OPTIMALITY CONDITIONS IN VECTOR OPTIMIZATION PROBLEMS

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1 Introduction, Notations and Preliminaries

Introduction

In vector optimization one investigate optimal elements of a nonempty set E of a partially ordered linear space Y.

Problems of this type can be found not only in mathematics but also in other fields such as engineering or economics.

In mathematics, vector optimization problems arise for example in

- Functional Analysis: the Hahn-Banach theorem, the lemma of Bishop-Phelps, Ekeland variational principle...
- Statistics: Bayes solutions, theory of test, minimal covariance matrices...
- Game Theory: cooperative n player differential games.
- Operational Research: multiobjective programming, multi-criteria decision making,...
- Approximation Theory: location theory, simultaneous approximation, solution of boundary value problems,...

In the last decade vector optimization has been extended to problems with set-valued maps (see [11]). This new field of research called set optimization seems to have important applications, and presents a new difficulty: in this case we are working without a linear structure.

The roots of vector optimization go back to the works by:

F.Y. Edgeworth (1845-1926). Mathematical Physics (1881).

V. Pareto (1848-1923). Manuale di Economia Politica (1896).

who already gave the definition of the standard optimality concepts in multiobjective optimization.

But in mathematics this branch of optimization has started with the legendary paper of Kuhn and Tucker (1951).

To introduce the minimality concepts we consider a partial ordering on Y as follows. Let Y be a real linear space.

Definition 1.1. A binary relation $\leq$ on Y is called a partial ordering on Y, if the following properties are satisfied (for arbitrary $x, y, z, u \in Y$ and $\alpha \in \mathbb{R}_+$):

(i) $x \leq x$

(ii) $x \leq y, y \leq z \Rightarrow x \leq z$

(iii) $x \leq y, u \leq z \Rightarrow x + u \leq y + z$

(iv) $x \leq y \Rightarrow \alpha x \leq \alpha y$.

A partial ordering is called antisymmetric if the following condition hold:
$x \leq y,\ y \leq x \ \Rightarrow \ x = y$.

Definition 1.2. A real linear space equipped with a partial ordering is called a partially ordered linear space.

Here we have a characterization of a partial ordering in a real linear space.

Theorem 1.3. *(i) If* $\leq$ *is a partial ordering on* Y*, then the set* $D = \{x \in Y : 0 \leq x\}$ *is a convex cone. If, in addition,* $\leq$ *is antisymmetric, then* D *is pointed.*
(ii) If D *is a convex cone in* Y*, then the binary relation*

$$x \leq_D y \Leftrightarrow y - x \in D$$

is a partial ordering on Y*. If, in addition,* D *is pointed, then* $\leq$ *is antisymmetric.*

Remark 1.4. Recall that:

(i) A nonempty set $D \subset Y$ is a cone if $x \in D, \alpha \geq 0 \Rightarrow \alpha x \in D$.

(ii) A cone D is pointed if $D \cap (-D) = \{0\}$.

(iii) A cone D is solid if $\mathrm{cor}\, D \neq \emptyset$, where the algebraic interior or core of E is the set

$$\mathrm{cor}\, E = \{\bar{x} \in E : \forall x \in X, \exists \bar{\lambda}, \text{ such that } \bar{x} + \lambda x \in E, \forall \lambda \in [0, \bar{\lambda}]\}.$$

If X is a topological linear space and C is a convex set, then $\mathrm{cor}(C) = \mathrm{int}(C)$, in particular if D is a convex cone, then $\mathrm{cor}(D) = \mathrm{int}(D)$.

Example 1.5. The most usual ordering cone in a finite dimensional space $\mathbb{R}^n$ is given by the positive orthant $\mathbb{R}^n_+$. This set is a pointed, closed and solid convex cone that define the componentwise partial ordering on $\mathbb{R}^n$, also called Pareto order.

One of the most important application of the vectorial optimization techniques is found in the study of vectorial mathematical programming problems, and as a particular case in multiobjective programming problems.

Multiobjective Program:

$$\text{Min } f(x) \text{ subject to } x \in M, \tag{1}$$

$$M = S \cap Q, \quad S = \{x \in \mathbb{R}^n : \ g(x) \leq_K 0, \ h(x) = 0\}, \quad Q \subset \mathbb{R}^n,$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^p, \quad g : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad h : \mathbb{R}^n \longrightarrow \mathbb{R}^r.$$

If $D = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$ we have a multiobjective Pareto program.

If $p = 1$ and $D = \mathbb{R}_+$ we have a scalar program.

We consider an extension of problem (1).

Let X, Y, Z and W be partially ordered linear spaces, let $D \subset Y$ be the ordering cone on Y, let $K \subset Z$ be the ordering cone on Z and let $M \subset X$ be a nonempty set.

Vectorial Program:

- General

$$\text{Min}\, f(x) \text{ subject to } x \in M. \tag{2}$$

- Constrained

$$\text{Min}\, f(x) \text{ subject to } x \in M, \tag{3}$$

$$M = S \cap Q, \quad S = \{x \in X : \ g(x) \in -K, \ h(x) = 0\}, \quad Q \subset X,$$

$$f : X \to Y, \quad g : X \to Z, \quad h : X \to W.$$

Now, we give the solution concepts for a vector optimization problem. We consider only efficient and weak-efficient solutions, but in vector optimization there are other solution concepts as ideal, strong, strict or proper efficient points.

Solution Concepts

Let (Y, D) be a linear ordered space, where D is a pointed convex cone, and let $E \subset Y$ be a nonempty set.

Definition 1.6. A point $y_0 \in E$ is said to be an efficient or minimal element of E if there is no $y \in E$, $y \neq y_0$, such that $y \leq y_0$. The set of efficient elements of E is denoted by $\mathrm{Min}(E, D)$.

This definition is equivalent to:

(i) there is no $y \in E$, such that $y_0 \in y + D \setminus \{0\}$.

(ii) $(E - y_0) \cap (-D) = \{0\}$.

If the condition is satisfied in a neighborhood of y_0, we have the concept of local efficient point.

Definition 1.7. A point $y_0 \in E$ is said to be a local efficient or local minimal element of E, if there exists a neighborhood V of y_0, such that $((E \cap V) - y_0) \cap (-D) = \{0\}$. We write $y_0 \in \mathrm{LMin}(E, D)$.

If we replace D for its interior, we have the concept of weak efficient point.

Definition 1.8. (i) A point $y_0 \in E$ is said to be a weak efficient or weak minimal element of E if $(E - y_0) \cap (-\operatorname{cor} D) = \emptyset$. We write $y_0 \in \mathrm{WMin}(E, D)$.
(ii) A point $y_0 \in E$ is said to be a local weak efficient or a local weak minimal element of E if there exists a neighborhood V of y_0, such that $(E \cap V - y_0) \cap (-\operatorname{cor} D) = \emptyset$. We write $y_0 \in \mathrm{LWMin}(E, D)$.

Note that for the definition of weak solution it is necessary to consider that D is a solid cone, that is $\operatorname{cor} D \neq \emptyset$, otherwise all element of E is a weak solution.

For a vectorial program we obtain the different concepts of solution making $E = f(M)$.

For example, a feasible point $x_0 \in M$ is an efficient solution, denoted by $x_0 \in \mathrm{Min}(f, M)$, if $f(x_0) \in \mathrm{Min}(f(M), D)$.

It is clear that $\mathrm{Min}(E, D) \subset \mathrm{WMin}(E, D)$, so it is usual to give the necessary conditions for the weak efficient points and the sufficient conditions for the efficient points.

From now on we consider normed spaces, and we will use the following tangent sets.

Tangent Sets

Definition 1.9. Let $M \subset X$ and $x_0, v \in X$.

(a) The tangent (Bouligand or contingent) cone to M at x_0 is

$$T(M, x_0) = \{u \in X : \exists t_n \to 0^+, \exists u_n \to u \text{ such that } x_0 + t_n u_n \in M \ \forall n \in \mathbb{N}\}.$$

(b) The interior tangent (Ursescu) cone to M at x_0 is

$$TI(M, x_0) = \{u \in X : \exists \delta > 0 \text{ such that } x_0 + tu' \in M \ \forall t \in (0, \delta] \ \forall u' \in B(u, \delta)\}.$$

(c) The second order tangent set to M at (x_0, v) is

$$T^2(M, x_0, v) = \left\{w \in X : \exists t_n \to 0^+, \exists w_n \to w \text{ such that } x_n := x_0 + t_n v + \tfrac{1}{2} t_n^2 w_n \in M \ \forall n \in \mathbb{N}\right\}.$$

(d) The asymptotic second order tangent (Penot) cone to M at (x_0, v) is

$$T''(M, x_0, v) = \big\{w \in X : \exists (t_n, r_n) \to (0^+, 0^+), \exists w_n \to w \text{ such that } t_n/r_n \to 0, x_n := x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in M \, \forall n \in \mathbb{N}\big\}.$$

The tangent cone T, the interior tangent cone TI and the second order tangent set T^2, that is not a cone, are well-known.

The asymptotic second order tangent cone T'' has been introduced by Penot and used by Penot [8] and Cambini, Martein and Vlach [1] in order to state optimality conditions in scalar optimization.

Here we well use this second order tangent cone in vector optimization problems and so we extend several results by Penot and Cambini et al..

Next propositions collects some properties of these first and second order tangent cones and sets.

Proposition 1.10. *Let* $C \subset X$ *be a convex set and* $x_0 \in \operatorname{cl} C$*, then we have*

i) $T(C, x_0) = \operatorname{cl}\operatorname{cone}(C - x_0)$.

If moreover $\operatorname{int} C \neq \emptyset$*, then*

ii) $TI(\operatorname{int} C, x_0) = TI(C, x_0) = \operatorname{int}\operatorname{cone}(C - x_0)$.

iii) $\operatorname{cl} TI(C, x_0) = T(C, x_0)$.

If moreover C *is a cone, then*

iv) $TI(C, 0) = TI(\operatorname{int} C, 0) = \operatorname{int} C$.

Proposition 1.11. *Let* M *be a subset of* X*, and let* $x_0 \in \operatorname{cl} M$*,* $v \in X$*.*

(i) $T^2(M, x_0, v)$ *and* $T''(M, x_0, v)$ *are closed sets contained in* $\operatorname{cl}\operatorname{cone}[\operatorname{cone}(M - x_0) - v]$ *and* $T''(M, x_0, v)$ *is a cone.*

(ii) If $v \notin T(M, x_0)$*, then* $T^2(M, x_0, v) = T''(M, x_0, v) = \emptyset$.

(iii) $T^2(M, x_0, 0) = T''(M, x_0, 0) = T(M, x_0)$.

Proposition 1.12. *Let* $C \subset X$ *be a convex set,* $x_0 \in C$, $v \in T(C, x_0)$. *Then*

(i) $T^2(C, x_0, v) + T(T(C, x_0), v) \subset T^2(C, x_0, v)$.

(ii) $T(T(C, x_0), v) = \mathrm{cl}\,\mathrm{cone}[\mathrm{cone}(C - x_0) - v]$.

(iii) If $T''(C, x_0, v) \neq \emptyset$ *(in particular, when* X *is finite dimensional), then* $T''(C, x_0, v) = \mathrm{cl}\,\mathrm{cone}[\mathrm{cone}(C - x_0) - v]$ *and* $T^2(C, x_0, v) \subset T''(C, x_0, v)$.

Directional Derivatives

Definition 1.13. Let $f : X \to Y$ and $x_0, v \in X$.

(a) The Hadamard derivative of f at x_0 in the direction v is

$$df(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(b) The Dini derivative of f at x_0 in the direction v is

$$Df(x_0, v) = \lim_{t\to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

(c) f is Hadamard (resp. Dini) derivable at x_0 if there exists $df(x_0, v)$ (resp. $Df(x_0, v)$) for all $v \in X$.

It is well-known that:

- If f is Fréchet differentiable at x_0 (we denote the Fréchet differential by $\nabla f(x_0)$) then $\nabla f(x_0)v = df(x_0, v)$.
- If there exists $df(x_0, v)$, then there exists $Df(x_0, v)$ and both derivatives are the same.
- In particular, if f is Lipschitz in a neighborhood of x_0 and there exists Dini derivative, then there exists Hadamard derivative.
- If f is Hadamard derivable at x_0, then f is continuous at x_0 and $df(x_0, \cdot)$ is continuous on X (see Demjanov and Rubinov [9]). This property is not true for a Dini type derivative.

Next we recall the notion of Dini subdifferential.

Definition 1.14. Let $f : X \to \mathbb{R}$ be Dini derivable at x_0. The Dini subdifferential of f at x_0 is

$$\partial_D f(x_0) = \{\xi \in X^* : \langle \xi, v \rangle \leq Df(x_0, v) \ \forall v \in X\}.$$

If $Df(x_0, \cdot)$ is a convex function, then there exists the Dini subdifferential.

If $Df(x_0, \cdot)$ is not a convex function, then $\partial_D f(x_0)$ can be the empty set.

2 Optimality Conditions in Vector Optimization

In this section we obtain very general first and second order optimality conditions for a point to be a local efficient element of a nonempty set $E \subset Y$, using the first and second order tangent sets.

Theorem 2.1. *If* $y_0 \in E \subset Y$ *is a local weak minimum of* E *(with respect to* D*), then the following conditions are satisfied:*

(i) $T(E, y_0) \cap TI(-D, 0) = \emptyset$,

(ii) $T^2(E, y_0, u) \cap TI(-\operatorname{int} D, u) = \emptyset$ *for all* $u \in T(E, y_0) \cap \operatorname{bd}(-D)$,

(iii) $T''(E, y_0, u) \cap TI(-\operatorname{int} D, u) = \emptyset$, *for all* $u \in T(E, y_0) \cap \operatorname{bd}(-D)$.

Proof. (i) From proposition 1.10 (iv) we have that $TI(-D,0) = TI(-\operatorname{int} D, 0)$.

Let us suppose that there exists $v \in T(E, y_0) \cap TI(-D, 0)$.

As $v \in T(E, y_0)$, there exist sequences $(y_n) \subset E$, $(y_n) \to y_0$ and $(t_n) \to 0^+$, such that $v_n = \frac{y_n - y_0}{t_n} \to v$, then

$$y_n = y_0 + t_n v_n \in E, \forall n \in \mathbb{N}. \tag{4}$$

On the other hand, since $v \in TI(-\operatorname{int} D, 0)$, there exists $\delta > 0$, such that $0 + tv' \in -\operatorname{int} D, \forall t \in (0, \delta], \forall v' \in B(v, \delta)$.

Now, for this $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $t_n \in (0, \delta]$ and $v_n \in B(v, \delta)$, for all $n \geq n_0$.

Making $t = t_n$ and $v' = v_n$, we have that $-d_n = t_n v_n \in -\operatorname{int} D,\ \forall n \geq n_0$ and taking into account (4), it follows that

$$y_n = y_0 - d_n \in E, \quad d_n \in \operatorname{int} D,$$

in contradiction with the local weak minimality of y_0.

Proofs of parts (ii) and (iii) are similar taking into account the definitions, so we only prove part (iii).

Suppose that there exists $z \in T''(E, y_0, u) \cap TI(-\operatorname{int} D, u)$. By the definition of the set $T''(E, y_0, u)$ there exist sequences $(t_n, r_n) \to (0^+, 0^+)$ and $z_n \to z$ such that $t_n/r_n \to 0$ and

$$y_n := y_0 + t_n u + \tfrac{1}{2} t_n r_n z_n \in E \quad \forall n \in \mathbb{N}. \tag{5}$$

On the other hand, as $z \in TI(-\operatorname{int} D, u)$, there exists $\delta > 0$ such that

$$u + \alpha z' \in -\operatorname{int} D \quad \forall \alpha \in (0, \delta),\ z' \in B(z, \delta).$$

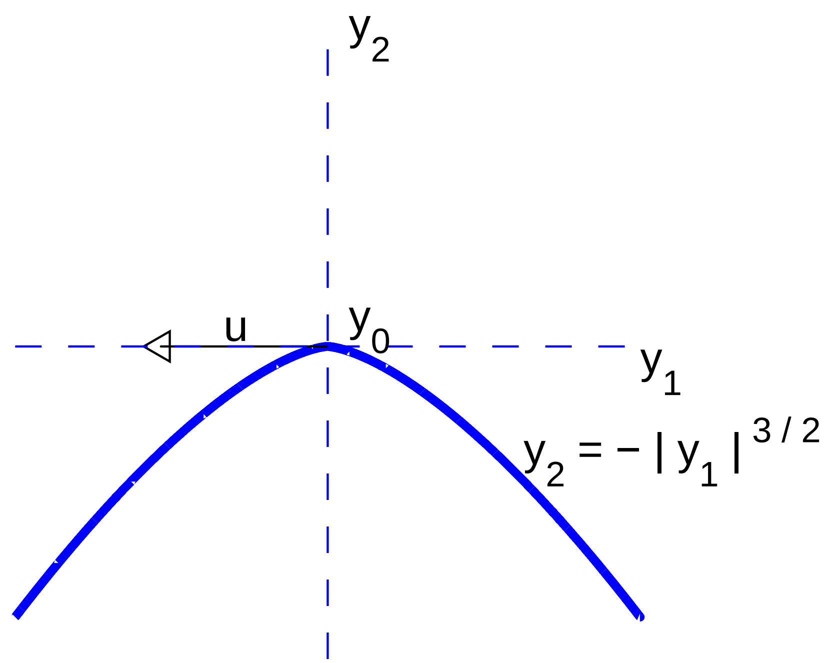
For this $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{2} r_n \in (0, \delta)$ and $z_n \in B(z, \delta)$ for all $n \geq n_0$. So, $u + \frac{1}{2} r_n z_n \in -\operatorname{int} D$, and consequently $-d_n := t_n u + \frac{1}{2} t_n r_n z_n \in -\operatorname{int} D$. Thus, (5) can be written

$$y_n = y_0 - d_n \text{ with } d_n \in \operatorname{int} D,$$

in contradiction to the local weak efficiency of y_0. $\square$

Even in the Paretian case, (i) and (ii) may fail to detect non-efficient points.

Example 2.2. $E = \{(y_1, y_2) : y_2 = -|y_1|^{3/2}\}$, $y_0 = (0,0)$, $u = (-1,0)$, $D = \mathbb{R}^2_+$.
Then $T(E, y_0) = \{(u_1, u_2) : \ u_2 = 0\}$, $T^2(E, y_0, u) = \emptyset$ and
$T''(E, y_0, u) = \{(z_1, z_2) : \ z_2 \leq 0\}$.
Thus (i) and (ii) are satisfied, but (iii) is false. So, $y_0 \notin \text{LWMin}(E, D)$.



The theorem that follows establishes sufficient conditions for local efficiency, and assuming that the space is finite dimensional shows that there is not gap with the necessary conditions established in Theorem 2.1.

Here we use the closure of D instead of the interior of D.

Theorem 2.3. *Let Y be a finite dimensional space and $y_0 \in E \subset Y$. If one of the following conditions holds:*

(i) $T(E, y_0) \cap \mathrm{cl}(-D) = \{0\}$.

(ii) For each $u \in T(E, y_0) \cap \mathrm{cl}(-D) \setminus \{0\}$ *we have*

$$T^2(E, y_0, u) \cap u^{\perp} \cap -\,\mathrm{cl}\,\mathrm{cone}(D + u) = \emptyset, \tag{6}$$

$$T''(E, y_0, u) \cap u^{\perp} \cap -\,\mathrm{cl}\,\mathrm{cone}(D + u) = \{0\}., \tag{7}$$

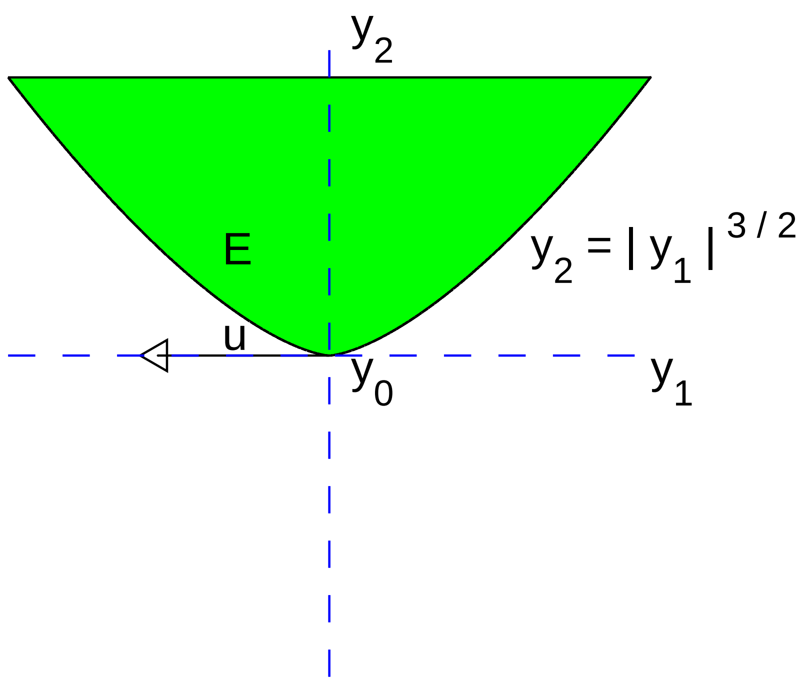
then y_0 is a local efficient element of E

Example 2.4. Let $E = \{(y_1, y_2) \in \mathbb{R}^2 : \; y_2 \geq |y_1|^{3/2}\}$, $y_0 = (0,0)$, $u = (-1,0)$, $D = \mathbb{R}^2_+$.

Then $T^2(E, y_0, u) = \emptyset$ and $T(E, y_0) = T''(E, y_0, u) = \mathbb{R} \times \mathbb{R}_+$.

Theorem 2.3(ii) applies and 2.3(i) is false.

So $y_0 \in \mathrm{LMin}(E, D)$, and since E is a convex set, $y_0 \in \mathrm{Min}(E, D)$.



3 Optimality Conditions in Nonsmooth Vectorial Programming Problems

In this section, we apply the obtained results to a vectorial mathematical programming problem with nonsmooth data. We suppose that f, g and h are Hadamard derivable at $x_0 \in M$.

In finite dimensional spaces, the linearized or critical cone (also called cone of descent directions) is defined by considering the active inequality constraints at x_0.

Here, we propose next critical cones.

For a feasible set given by $S = \{x \in X : \ g(x) \in -K, \ h(x) = 0\}$ the critical cone to S at x_0 is defined by:

$$C(S, x_0) = \{v \in X : \ dg(x_0, v) \in \operatorname{cl}\operatorname{cone}(-K - g(x_0)), \ dh(x_0, v) = 0\}.$$

For the objective function f of problem (2), we define the critical cone and the strict critical cone to f at x_0 as follows:

$$C(f, x_0) = \{v \in X : \ df(x_0, v) \in -D\},$$

$$C_0(f, x_0) = \{v \in X : \ df(x_0, v) \in -\operatorname{int} D\}.$$

In the following lemma we prove an interesting property of the Hadamard derivative.

Lemma 3.1. *Let $M \subset X$, $x_0 \in M$ and $v \in X$.*

i) If there exist $df(x_0, v)$ and $v = \lim_{n\to\infty} \frac{x_n - x_0}{t_n}$, with $t_n \to 0^+$ and $x_n \in M$, then $\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{t_n} = df(x_0, v)$.

ii) If f is Hadamard derivable at x_0, then $df(x_0, \cdot)\big(T(M, x_0)\big) \subset T\big(f(M), f(x_0)\big)$.

Proof. If $v \in T(M, x_0)$, then there exist sequences $(x_n) \to x_0$, $(x_n) \subset M$ and $(t_n) \to 0^+$ such that $v_n = \frac{x_n - x_0}{t_n} \to v$, so

$$df(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{n\to\infty} \frac{f(x_0 + t_n v_n) - f(x_0)}{t_n} = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{t_n}$$

and part (i) follows.

Since f is continuous at x_0 we have that $(f(x_n)) \to f(x_0)$. Taking into account that $f(x_n) \in f(M)$ and that $(t_n) \to 0^+$, we deduce that $df(x_0, v) \in T(f(M), f(x_0))$, and the proof is finished. □

As an application of Theorem 2.1 we prove the following first order necessary optimality condition.

Theorem 3.2. *Let* $f : X \to Y$ *be Hadamard derivable at* $x_0 \in M$*. If* $x_0 \in \mathrm{LWMin}(f, M)$ *for problem* (2)*, then* $T(M, x_0) \cap C_0(f, x_0) = \emptyset$.

Proof. If $x_0 \in \mathrm{LWMin}(f, M)$ there exists a neighborhood U of x_0 such that $f(x_0) = y_0 \in \mathrm{WMin}(f(M \cap U), D)$, and from theorem 2.1, we have that

$$T(f(M \cap U), y_0) \cap TI(-D, 0) = \emptyset,$$

equivalent to $T(f(M \cap U), y_0) \cap (-\operatorname{int} D) = \emptyset$. Since $T(M \cap U, x_0) = T(M, x_0)$, from lemma 3.1 it follows that

$$df(x_0, \cdot)(T(M, x_0)) = df(x_0, \cdot)(T(M \cap U, x_0)) \subset T(f(M \cap U), y_0),$$

consequently $df(x_0, \cdot)(T(M, x_0)) \cap (-\operatorname{int} D) = \emptyset$. Using the inverse of $df(x_0, \cdot)$, we conclude that $T(M, x_0) \cap C_0(f, x_0) = \emptyset$ and the proof is finished. □

Next, we obtain, as corollaries of this theorem, several well-known necessary optimality conditions for scalar and multiobjetive optimization problems.

First we consider a nonsmooth scalar problem.

Corollary 3.3. *Let us consider problem* (2) *with* $Y = \mathbb{R}$, $D = \mathbb{R}^+$ *and* f *Hadamard derivable.*
If x_0 *is a local minimum, then* $df(x_0, v) \geq 0$ *for all* $v \in T(M, x_0)$.

Proof. The proof is an easy consequence of our theorem 3.2, there is not $v \in T(M, x_0)$ such that $df(x_0, v) < 0$, therefore

$$df(x_0, v) \geq 0, \quad \forall v \in T(M, x_0).$$

□

Corollary 3.4. *Let us consider problem* (2) *with* $Y = \mathbb{R}$, $D = \mathbb{R}^+$, *and let* f *be Fréchet differentiable at* $x_0 \in \operatorname{int} M$.
If x_0 *is a local minimum, then* $\nabla f(x_0) = 0$.

Proof. Because $x_0 \in \operatorname{int} M$, we have that $T(M, x_0) = X$ and from theorem 3.2 it is follows that

$$df(x_0, v) \geq 0, \quad \forall v \in X.$$

Since f is Fréchet differentiable at x_0 that condition is equivalent to

$$\nabla f(x_0)(v) \geq 0, \quad \forall v \in X.$$

As $\nabla f(x_0)$ is linear, we conclude that $\nabla f(x_0) = 0$. □

Corollary 3.5. *Let us consider problem* (2), *with* $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $D = \mathbb{R}^+$, $M = \{x \in \mathbb{R}^n : h(x) = 0\}$ *given by equality constraint, where* $h : \mathbb{R}^n \to \mathbb{R}^r$ *is Fréchet differentiable at* $x_0 \in M$ *and continuous in a neighborhood of* x_0, *such that* $\nabla h_1(x_0), \nabla h_2(x_0), \ldots, \nabla h_r(x_0)$ *are linearly independent.*
If x_0 *is a local minimum, then* $\nabla f(x_0) + \sum_{k=1}^r \nu_k \nabla h_k(x_0) = 0$.

Proof. Under these hypotheses, from the Lyusternik theorem, we have that

$$T(M, x_0) = \mathrm{Ker}\, \nabla h(x_0).$$

This set is a linear subspace and now the result of theorem 3.2 is $\nabla f(x_0)v \geq 0,\ \forall v \in \mathrm{Ker}\, \nabla h(x_0)$, therefore $\nabla f(x_0)v = 0,\ \forall v \in \mathrm{Ker}\, \nabla h(x_0)$, equivalent to:

$$\nabla f(x_0) \in (\mathrm{Ker}\, \nabla h(x_0))^{\perp} = \left\{ \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) :\ \nu_k \in \mathbb{R} \right\},$$

consequently, there exist multipliers $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{R}$ such that

$\nabla f(x_0) + \sum_{k=1}^r \nu_k \nabla h_k(x_0) = 0.$ □

Corollary 3.6. *Let us consider problem* (2) *with* $X = \mathbb{R}^n$, $Y = \mathbb{R}$ *and* $D = \mathbb{R}^+$. *Let us suppose that* M *is given by inequality constraints,* $M = \{x \in \mathbb{R}^n : g(x) \leq 0\}$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is Fréchet differentiable at* $x_0 \in M$ *and next constraint qualification is verified:*

$$(MFCQ) \text{ There exists } \bar{v} \in \mathbb{R}^n \text{ such that } \nabla g_j(x_0)\bar{v} < 0 \ \forall j \in J_0,$$

where $J_0 = \{j \in \{1, \ldots, m\} : \ g_j(x_0) = 0\}$.
If x_0 *is a local minimum, then there there exist* $\mu_1, \ldots, \mu_m \in \mathbb{R}$ *such that*

$$\mu_j \geq 0, \ \mu_j g_j(x_0) = 0, \ j = 1, \ldots, m,$$
$$\nabla f(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0.$$

Proof. In these conditions, it is easy to prove that

$$T(M, x_0) = C(M, x_0) = \{v \in \mathbb{R}^n : \ \nabla g_j(x_0)v \leq 0 \ \forall j \in J_0\}.$$

From corollary 3.3, we have that $\nabla f(x_0)v \geq 0, \ \forall v \in C(M, x_0)$ and, consequently, next system

$$\begin{cases} \nabla f(x_0)v < 0 \\ \nabla g_j(x_0)v \leq 0 \ \forall j \in J_0 \end{cases}$$

is incompatible in $v \in \mathbb{R}^n$. Using the Farkas lemma, there exist $\mu_j \geq 0$, $j \in J_0$ such that $\nabla f(x_0) + \sum_{j \in J_0} \mu_j \nabla g_j(x_0) = 0$.

Finally, choosing $\mu_j = 0 \ \forall j \in \{1, \ldots, m\} \setminus J_0$ the conclusion is obtained. $\square$

Corollary 3.7. *Let us consider problem* (2) *with* $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$ *and* $D = \mathbb{R}^p_+$. *Let us suppose that* $M = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$ *is given by equality and inequality constraints, the involved functions* $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *and* $h : \mathbb{R}^n \longrightarrow \mathbb{R}^r$ *are Fréchet differentiable at* $x_0 \in M$ *and next constraint qualification is satisfied:*

$$(MFCQ) \quad \begin{cases} \nabla h_k(x_0), k = 1, \ldots, r, \text{ are linearly independent,} \\ \exists \bar{v} \in \mathbb{R}^n \text{ such that } \nabla g_j(x_0)\bar{v} < 0 \ \forall j \in J_0, \nabla h_k(x_0)\bar{v} = 0, k = 1, \ldots, r, \end{cases}$$

where $J_0 = \{j \in \{1, \ldots, m\} : \ g_j(x_0) = 0\}$.

If $x_0 \in \mathrm{LWMin}(f, M)$, *then there exist* $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_r \in \mathbb{R}$ *such that*

$$\lambda_i \geq 0, \ i = 1, \ldots, p, \quad \lambda \neq 0,$$

$$\mu_j \geq 0, \ \mu_j g_j(x_0) = 0, \ j = 1, \ldots, m,$$

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) = 0.$$

4 A Multiplier Rule for a Nonsmooth Multiobjective Pareto Program

We consider next multiobjective Pareto program with equality and inequality constraints:

$$\text{Min}\, f(x) \text{ subject to } x \in S, \tag{8}$$

where

$$S = \{x \in \mathbb{R}^n : \ g(x) \leq 0, \ h(x) = 0\}, \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}^p, \quad g : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad h : \mathbb{R}^n \longrightarrow \mathbb{R}^r.$$

We denote:

f_i, $i \in I = \{1, 2, \ldots, p\}$, g_j, $j \in J = \{1, 2, \ldots, m\}$, h_k, $k \in K = \{1, 2, \ldots, r\}$, the component functions of f, g and h, respectively

$J_0 = \{j \in J : \ g_j(x_0) = 0\}$ the set of active indexes of g at x_0

$G = \{x \in \mathbb{R}^n : \ g(x) \leq 0\}$

$H = \{x \in \mathbb{R}^n : \ h(x) = 0\}$

$S = G \cap H$,

and we consider the following conditions:

(H1) f and g are Hadamard derivable with convex derivative.

(H2) h is Fréchet differentiable, such that $\nabla h(x_0)$ has maximal rank (linearly independent $\nabla h_k(x_0),\ k \in K$).

In these conditions, the strict critical cones to the objective function and to the set given by the inequality constrains are, respectively:

$$C_0(f, x_0) = \{v \in \mathbb{R}^n : \, df(x_0, v) \in -\operatorname{int} \mathbb{R}^p_+\} = \{v \in \mathbb{R}^n : \, df_i(x_0, v) < 0, \ \forall i \in I\},$$

$$C_0(G, x_0) = \{v \in \mathbb{R}^n : \, dg_j(x_0, v) < 0, \ \forall j \in J_0\}.$$

For the proof we need two previous results.

Theorem 4.1. (Jiménez, Novo [4]). *Under the hypotheses (H1) and (H2), we have that*

$$C_0(G, x_0) \cap \operatorname{Ker} \nabla h(x_0) \subset T(S, x_0).$$

Theorem 4.2. (Jiménez, Novo [3]). *Let us suppose that* $\varphi_1, \varphi_2, \ldots, \varphi_q : \mathbb{R}^n \to \mathbb{R}$ *are sublinear functions and* $\psi_1, \psi_2, \ldots, \psi_r : \mathbb{R}^n \to \mathbb{R}$ *are linear functions given by* $\psi_k(u) = \langle c_k, u \rangle$, $k \in K = \{1, 2, \ldots, r\}$.

Then one and only one of the following assertions are true

(a) There exist $v \in \mathbb{R}^n$ *such that*

$$\begin{cases} \varphi_i(x_0, v) < 0 \ \forall i = 1, 2, \ldots q \\ \psi_k(v) = 0 \ \forall k = 1, 2, \ldots k. \end{cases}$$

(b) There exists $(\xi, \nu) = (\xi_1, \xi_2, \ldots, \xi_q, \nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{R}^{q+r}$, $\xi \neq 0$, $\xi \geq 0$, *such that*

$$0 \in \sum_{i=1}^{q} \xi_i \partial \varphi_i(0) + \sum_{k=1}^{r} \nu_k c_k.$$

Theorem 4.3. *Let us consider the multiobjective Pareto program with the preceding conditions (H1) and (H2).*
If $x_0 \in \mathrm{LWMin}(f, M)$, *then there exists* $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ *such that*

$$(\lambda, \mu) \geq 0, \ (\lambda, \mu) \neq 0, \tag{9}$$

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_D f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_D g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0), \tag{10}$$

$$\mu_j g_j(x_0) = 0, \ j = 1, \ldots, m. \tag{11}$$

If, in addition, $C_0(S, x_0) \neq \emptyset$, *then* $\lambda \neq 0$.

Proof. As $x_0 \in \mathrm{LWMin}(f, M)$, from theorem 3.2 we have that

$$T(S, x_0) \cap C_0(f, x_0) = \emptyset, \tag{12}$$

but, since in that case $C_0(f, x_0) = \{v \in \mathbb{R}^n : df_i(x_0, v) < 0, \forall i \in I\}$, condition (12) means that there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} df_i(x_0, v) < 0 \ \forall i \in I \\ v \in T(S, x_0). \end{cases} \tag{13}$$

Now, from theorem 4.1 we have that

$$C_0(G, x_0) \cap \mathrm{Ker}\, \nabla h(x_0) \subset T(S, x_0).$$

So, taking into account (13), there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} df_i(x_0, v) < 0 \ \forall i \in I \\ dg_j(x_0, v) < 0 \ \forall j \in J_0 \\ \nabla h_k(x_0)v = 0 \ \forall k \in K, \end{cases} \tag{14}$$

and using theorem 4.2 the conclusion follows choosing $\mu_j = 0$ for all $j \in J \setminus J_0$.

For the second part, let us suppose that $C_0(S, x_0) \neq \emptyset$, that is, there exists $w \in \mathbb{R}^n$ such that

$$dg_j(x_0, w) < 0, \ \forall j \in J_0, \ \nabla h_k(x_0)(w) = 0, \ \forall k \in K. \tag{15}$$

Assume that $\lambda = 0$. Then conditions (9)-(11) imply that

$$\sum_{j \in J_0} \mu_j dg_j(x_0, u) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)(u) \geq 0, \ \forall u \in \mathbb{R}^n$$

with $\mu \neq 0$. For $u = w$ we have a contradiction since from (15) it follows that

$$\sum_{j \in J_0} \mu_j dg_j(x_0, w) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)(w) < 0.$$

Consequently $\lambda \neq 0$. $\square$

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