T x - x^T(\frac{C}{\lambda_{min}(C)} + I_n)x : x \geq 0\}$ and C ranges over all weighted adjacency matrices of G which are real symmetric matrices $C = (c_{ij})$ such that $c_{ij} = 0$ if $i = j$ or $ij \notin E(G)$.

- Then, for every graph G, $\vartheta(G) \leq \upsilon(G)$.

3.Polynomial-time upper bounds(cont.)

• **Additionally, according to the sandwich theorem,**

1. $\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$,
   where $\bar{\chi}(G)$ denotes the cardinality of a minimum clique cover of G (that is, a minimum vertex set partition such that each subset is a clique);
2. if G is a perfect graph then $\alpha(G) = \vartheta(G)$ (note that a perfect graph is a graph G such that $\alpha(H) = \bar{\chi}(H)$ for every induced subgraph H of G).

3. Polynomial-time upper bounds (cont.)

- It follows two additional polynomial-time upper bounds on the stability number.

1. [Cvetković, 1971] Let us denote by p_-, p_0, p_+ the number of eigenvalues of A_G smaller than, equal to, and greater than zero, respectively. Then

$$\alpha(G) \leq p_0 + \min\{p_-, p_+\}. \qquad (10)$$

2. [Haemers, 1980] If G is a graph of order n and smallest degree $\delta(G)$, then

$$\alpha(G) \leq \frac{-n\lambda_{min}(A_G)\lambda_{max}(A_G)}{\delta(G)^2 - \lambda_{min}(A_G)\lambda_{max}(A_G)}. \qquad (11)$$

3.Polynomial-time upper bounds(cont.)

- **Computational experiments:**

Graph	$\|V(G)\|$	$\alpha(\bar{G})$	(7)	(9)	(10)	(11)
brock200-1.clq	200	21	40	27	98	69
hamming6-2.clq	64	32	32	32	42	32
hamming6-4.clq	64	4	13	5	28	13
hamming8-2.clq	256	128	128	128	163	128
johnson8-2-4.clq	28	4	4	4	8	4
johnson8-4-4.clq	70	14	14	14	28	14
johnson16-2-4.clq	120	8	8	8	16	8
MANN-a9.clq	45	16	19	17	20	20

3. Polynomial-time upper bounds (cont.)

Graph	$\|V(G)\|$	$\alpha(\bar{G})$	(7)	(9)	(10)	(11)
MANN-a27.clq	378	126	230	132	143	252
san200-0.7-1.clq	200	30	93	30	95	120
san200-0.7-2.clq	200	18	108	18	77	138
san200-0.9-1.clq	200	70	113	70	98	171
san200-0.9-2.clq	200	60	95	60	98	147
san200-0.9-3.clq	200	44	84	44	96	132
sanr200-0.7.clq	200	18	36	23	97	69
sanr200-0.9.clq	200	42	64	49	99	128

4.Graphs whose stability number is easily determined

- The graphs G such that $\alpha(G) = \upsilon(G)$ are called **graphs with convex-QP stability number**, where QP means quadratic program. The class of these graphs is denoted by $\mathcal{Q}$ and its elements called $\mathcal{Q}$-graphs.

- **[Lozin and C, 2001]** The class $\mathcal{Q}$ is not hereditary. However, if $G \in \mathcal{Q}$ and $\exists U \subseteq V(G)$ such that $\alpha(G) = \alpha(G - U)$, then $G - U \in \mathcal{Q}$.

4. Graphs whose stability number is easily determined (cont.)

• **[C, 2001]** **There exists an infinite number of $\mathcal{Q}$-graphs.**

1. **A connected graph with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph is a $\mathcal{Q}$-graph.**
2. **If each component of G has a no zero even number of edges then $L(L(G))$ is a $\mathcal{Q}$-graph.**
3. **There are several famous $\mathcal{Q}$-graphs. For instance, the Petersen graph and the Hoffman-Singleton graph.**

4. Graphs whose stability number is easily determined (cont.)

[C, 2001] Related with the recognition of $\mathcal{Q}$-graphs, we may refer the following results:

A graph G belongs to $\mathcal{Q}$ if and only if each of its components belongs to $\mathcal{Q}$.

Every graph G has an induced subgraph $H \in \mathcal{Q}$.

If $\exists U \subseteq V(G)$ such that $\upsilon(G) = \upsilon(G - U)$ and $\lambda_{min}(A_G) < \lambda_{min}(A_{G-U})$, then $G \in \mathcal{Q}$.

4. Graphs whose stability number is easily determined (cont.)

If $\exists v \in V(G)$ **such that**

$$\upsilon(G) \neq \max\{\upsilon(G-v), \upsilon(G-N_G(v))\},$$

then $G \notin \mathcal{Q}$.

Consider that $\exists v \in V(G) \;\; \upsilon(G-v) \neq \upsilon(G-N_G(v))$.

- **If** $\upsilon(G) = \upsilon(G-v)$ **then**

$$G \in \mathcal{Q} \Leftrightarrow G-v \in \mathcal{Q};$$

- **If** $\upsilon(G) = \upsilon(G-N_G(v))$ **then**

$$G \in \mathcal{Q} \Leftrightarrow G-N_G(v) \in \mathcal{Q}.$$

5. Recognition of Q-graphs

The above results allows the recognition of $\mathcal{Q}$-graphs, except for adverse graphs, which are graphs having an induced subgraph G, without isolated vertices, such that $\upsilon(G)$ and $\lambda_{min}(A_G)$ are both integers, for which the following conditions hold:

- $\forall v \in V(G), \;\; \upsilon(G) = \upsilon(G - N_G(v))$.
- $\forall v \in V(G), \;\; \lambda_{min}(A_G) = \lambda_{min}(A_{G-N_G(v)})$.

5. Recognition of Q-graphs (cont.)

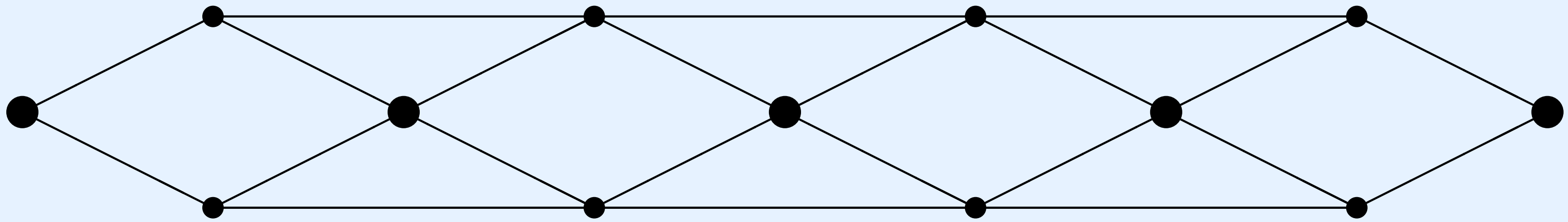
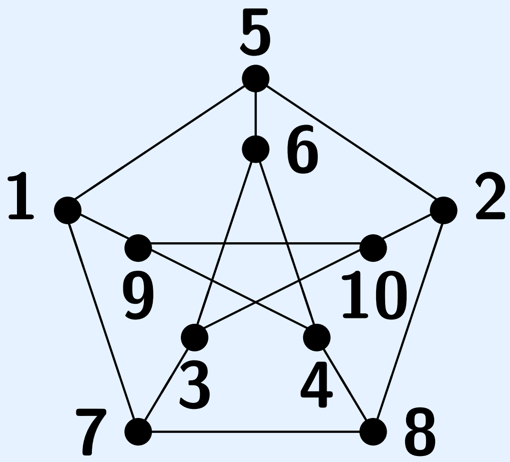


Figure 6: Adverse graph G, where $\lambda_{min}(A_G) = -2$ and $\upsilon(G) = \alpha(G) = 5$.

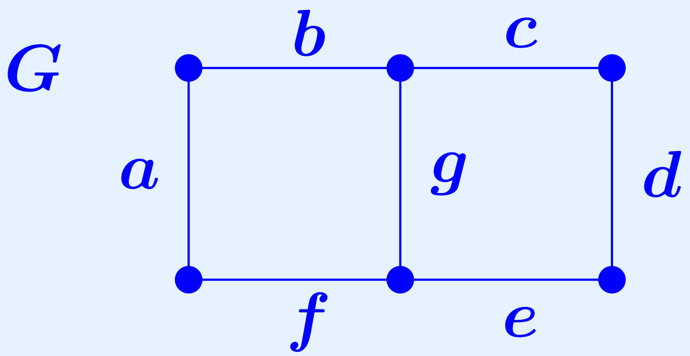
5. Recognition of $\mathcal{Q}$-graphs

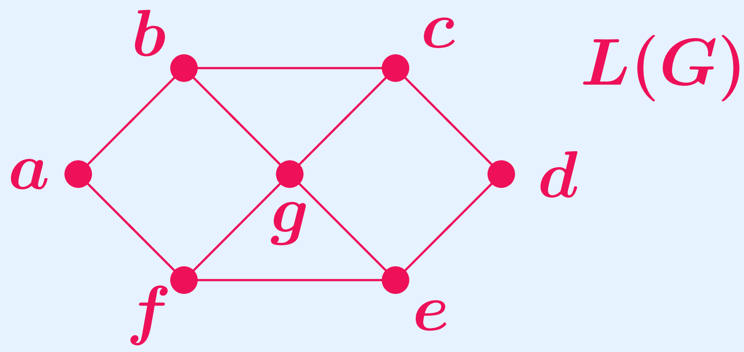
- **A vertex subset $S \subseteq V(G)$ is (k,τ)-regular if induces a k-regular subgraph and $\forall v \notin S \ \ |N_G(v) \cap S| = \tau$.**



Considering the Petersen graph, $S_1 = \{1,2,3,4\}$ is $(0,2)$-regular, $S_2 = \{5,6,7,8,9,10\}$ is $(1,3)$-regular and $S_3 = \{1,2,5,7,8\}$ is $(2,1)$-regular.

5. Recognition of $\mathcal{Q}$-graphs (cont.)





- **Each Hamilton cycle of an Hamiltonian graph G defines a $(2,4)$-regular set in $L(G)$.**
- **In the graph G, depicted above, the edge set $\{a,b,c,d,e,f\} \subset E(G)$ defines a $(2,4)$-regular set in $L(G)$.**

5. Recognition of $\mathcal{Q}$-graphs (cont.)

[Thompson, 1981] A p-regular graph has a (k, τ)-regular set S, with $\tau > 0$, if and only if $k - \tau$ is an adjacency eigenvalue and

$$(p - k + \tau)x(S) - \tau\hat{e}$$

belongs to the corresponding eigenspace.

[C and Rama, 2004] A graph G has a (k, τ)-regular set S if and only if the characteristic vector x of S is a solution for the linear system

$$(A_G - (k - \tau)I)x = \tau\hat{e}.$$

5. Recognition of $\mathcal{Q}$-graphs (cont.)

[C, 2003] If G is an adverse graph then $G \in \mathcal{Q}$ if and only if $\exists S \subseteq V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

- **If G is a regular graph then $G \in \mathcal{Q}$ if and only if $\exists S \subseteq V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.**

- **Therefore, for regular graphs G, the Hoffman bound is attained if and only if G includes a $(0, \tau)$-regular set, with $\tau = -\lambda_{min}(A_G)$.**

5. Recognition of $\mathcal{Q}$-graphs (cont.)

There are several families of graphs for which we may recognize (in polynomial-time) $\mathcal{Q}$-graphs. For instance,

- Bipartite graphs.
- Dismantable graphs, that is, graphes with the following recursive definition:
  - One-vertex graph is dismantable and a graph G with at least two vertices is dismantable if $\exists x, y \in V(G)$ such that $N_G[x] \subseteq N_G[y]$ and $G - \{x\}$ is dismantable.

5. Recognition of $\mathcal{Q}$-graphs (cont.)

[C, 2003] Given a graph G and $\tau > 1$, if $\exists p, q \in V(G)$ such that $N_G[q] \subseteq N_G[p]$ then $\upsilon_G(\tau) > \upsilon_{G-N_G(p)}(\tau)$.

- **Graphs with low Dilworth number (note that given two vertices $x, y \in V(G)$, if $N_G(y) \subseteq N_G[x]$ then we say that the vertices x and y are comparable, and then the Dilworth number of a graph is the largest number of pairwise incomparable vertices of).**

[C, 2003] Let G be a not complete graph. If $\text{dilw}(G) < \omega(G)$ then G is not adverse.

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