A contribution to duality theory, applied to the measurement of risk aversion

J.E. Martínez-Legaz
Universitat Autònoma de Barcelona

John K.H. Quah
University of Oxford

Iberian Conference in Optimization
Coimbra, 16-18 November, 2006

R^n_+ the commodity space

$u : R^n_+ \to R$ Bernoulli utility function:
$u\left(t \bullet x' \oplus (1-t) \bullet x''\right) = tu\left(x'\right) + (1-t)u\left(x''\right)$

An agent is *risk averse in consumption space* if she prefers the sure bundle $tx' + (1-t)x''$ to the lottery $t \bullet x' \oplus (1-t) \bullet x''$

Risk aversion in commodity space: u concave

u quasiconcave, u.s.c., has no local maximum

$v : R^n_{++} \times R_+ \to R$
$v\left(p, y\right) = \max\left\{u\left(x\right) \mid p \cdot x \leq y\right\}$

$v\left(p, t \bullet y' \oplus (1-t) \bullet y''\right) = tv(p, y') + (1-t)v(p, y'')$

Risk aversion in income: $v\left(p, \cdot\right)$ concave

u concave $\Leftrightarrow$ $v\left(p, \cdot\right)$ concave $\forall\, p \in R^n_{++}$

<u>CHARACTERIZING RISK AVERSION OVER INCOME</u>

I open interval of the real line R

f, $g : I \longrightarrow R$ $\qquad g$ increasing

f is *more concave than* $g \quad \Leftrightarrow \quad f \circ g^{-1}$ is concave.

If f and g are C^2 with positive first derivatives and $\mathcal{A}_f(y) \equiv -\dfrac{f''(y)}{f'(y)}$,
f is *more concave than* $g \Leftrightarrow \mathcal{A}_f(y) \geq \mathcal{A}_g(y) \quad \forall\, y \in I$

$\bar{g} : I \longrightarrow R$ is a *support function of* f *at* y^* $\qquad \Leftrightarrow$
$\bar{g}(y^*) = f(y^*)$ and $\bar{g}(y) \geq f(y) \quad \forall\, y \in I$

$g : I \longrightarrow R$ is a *capping function of* f $\qquad \Leftrightarrow$
$\forall y^* \in I$, $\exists\, r, r' \in R$ such that $rg + r'$ is a support function of f at y^*.

THEOREM. *Let f and g be two real-valued and continuous functions defined on an open interval I, with g increasing. Then the following are equivalent:*
i. f is more concave than g;
ii. g is a capping function of f;
iii. the function f has the representation

$$f(y) = \min_{r \in U} \{\phi(r) + rg(y)\},$$

where $U \subset R$ and $\phi : U \to R$.

$$f(y) \leq f(y^*) + \frac{f'(y^*)}{g'(y^*)}(g(y) - g(y^*)) \quad \forall\, y^*, y \in R$$

$\sigma > 0$, $y \geq 0, t \in [0,1]$
$z', z'' \in (0, e^{\sigma y})$ such that $tz' + (1-t)z'' = 1$

$L_A(\sigma, y, t, z')$ the lottery

$$t \bullet \left(y - \frac{1}{\sigma} \ln z'\right) \oplus (1-t) \bullet \left(y - \frac{1}{\sigma} \ln z''\right)$$

Mean income $\qquad y - \dfrac{t \ln z' + (1-t) \ln z''}{\sigma}$

$v : R_{++} \to R$ nondecreasing Bernoulli utility function

$$v\left(L_A(\sigma, y, t, z')\right) = tv(y - \frac{1}{\sigma}\ln z') + (1-t)v(y - \frac{1}{\sigma}\ln z'')$$

v is said to be of *type* A_σ if

$$v(y) \geq tv(y - \frac{1}{\sigma}\ln z') + (1-t)v(y - \frac{1}{\sigma}\ln z'')$$

LEMMA. *Suppose $\sigma > \bar{\sigma}$. Then for every lottery $L_A(\bar{\sigma}, y, t, \bar{z}')$ with $\bar{z}' \neq 1$, there is a lottery $L_A(\sigma, y, t, z')$ such that*

$$\begin{aligned} y - \frac{1}{\sigma}\ln z' &> y - \frac{1}{\sigma}\ln \bar{z}', \\ y - \frac{1}{\sigma}\ln z'' &> y - \frac{1}{\sigma}\ln \bar{z}''. \end{aligned}$$

PROPOSITION. *v is of type A_σ if and only if it is of type $A_{\bar{\sigma}}$ for all $\bar{\sigma} \leq \sigma$.*

PROPOSITION. *Suppose that* v *is* C^2 *with* $v' > 0$. *Then*

$$\mathcal{A}_v \geq \sigma \text{ for all } y > 0 \qquad \Longleftrightarrow \qquad v \text{ is of type } A_\sigma.$$

PROPOSITION. *Suppose that* v *is* C^2 *with* $v' > 0$. *Then* $\mathcal{A}_v(y^*) = \sigma$ *if and only if the following holds:*
(a) for each $\tilde{\sigma} > \sigma$, *there is a neighborhood of 1 such that whenever* z' *and* z'' *are in that neighborhood,* $v(y^*) \geq v(L_A(\tilde{\sigma}, t, y^*, z'))$.
(b) for each $\tilde{\sigma} < \sigma$, *there is a neighborhood of 1 such that whenever* z' *and* z'' *are in that neighborhood,* $v(L_A(\tilde{\sigma}, t, y^*, z')) \geq v(y^*)$.

PROPOSITION. *For a nondecreasing utility function* v, *the following are equivalent:*
i. v *is of type* A_σ,
ii. the function g_σ *given by* $g_\sigma(y) = -e^{-\sigma y}$ *is a capping function of* v,
iii. v *has the representation* $v(y) = \min_{r \in U} \left\{ \phi(r) - re^{-\sigma y} \right\}$, *where* $U \subset R$ *and* $\phi : U \to R$.

$\theta \geq 0$, $\theta \neq 1$, $y \geq 0, t \in [0, 1]$
$z', z'' > 0$ such that $tz' + (1-t)z'' = 1$

$L_R(\theta, y, t, z')$ the lottery

$$t \bullet z'^{1/(1-\theta)} y \oplus (1-t) \bullet z''^{1/(1-\theta)} y,$$

$L_R(1, y, t, z')$ the lottery

$$t \bullet e^{z'} y \oplus (1-t) \bullet e^{z''} y,$$

with $z', z'' > 0$ such that $tz' + (1-t)z'' = 0$

v is said to be of *type* R_θ if

$$v(y) \geq v\left(L_R(\theta, y, t, z')\right)$$

Coefficient of *relative* risk aversion at y:

$$\mathcal{R}_v(y) = -\frac{y v''(y)}{v'(y)}.$$

PROPOSITION. *Suppose that* v *is* C^2 *with* $v' > 0$. *Then*

$\mathcal{R}_v(y) \geq \theta$ *for all* $y > 0$ *if and only if* v *is of type* R_θ.

PROPOSITION. *Suppose that* v *is* C^2 *with* $v' > 0$. *Then* $\mathcal{R}_v(y^*) = \theta$ *if and only if, for an agent with utility* v*, the following holds:*
(a) for each $\tilde{\theta} > \theta$*, there is a neighborhood of 1 such whenever* z' *and* z'' *are in that neighborhood,* $v\left(L_R(\widetilde{\theta}, y^*, t, z')\right) \geq v\,(y^*)$.
(b) for each $\tilde{\theta} < \theta$*, there is a neighborhood of 1 such whenever* z' *and* z'' *are in that neighborhood,* $v\,(y^*) \geq v\left(L_R(\widetilde{\theta}, y^*, t, z')\right)$.

PROPOSITION. *A nondecreasing utility function* v *is of type* R_θ *if and only if it is of type* $R_{\bar{\theta}}$ *for all* $\bar{\theta} \leq \theta$.

PROPOSITION. *For a nondecreasing function* v*,* v *is of type* R_θ $\iff$ v *has the representation* $v(y) = \min_{r \in U} \{\phi(r) + r\hat{g}_\theta(y)\}$*, where* $U \subset R$ *and* $\phi : U \to R$

RELATING RISK AVERSION OVER INCOME AND RISK AVERSION OVER COMMODITIES

$p \in R^n_{++}$, $y > 0$

The *budget set* at (p, y) :

$$B(p, y) = \{x \in R^n_{++} : p \cdot x \leq y\}$$

The *demand at* (p, y) : $\bar{x}(p, y) = \text{argmax}_{x \in B(p,y)} u(x)$

u is *well behaved* if:

(a) $\bar{x}(p, y) \neq \emptyset \quad \forall\, (p, y) \in R^n_{++} \times R_{++}$ and $p \cdot x' = y$ for x' in $\bar{x}(p, y)$

(b) $\forall\, x \in R^n_{++}$, $\exists\, p$ such that $x \in \bar{x}(p, 1)$.

u is *very well behaved* if, in addition to (a) and (b), the demand set $\bar{x}(p, y)$ is a singleton at all (p, y) and the function $\bar{x}$ is continuous.

u is *regular* if it is increasing, continuous, quasiconcave, and $\{x \in R^n_{++} : u(x) \geq \bar{u}\}$ is a closed set in R^n for any $\bar{u}$.

u is *very regular* if it is regular and strictly quasiconcave

For $\omega \in R^n_+ \setminus \{0\}$, the *normalized price set*:

$$Q^\omega = \{p \in R^n_{++} : p \cdot \omega = 1\}$$

$\omega \in R^n_+ \setminus \{0\}$, $\sigma > 0$.

$u : R^n_{++} \to R$ is of type A^ω_σ if
$u\left(tx' + (1-t)x''\right) \geq$
$u\left(t \bullet \left(\frac{1}{\alpha'}\, x' - \frac{\ln \alpha'}{\sigma}\, \omega\right) \oplus (1-t) \bullet \left(\frac{1}{\alpha''}\, x'' - \frac{\ln \alpha''}{\sigma}\, \omega\right)\right)$
$\forall\, t \in [0,1]$, $\forall\, \alpha', \alpha'' > 0$ such that $t\alpha' + (1-t)\alpha'' = 1$,
$\forall\, x', x'' \in R^n$ such that

$$\frac{1}{\alpha'}\, x' - \frac{\ln \alpha'}{\sigma}\, \omega, \frac{1}{\alpha''}\, x'' - \frac{\ln \alpha''}{\sigma}\, \omega \in R^n_{++}$$

THEOREM. *Suppose $u : R^n_{++} \to R$ is very well behaved and generates the indirect utility function $v : R^n_{++} \times R_{++} \to R$. Then the following are equivalent:*

a. $v(p, \cdot)$ is of type A_σ for all p in the normalized price set Q^ω;

b. u has the representation

$$u(x) = \min_{(q,r) \in \bar{U}} \left\{ \phi(q, r) - re^{-\sigma(q \cdot x)} \right\},$$

where $\overline{U} \subset Q^\omega \times R$ and $\phi : \overline{U} \to R$;

c. u is of type A^ω_σ.

Suppose that $u : R^n_{++} \to R$ is well behaved

$\theta \geq 0$, $\theta \neq 1$

u is of type R_θ if
$u\left(tx' + (1-t)x''\right) \geq$
$u\left(t \bullet \left(\alpha'^{\theta/(1-\theta)} x'\right) \oplus (1-t) \bullet \left(\alpha''^{\theta/(1-\theta)} x''\right)\right)$
$\forall\, t \in [0,1]\ \forall\, \alpha',\alpha'' > 0$ such that $t\alpha' + (1-t)\alpha'' = 1$,
$\forall\, x',x'' \in R^n_{++}$

u is of type R_1 if

$$u\left(tx' + (1-t)x''\right) \geq u\left(t \bullet \left(e^{\alpha'} x'\right) \oplus (1-t) \bullet \left(e^{\alpha''} x''\right)\right)$$

$\forall\, t \in [0,1],\ \forall\, \alpha',\alpha'' > 0$ such that $t\alpha' + (1-t)\alpha'' = 0$,
$\forall\, x',x'' \in R^n_{++}$

THEOREM. *Suppose $u : R^n_{++} \to R$ is well behaved and generates the indirect utility function $v : R^n_{++} \times R_{++} \to R$. Then*

$v(p,\cdot)$ *is of type R_θ for all $p \in R^n_{++} \Longleftrightarrow u$ is of type R_θ*

θ-CONCAVE FUNCTIONS OF ONE REAL VARIABLE

Let $\theta \in R \setminus [0,1)$.

A nondecreasing function $F : R_+ \longrightarrow R$ is θ-*concave* if $R_{++} \ni y \longmapsto F(y^\theta)$ is concave.

PROPOSITION. If $F : R_+ \longrightarrow R$ is θ-concave then it is α-concave for all $\alpha \in R \setminus [0,1)$ such that $\dfrac{1}{\alpha} \geq \dfrac{1}{\theta}$ (that is, for $1 \leq \alpha \leq \theta$ if $\theta \geq 1$ and for all $\alpha \leq \theta$ and all $\alpha \geq 1$ if $\theta < 0$).

In particular, every θ-concave function is concave.

PROPOSITION. Suppose $F : R_{+} \rightarrow R$ is a non-decreasing function and let $\theta \in R \setminus [0,1)$. Then the following statements are equivalent:

(i) The function F is θ-concave.

(ii) There exists a set $U \subseteq R_{++}$ and a map $g : U \rightarrow R$ such that, for any $x \in R_{++}$,

$$F(x) = \min_{r \in U} \left\{ g(r) + s(\theta)(rx)^{\frac{1}{\theta}} \right\}, \text{ where } s(\theta) = \frac{\theta}{|\theta|}.$$

(iii) For any $t \in [0,1]$ and $x', x'' \in R_{++}$, we have

$$F\left(tx' + (1-t)x''\right) \geq tF\left(\frac{x'^{\ \theta}}{(tx' + (1-t)x'')^{\theta-1}}\right) + (1-t)F\left(\frac{x''^{\ \theta}}{(tx' + (1-t)x'')^{\theta-1}}\right).$$

PROPOSITION. Suppose $F : R_+ \to R$ is a nondecreasing function and let $\theta \in R \setminus [0,1)$. If F is θ-concave and differentiable at $x \in R_{++}$ then

$$F(y) \leq F(x) + \theta F'(x)\left(\left(x^{\theta-1}y\right)^{\frac{1}{\theta}} - x\right) \qquad (1)$$

for all $y \in R_{++}$.

Conversely, if F is differentiable on R_{++} and satisfies (1) for all $x, y \in R_{++}$ then it is θ-concave.

PROPOSITION. Suppose $F : R_+ \to R$ is increasing, C^2 on R_{++} and satisfies $F'(y) > 0$ for all $y \in R_{++}$ and let $\theta \in R \setminus [0,1)$. Then F is θ-concave if and only if the function $K_F : R_{++} \to R$ given by

$$K_F(y) = -\frac{yF''(y)}{F'(y)}$$

satisfies $K_F(y) \geq 1 - \dfrac{1}{\theta}$ for all $y \in R_{++}$.

θ-CONCAVE UTILITY FUNCTIONS

A function $u : R^l_+ \to R$ is called a *utility function* if it has the following properties:

(i) u is nondecreasing along rays, i.e., $u(\lambda x) \geq u(x)$ for any scalar $\lambda \geq 1$ and $x \in R^l_+$;

(ii) u is locally non-satiated, i.e., for any x, there is x' arbitrarily close to x such that $u(x') > u(x)$;

(iii) for any (p, y) in $R^l_{++} \times R_+$, there is $\bar{x} \in R^l_+$ that maximizes $u(x)$ in $B(p, y) = \{x \in R^l_+ : p \cdot x \leq y\}$.

$$f(p, y) = \left\{\bar{x} \in R^l_+ \mid \bar{x} \text{maximizes } u(x) \text{ in } B(p, y)\right\}$$

$u : R^l_+ \to R$ is *θ-concave at* $p \in R^l_{++}$ if

$$u(x) \geq tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x''),$$

whenever $x \in f(p, 1)$, $0 \leq t \leq 1$, $x', x'' \in R^l_+ \setminus \{0\}$, and $p \cdot (tx' + (1-t)x'') = 1$.

PROPOSITION. Suppose $F : R_+ \to R$ is an increasing function and let $\theta \in R \setminus [0, 1)$. Then F is θ-concave if and only if it is θ-concave at p for all $p \in R_{++}$.

PROPOSITION. If a utility function $u : R^l_+ \to R$ is θ-concave at p then it is α-concave at p for all $\alpha \in R \setminus [0, 1)$ such that $\frac{1}{\alpha} \geq \frac{1}{\theta}$ (that is, for $1 \leq \alpha \leq \theta$ if $\theta \geq 1$ and for all $\alpha \leq \theta$ and all $\alpha \geq 1$ if $\theta < 0$).

In particular, every θ-concave function is concave.

$v(p, \cdot)$ is θ-concave $\Longleftrightarrow u$ is θ-concave at $\lambda p \; \forall \, \lambda > 0$

$u : R^l_+ \to R$ has the *supporting price property* if at every $x \in R^l_+ \setminus \{0\}$, there is $p \in R^l_{++}$ such that $x \in f(p, 1)$.

THEOREM. Suppose $u : R^l_+ \to R$ is a utility function with the supporting price property and let $\theta \in R \setminus [0,1)$. Then the following statements are equivalent:

(i) The function u is θ-concave at all prices.

(ii) There exist a set $U \subseteq R^l_{++}$ and a map $g : U \to R$ such that, for any $x \in R^l_+ \setminus \{0\}$,

$$u(x) = \min_{r \in U} \{g(r) + s(\theta)(r \cdot x)^{\frac{1}{\theta}}\}, \text{ where } s(\theta) = \frac{\theta}{|\theta|}.$$

(iii) For any $p \in R^l_{++}$, $t \in [0,1]$ and $x', x'' \in R^l_+ \setminus \{0\}$ satisfying $p \cdot (tx' + (1-t)x'') = 1$, we have

$$\begin{aligned} u(tx' + (1-t)x'') \geq & \; tu((p \cdot x')^{\theta-1}x') \\ & +(1-t)u((p \cdot x'')^{\theta-1}x''). \end{aligned}$$

(iv) For any $p \in R^l_{++}$, $t \in [0,1]$ and $x', x'' \in R^l_+ \setminus \{0\}$, we have

$$u(tx' + (1-t)x'') \geq tu((\frac{p \cdot x'}{tp \cdot x' + (1-t)\, p \cdot x''})^{\theta-1} x') \\ +(1-t)u((\frac{p \cdot x''}{tp \cdot x' + (1-t)\, p \cdot x''})^{\theta-1} x'').$$

(v) For any $p \in R^l_{++}$, $t \in [0,1]$, $x', x'' \in R^l_+ \setminus \{0\}$ and $\alpha, \beta \in R_{++}$ satisfying $t\alpha + (1-t)\,\beta = 1$ and $\alpha x'' - \beta x' \notin \left(R^l_+ \cup \left(-R^l_+\right)\right) \setminus \{0\}$, we have

$$u(tx' + (1-t)x'') \geq tu(\alpha^{\theta-1} x') + (1-t)u(\beta^{\theta-1} x'').$$

PROPOSITION. If $u : R^l_+ \to R$ is a θ-concave utility function, with $\theta \in R \backslash [0, 1)$, satisfying the supporting price property and being differentiable at $x \in R^l_{++}$ then

$$u(y) \leq u(x) + \theta \left(\left((\nabla u(x) \cdot x)^{\theta - 1} \nabla u(x) \cdot y \right)^{\frac{1}{\theta}} - \nabla u(x) \cdot x \right) \quad (2)$$

for all $y \in R^l_+ \backslash \{0\}$.

Conversely, if $u : R^l_{++} \to R$ is differentiable and satisfies $\nabla u(x) \in R^l_{++}$ and (2) for all $x, y \in R^l_{++}$ then it admits an extension as a θ-concave utility function on R^l_+.

PROPOSITION. If $u : R^l_+ \to R$ is a θ-concave utility function, with $\theta \in R\setminus[0,1)$, having the supporting price property and being C^2 on R^l_{++} then the function $K_u : R^l_{++} \to R$ given by

$$K_u(x) = \begin{cases} -\dfrac{\nabla u(x) \cdot x}{\nabla u(x) \cdot \left(\nabla^2 u(x)\right)^{-1} \nabla u(x)} & \text{if } \nabla^2 u(x) \text{ is nonsingular} \\ 0 & \text{otherwise} \end{cases}$$

satisfies $K_u(x) \geq 1 - \dfrac{1}{\theta}$ for all $x \in R^l_{++}$.

Conversely, if $u : R^l_{++} \to R$ is a C^2 concave function satisfying $\nabla u(x) \in R^l_{++}$ and $K_u(x) \geq 1 - \dfrac{1}{\theta} \geq 0$, with $\theta \in R\setminus[0,1)$, for all $x \in R^l_{++}$ then it admits an extension as a θ-concave utility function on R^l_+.