Jornada Matemática SPM/CIM em Epidemiologia Teórica
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Stochastic models of infection dynamics

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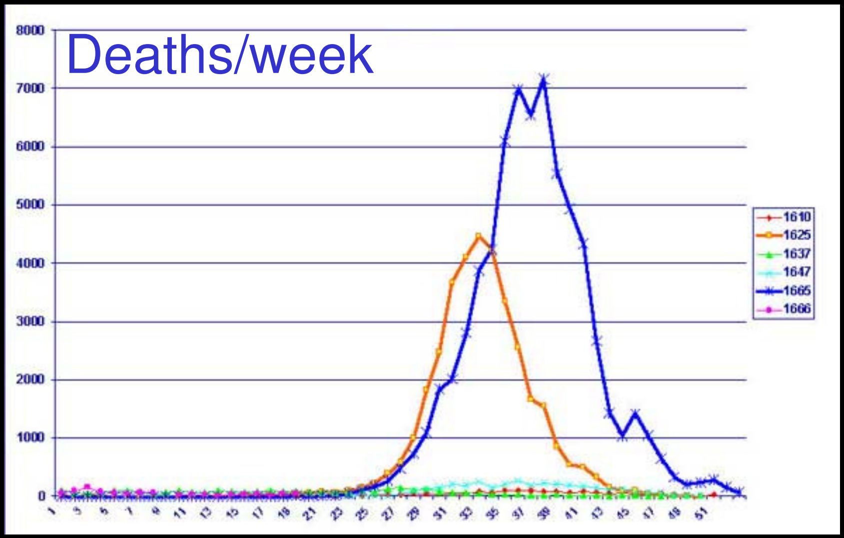
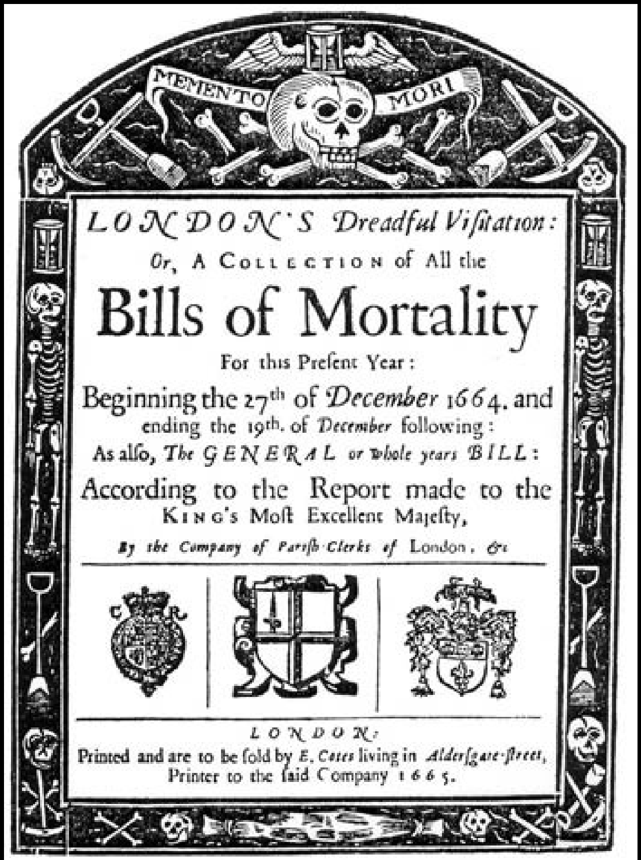
Outline

- Deterministic models and phenomenology
- Resonant amplification of stochastic fluctuations





- The effect of correlations
- Conclusions and perspectives

Deterministic models & phenomenology

SIR dynamics

Population N, compartments S, I, R

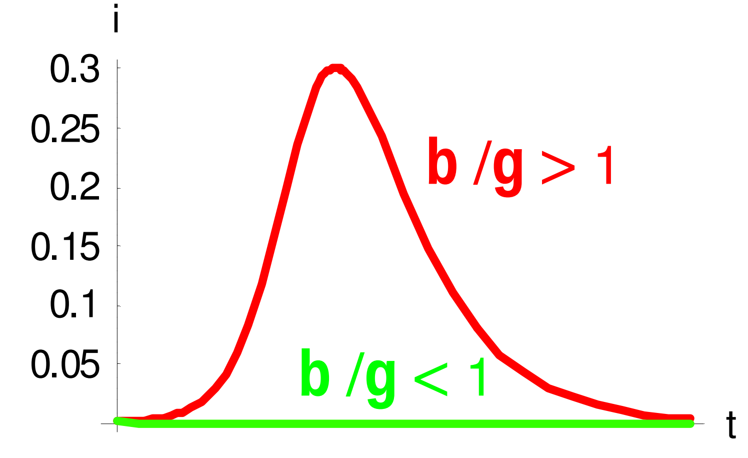
$$S' = -\mathbf{b}\, S\, I$$
$$I' = \mathbf{b}\, S\, I - \mathbf{g}\, I$$
$$R' = \mathbf{g}\, I$$

Kermack – McKendrick, 1927

N constant, S+I+R=N, N >> 1

$$s' = -\mathbf{b}\, s\, i$$
$$i' = \mathbf{b}\, s\, i - \mathbf{g}\, i$$

s,i densities



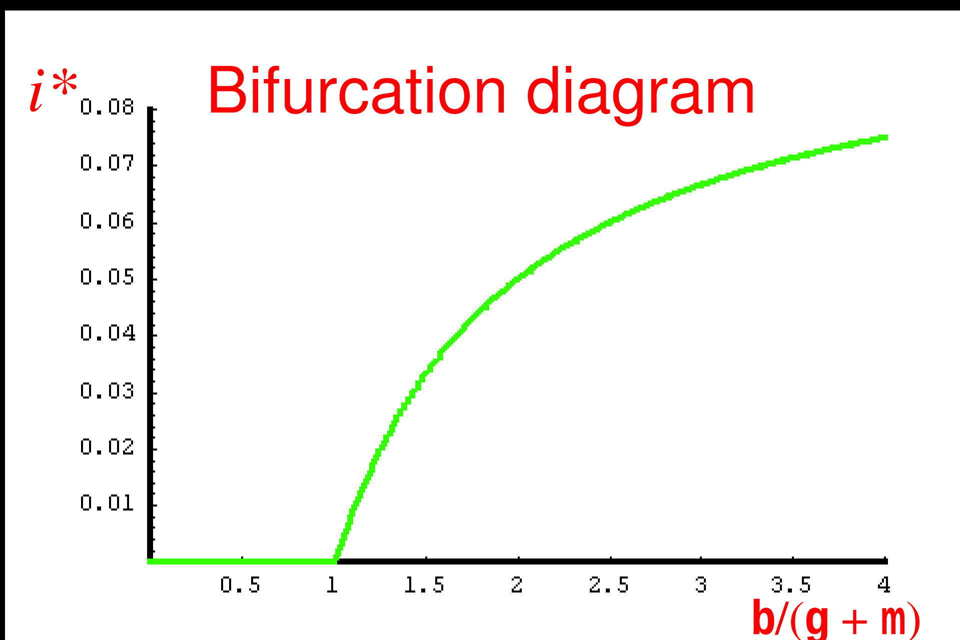
Deterministic models & phenomenology

SIR(S) dynamics

$$s' = \mathbf{m}\,(1 - s) - \mathbf{b}\, s\, i$$
$$i' = \mathbf{b}\, s\, i - (\mathbf{g} + \mathbf{m})\, i$$

$$s' = \mathbf{m}\,(1 - s - i) - \mathbf{b}\, s\, i$$
$$i' = \mathbf{b}\, s\, i - \mathbf{g}\, i$$

m birth/death or immunity waning rate



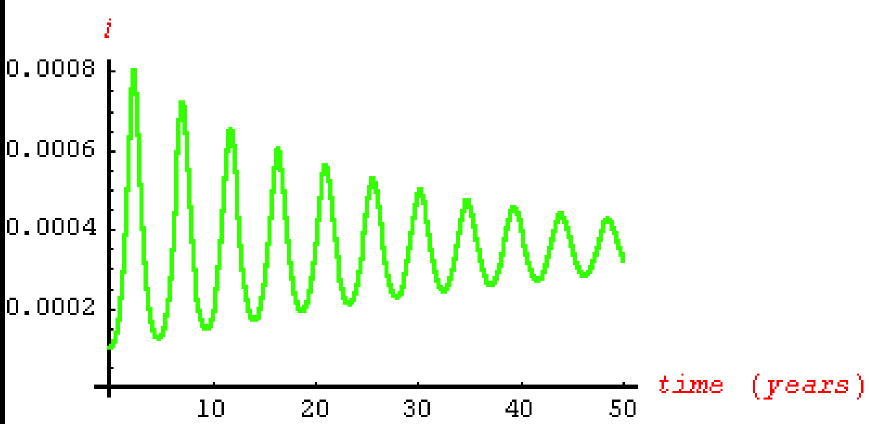
Epidemic/endemic threshold

$R_0 = 1$,

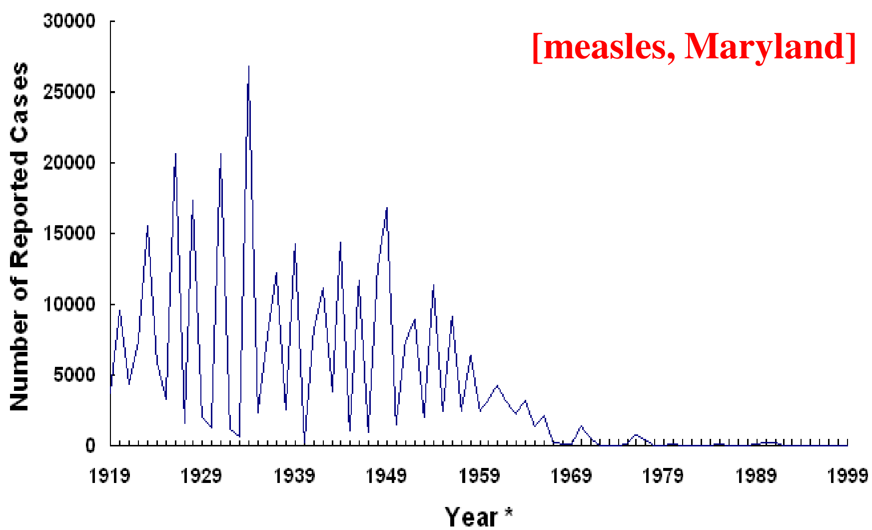
$R_0 = \mathbf{b}/(\mathbf{g} + \mathbf{m})$ - dSIR,

$R_0 = \mathbf{b}/\mathbf{g}$ - SIRS,

Deterministic models & phenomenology
SIR(S) dynamics



Damped oscillations

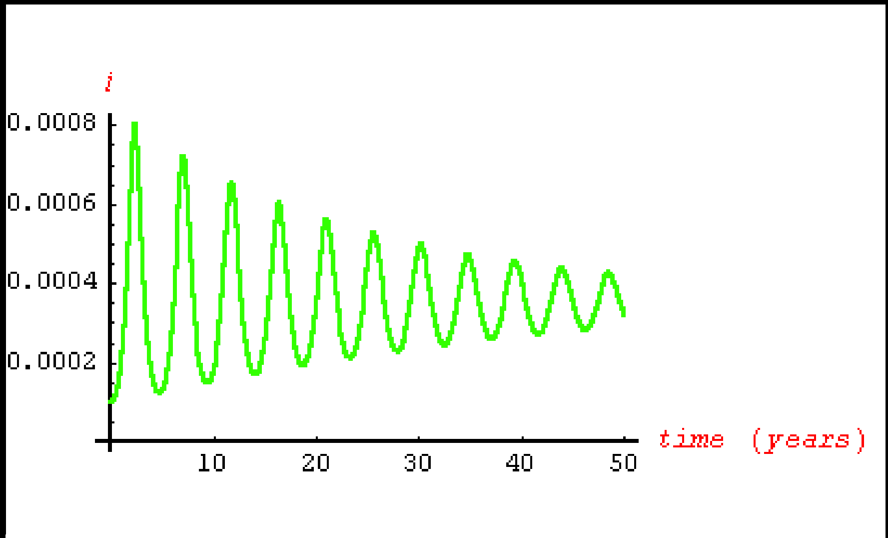


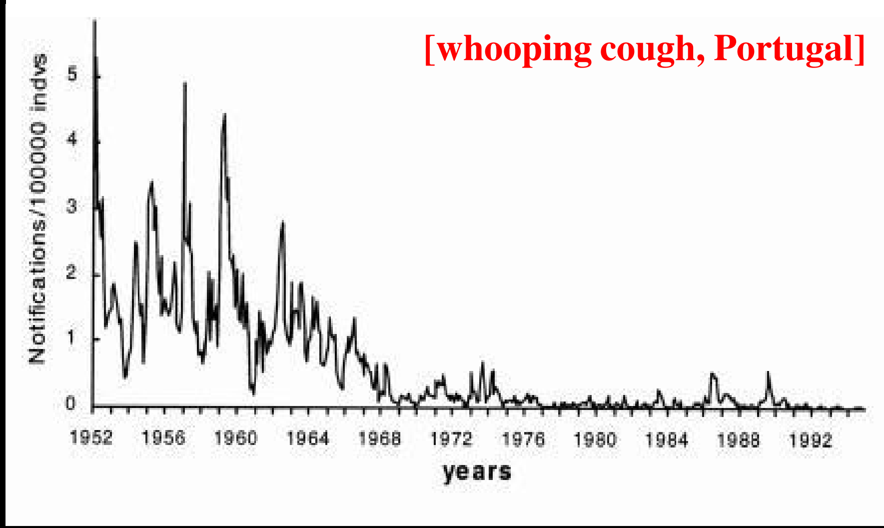
Recurrent epidemics

Diverse incidence patterns

Extinctions

Deterministic models & phenomenology
SIR(S) dynamics





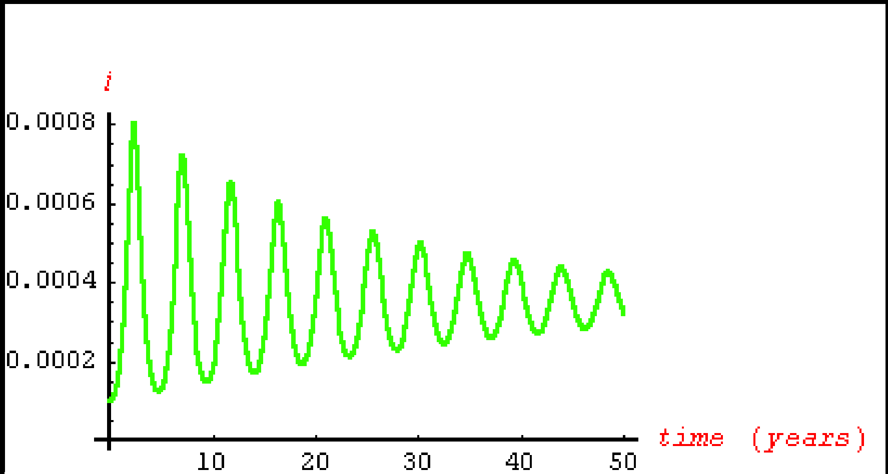
Damped oscillations

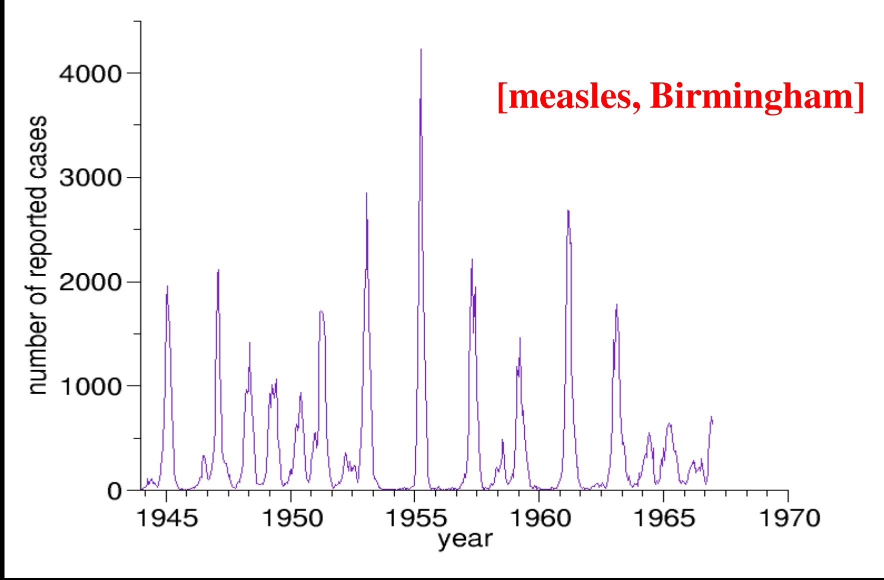
Recurrent epidemics

Diverse incidence patterns

Extinctions

Deterministic models & phenomenology
SIR(S) dynamics





Damped oscillations

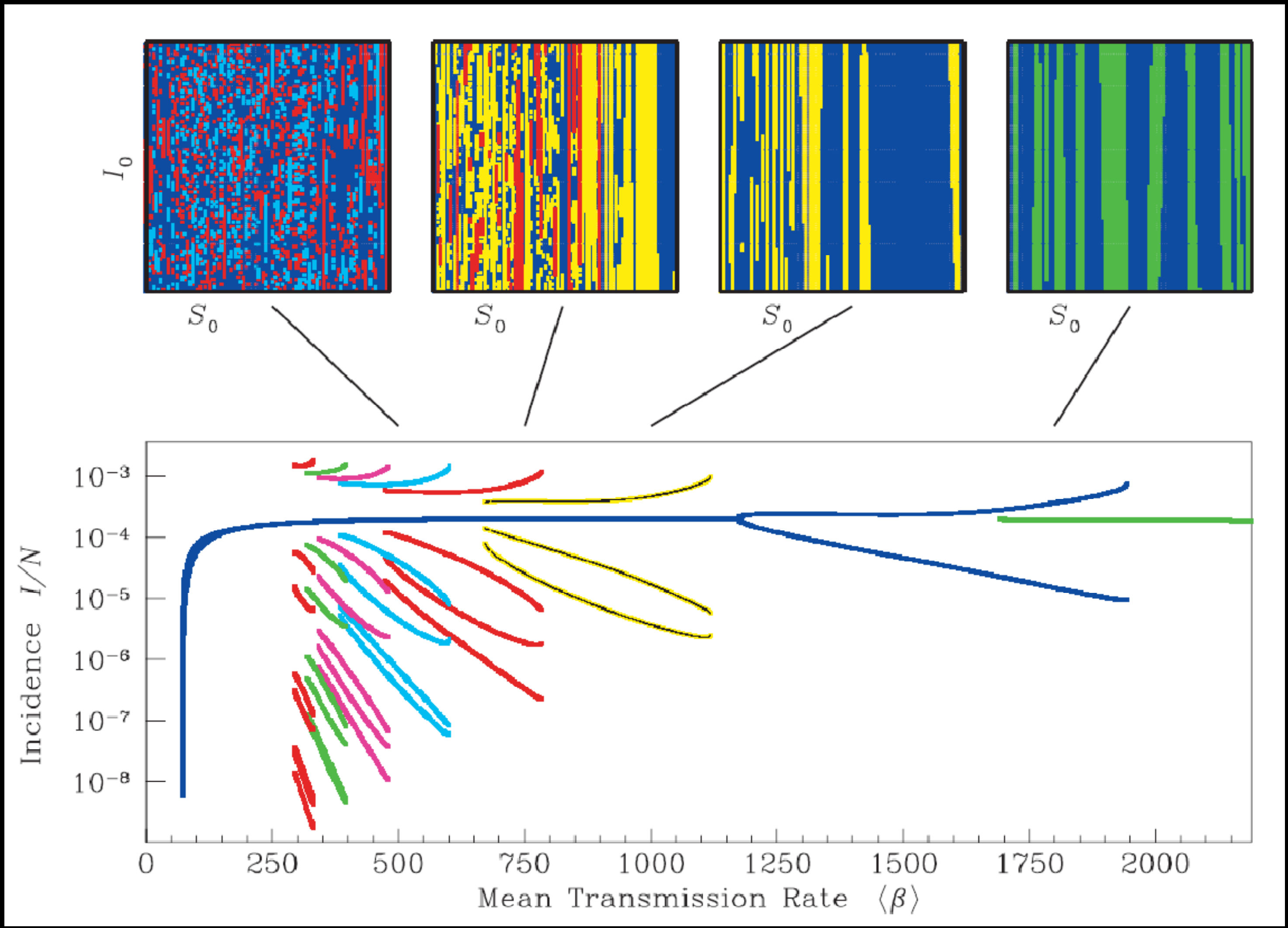
Recurrent epidemics

Diverse incidence patterns

Extinctions

Deterministic models & phenomenology

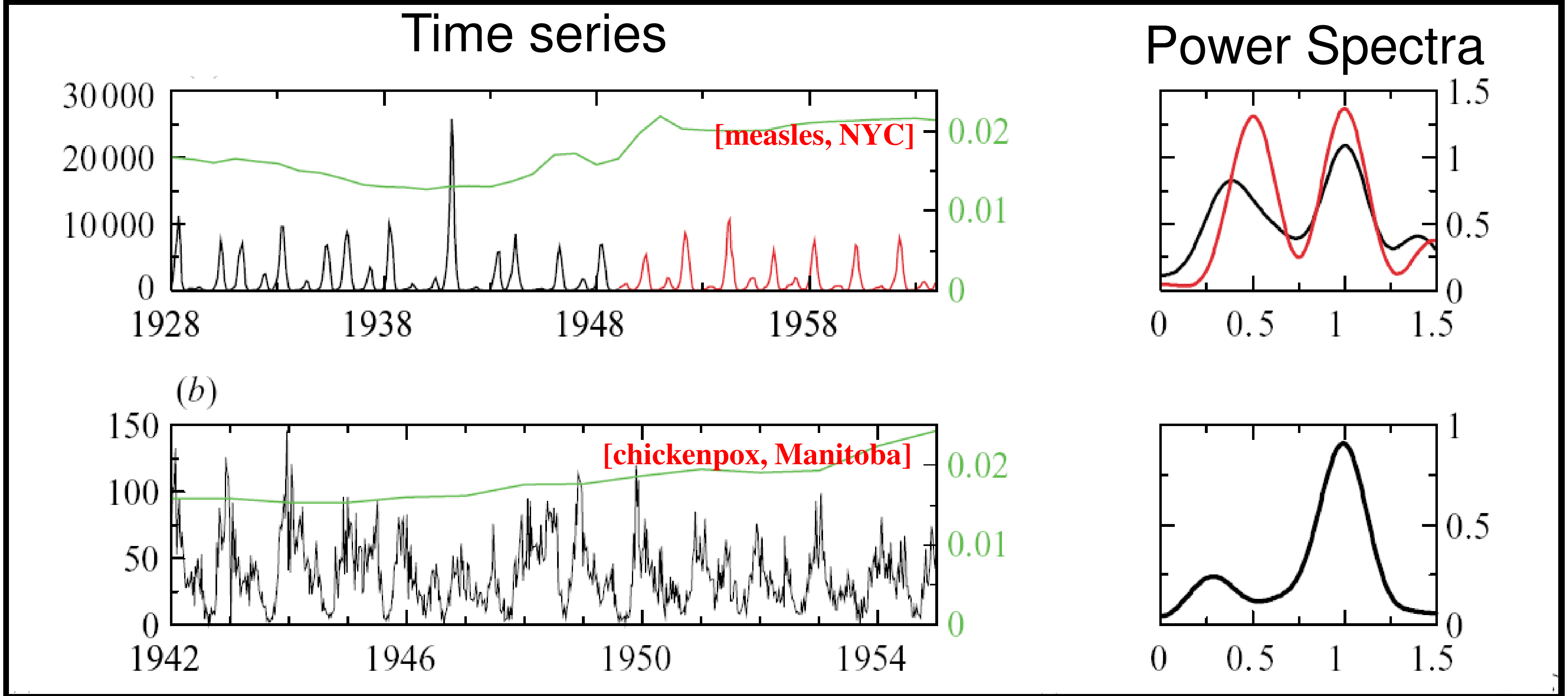
Periodic forcing



[1] Earn et al., Science 287, 667-670 (2000)

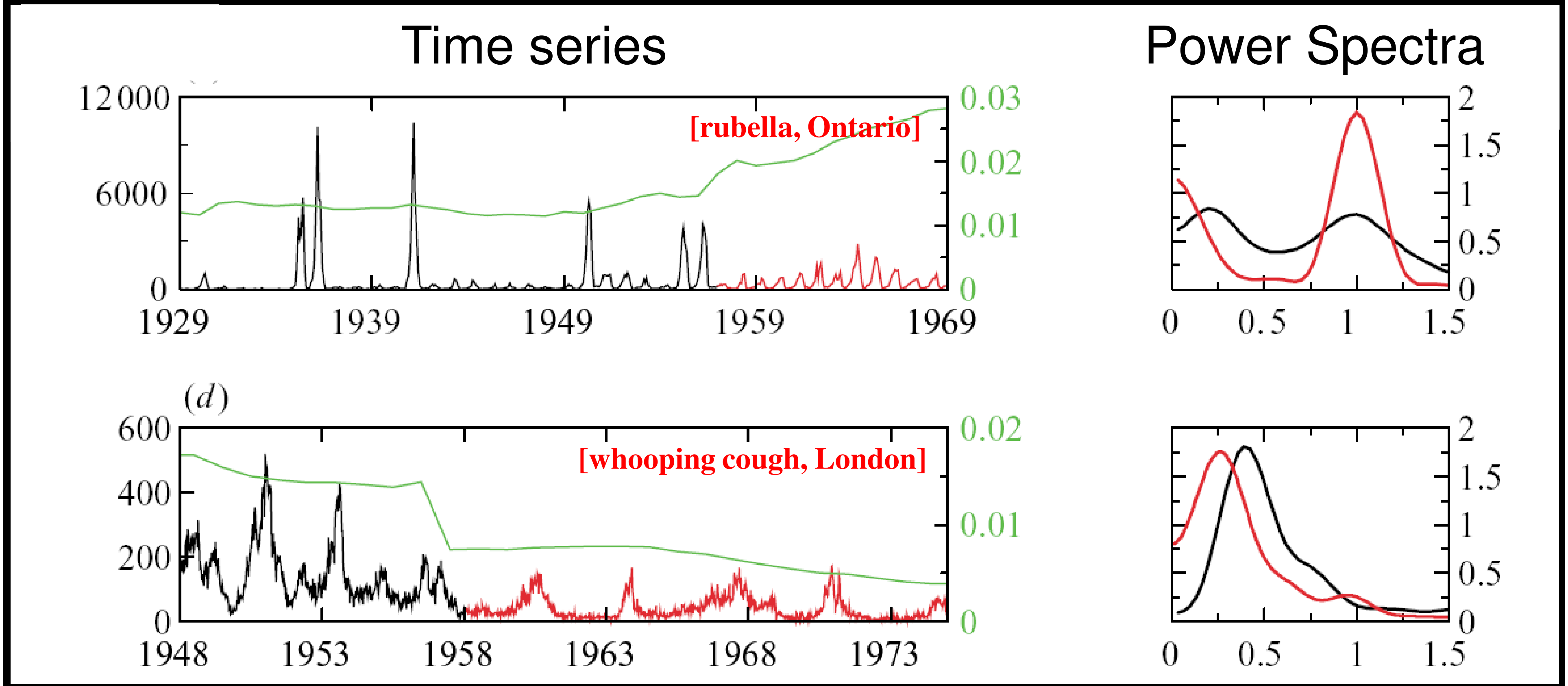
Deterministic models & phenomenology

Periodic forcing



[2] Bauch and D. J. D. Earn, Proc. R. Soc. Lond. B 270, 1573-1578 (2003)

Deterministic models & phenomenology
Periodic forcing



[2] Bauch and D. J. D. Earn, Proc. R. Soc. Lond. B 270, 1573-1578 (2003)

Deterministic models & phenomenology
Periodic forcing

Seasonality versus stochasticity debate:

- Forced nonlinear dynamics explains the data, and stochasticity has a secondary role (switching between attractors and sustaining small amplitude fluctuations around the deterministic system's equilibrium)

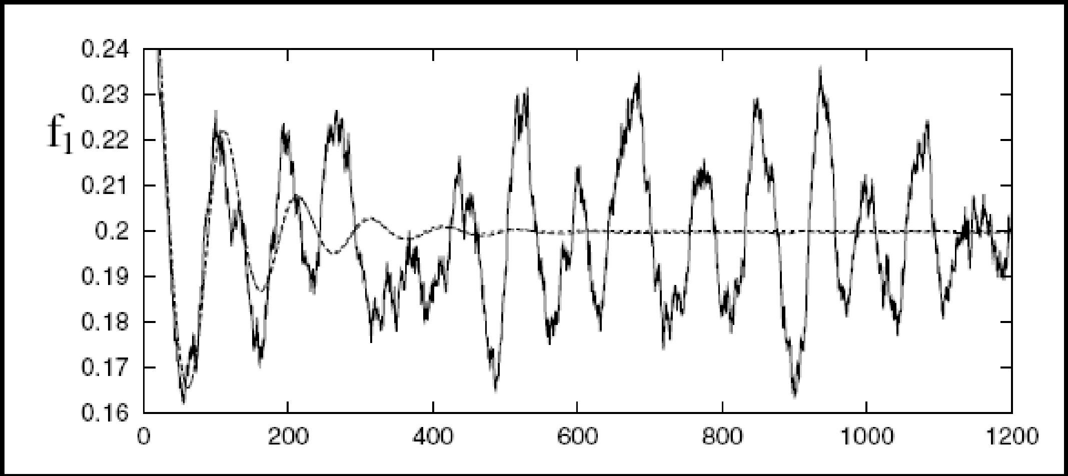
- Many data records cannot be understood in a deterministic framework, fluctuations can be dominant

Resonant amplification of stochastic fluctuations

Predator-prey dynamcis

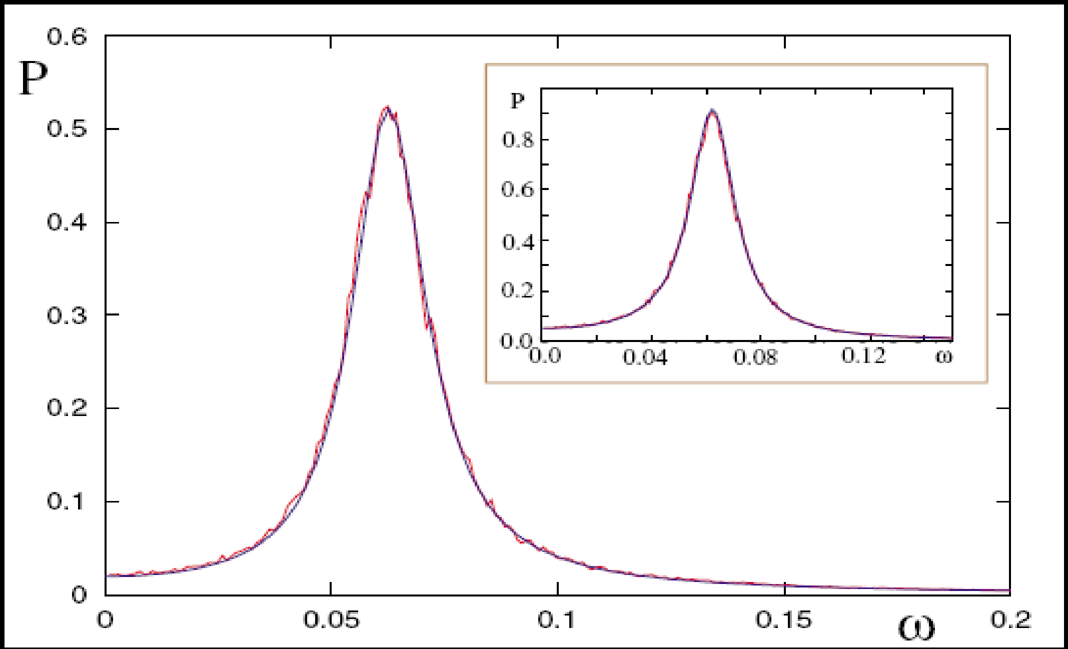
[3] A J McKane and TJ Newman, Phys. Rev. Lett. 94 (2005), 218102

Mean field and average densities exhibit oscillatory transients.



$$P(\omega) = \frac{\alpha + \beta\omega^2}{[(\omega^2 - \Omega_0^2)^2 + \Gamma^2\omega^2]},$$

Sustained oscillatory patterns arise through resonant amplification of internal noise



Resonant amplification of stochastic fluctuations

Stochastic infection dynamics in a well mixed population,
N = S + I + R

	Process	Rate
Infection	$S + I \rightarrow I + I$	$\lambda S I/N$
Recovery	$I \rightarrow R$	$d I$
Loss of immunity	$R \rightarrow S$	γR

[4] D Alonso, A J McKane and M Pascual, J R Soc Interface 4, 575-82 (2007)

Resonant amplification of stochastic fluctuations

Master Equation

$$\frac{dP(\sigma;t)}{dt} = \sum_{\sigma' \neq \sigma} T(\sigma|\sigma')P(\sigma';t) - \sum_{\sigma' \neq \sigma} T(\sigma'|\sigma)P(\sigma;t), \quad \sigma = (S, I)$$

$$\begin{aligned}
T(S-1, I+1|S, I) &= \lambda\, S\, I/\, N, \\
T(S, I-1|S, I) &= \delta\, I\ , \\
T(S+1, I|S, I) &= \gamma\, (N - S - I)
\end{aligned}$$

van Kampen's system size expansion

$$\begin{aligned}
S &= N\phi + \sqrt{N}x_0 \\
I &= N\psi + \sqrt{N}x_1
\end{aligned}$$

Resonant amplification of stochastic fluctuations

van Kampen's system size expansion

$$\phi' = \mathbf{d}\,(1-\phi-\psi) - \mathbf{b}\,\phi\,\psi$$
$$\psi' = \mathbf{b}\,\phi\,\psi - \mathbf{g}\,\psi$$

Leading order terms
MF equations for the densities

Next to leading order
Linear Fokker-Planck or Langevin equations for the normalized fluctuations

$$\dot{x}_i(t) = \sum_{j=1}^{2} J_{ij} x_j(t) + L_i(t), \;\; i = 1, 2$$

Resonant amplification of stochastic fluctuations

Fluctuation power spectrum around the MF equilibrium

$$P_i(\omega) \equiv \langle |\tilde{x}_i(\omega)|^2 \rangle = \sum_{j,k} M_{ik}^{-1}(\omega) B_{kj} M_{ji}^{-1}(-\omega)$$

$$M_{ik}(\omega) = \mathrm{i}\omega\delta_{ik} - J_{ik} \text{ and } \langle \tilde{L}_i(\omega)\tilde{L}_j(\omega')\rangle = B_{ij}\delta(\omega + \omega')$$

$$P_{\mathrm{S}} = \frac{B_{11}(J_{12}^2 + \omega^2)}{(D - \omega^2)^2 + T^2\omega^2} ,$$

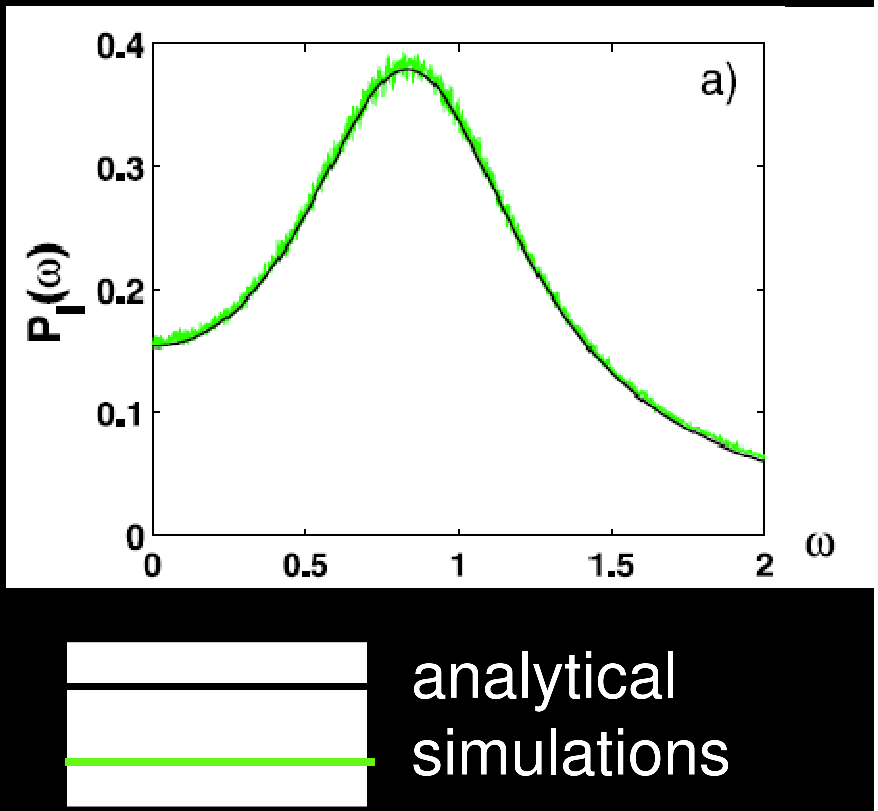
$$P_{\mathrm{I}} = \frac{B_{11}(J_{11}^2 + J_{11}J_{21} + J_{21}^2 + \omega^2)}{(D - \omega^2)^2 + T^2\omega^2}$$

Resonant amplification of stochastic fluctuations

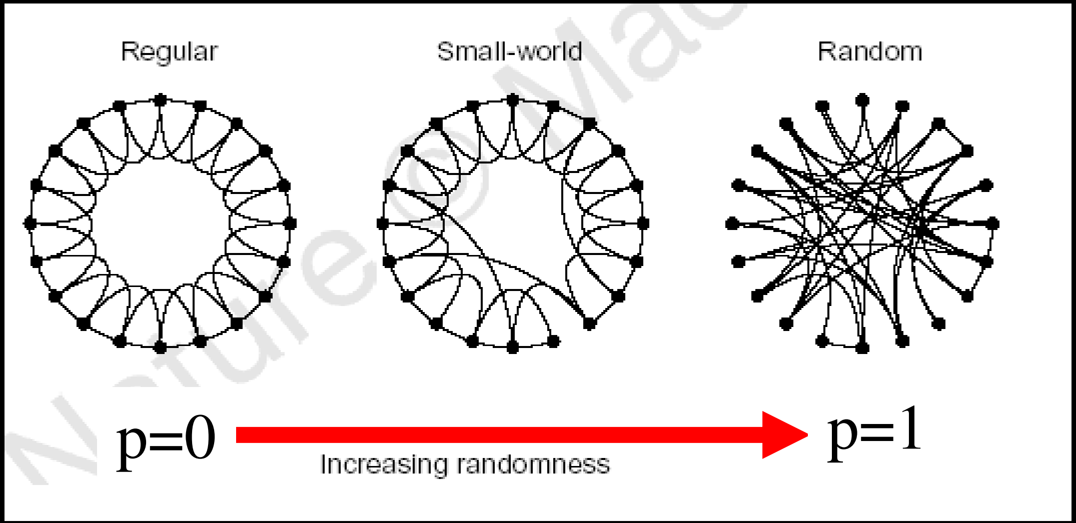
Power spectrum of the incidence fluctuations for the stochastic SIRS model

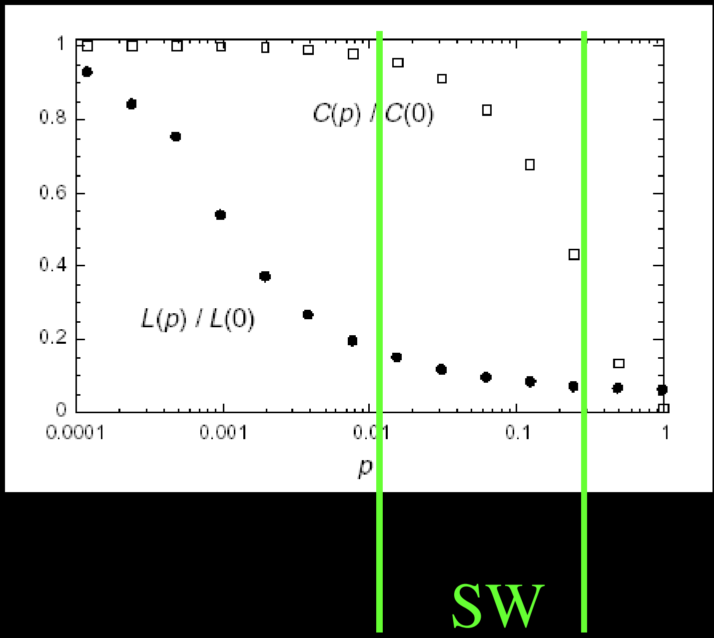
$\lambda = 10$, $\gamma = 0.1$, $N = 10^6$

Finite N, sustained noisy oscillations



The effect of correlations
in space





D Watts e S Strogatz, Nature 393 (1998), 440-442

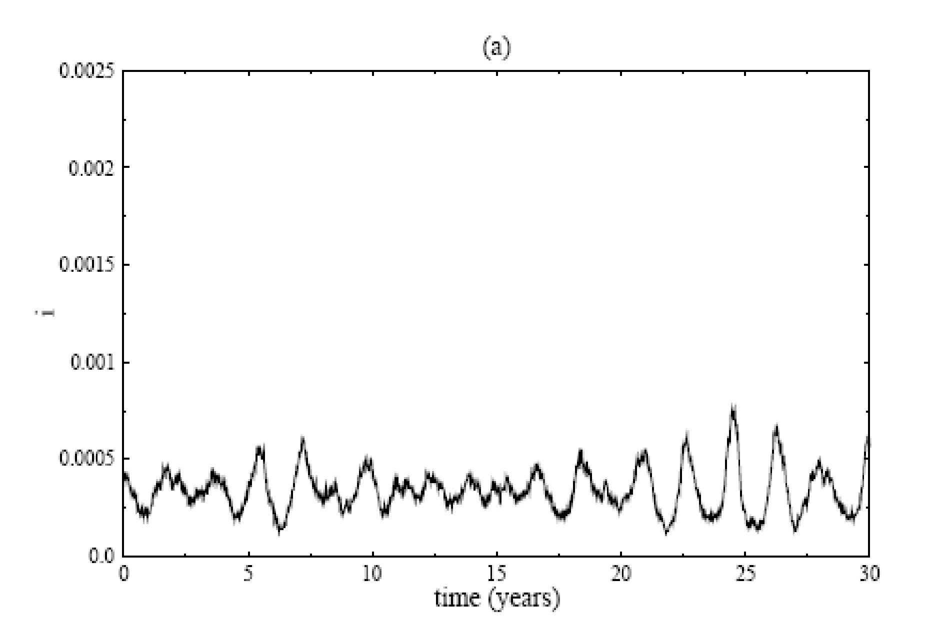
Simulations on small world networks may be used to assess the effect of spatial correlations on the stochastic fluctuations

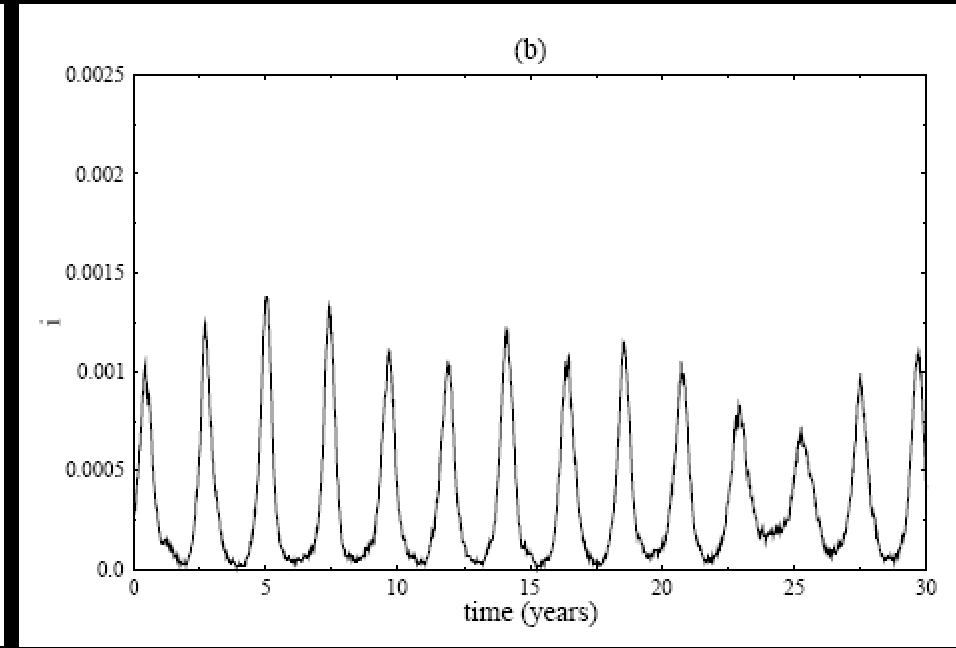
The effect of correlations in space

SEIR stochastic simulations, measles, $N=10^6$

a) p=1

b) p=0.2

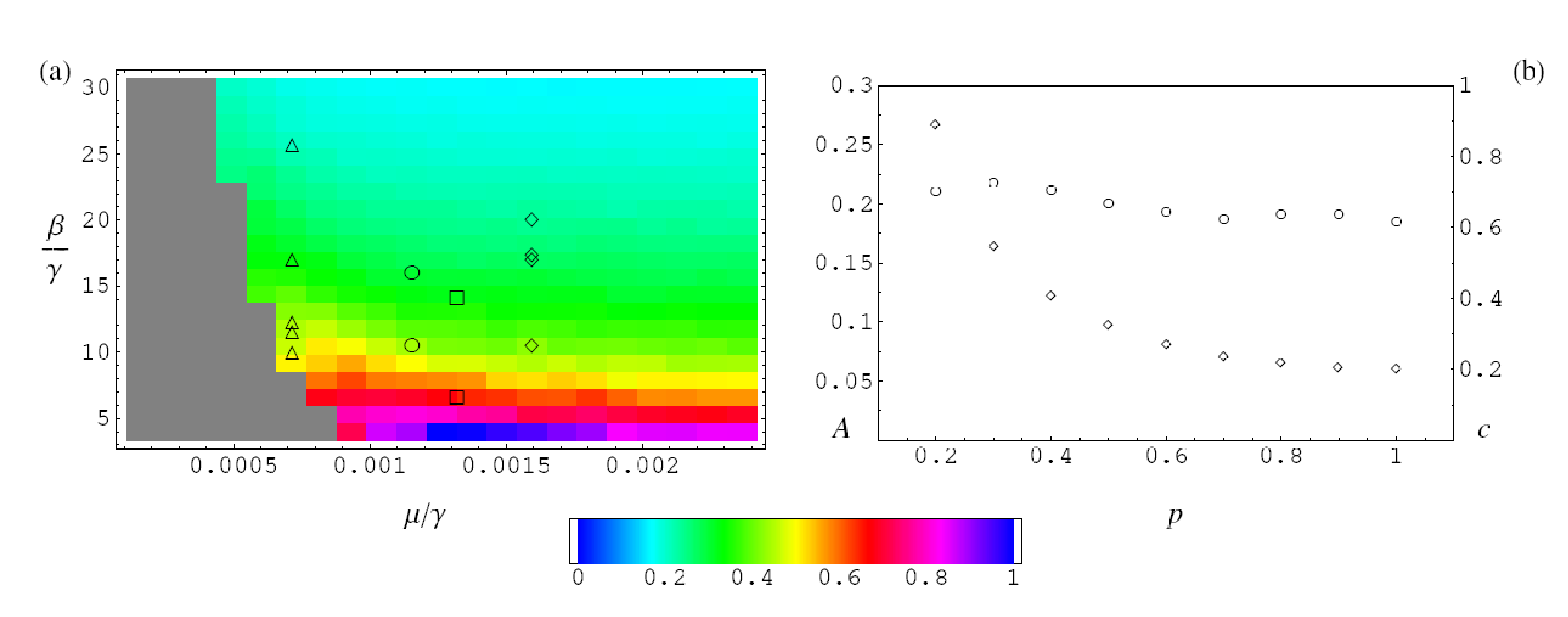




Verdasca et al., J Theor. Biol. 233 (2005), 553-561

The effect of correlations
in space

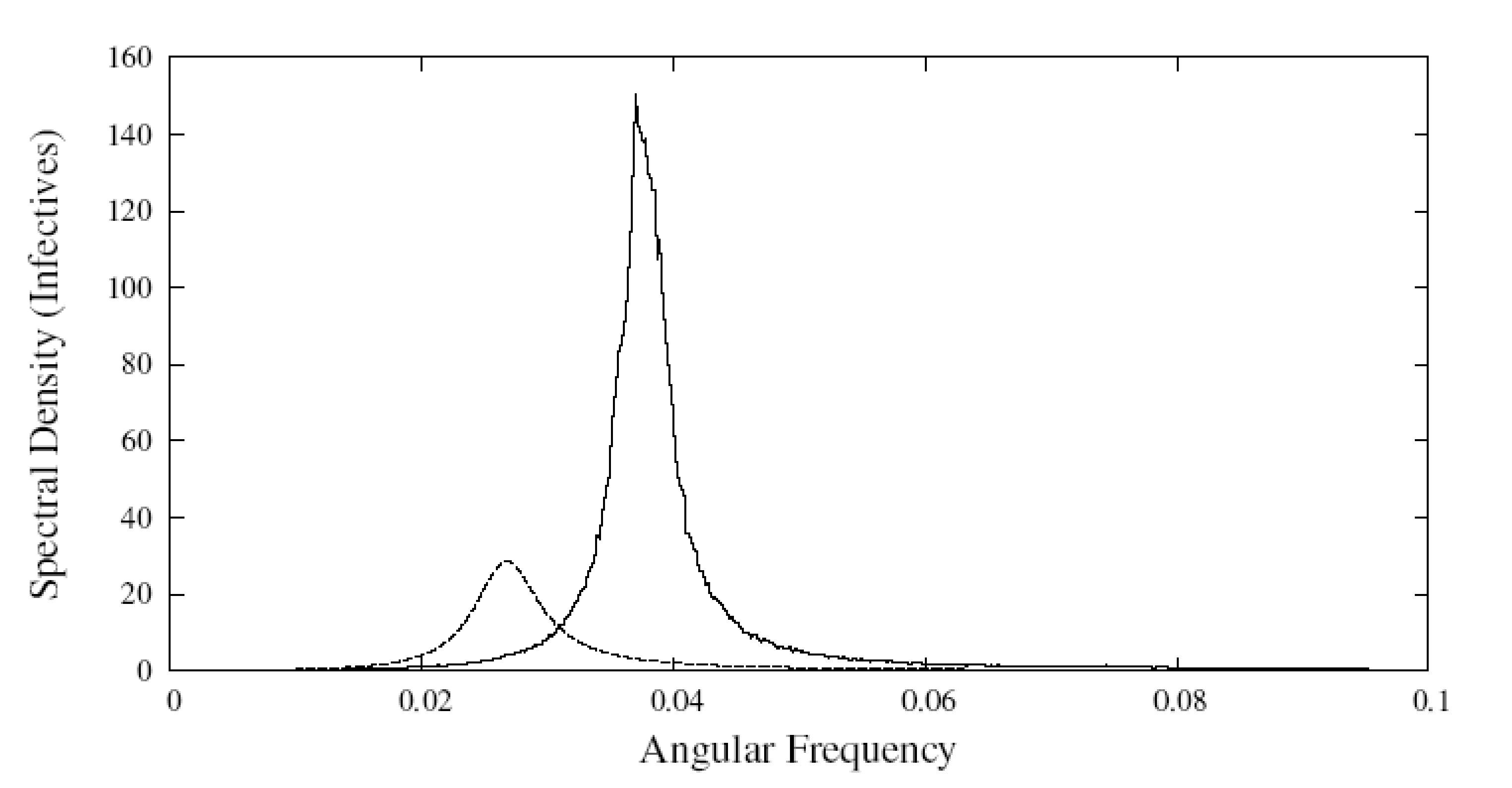
p=0.6 **Amplification and coherence**



Symbols correspond to published parameter values for childhood infectious diseases: measles (triangles), chickenpox (circle), rubella (squares) and whooping cough (diamonds), N=10^6

Simões et al., J R Soc Interface 5 (2008), 555-566

The effect of correlations in time



Fluctuation power spectrum for a stochastic dSIR model with the two extreme types of recovery period distribution: constant recovery rate (dashed), and fixed infectious period (full).

Simões et al., J R Soc Interface 5 (2008), 555-566

The effect of correlations
analytical description

Pair approximation for SIRS (k neighbours)

The rates of change of the pair densities [xy], x,y $\in$ {s,i,r}, depend on [xy] e [xyz]

The standard pair approximation...

triplets are formed by uncorrelated pairs

$$\Sigma\,[asi] = (k\text{-}1)([as][si])/[s],\ \ldots$$

... closes the equations for the pair densities

$$[si]' = \mathbf{g}[ir] - (\lambda + \mathbf{d})\,[si] + \lambda\,(\Sigma\,[ssi] - \Sigma\,[isi]\,),\ \ldots$$

and the pair densities determine the singlet densities

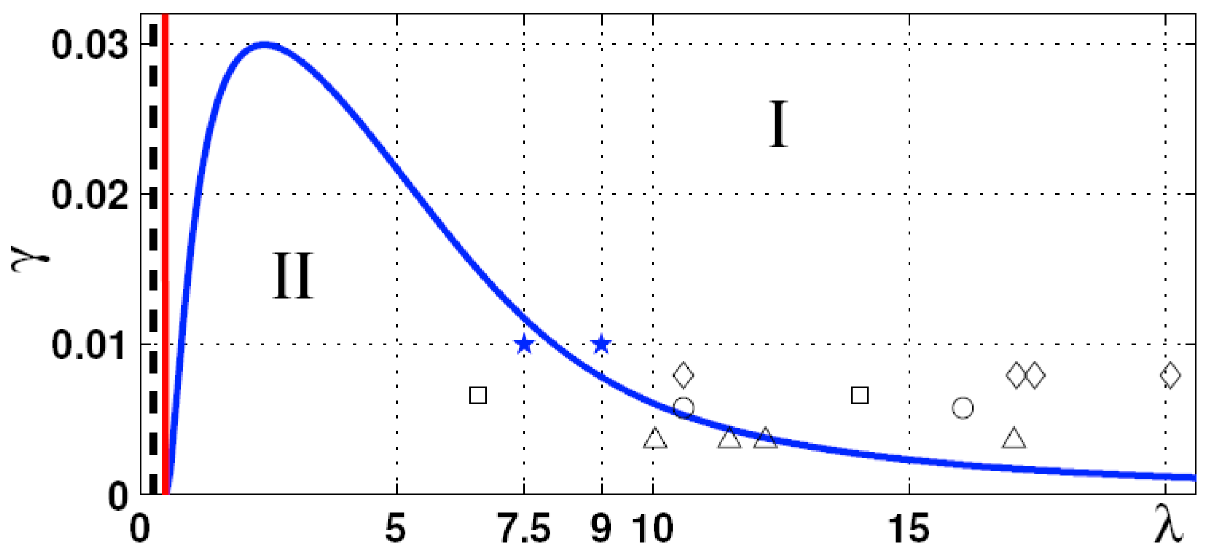
$$[s] = [si]+[ss]+[sr],\ \ldots$$

The effect of correlations
analytical description

$$\frac{ds}{dt} = \gamma\,(1-i-s) - k\lambda\, si\,,$$
$$\frac{di}{dt} = k\lambda\, si - \delta\, i\,,$$
$$\frac{d\,si}{dt} = \gamma\, ri - (\lambda+\delta)\, si + \frac{(k-1)\lambda\, si}{s}(s - sr - 2si)\,,$$
$$\frac{d\,sr}{dt} = \delta\, si + \gamma\,(1-s-i-ri-2sr) - \frac{(k-1)\lambda\, si\, sr}{s}\,,$$
$$\frac{d\,ri}{dt} = \delta\,(i - si) - (\gamma+2\delta)\, ri + \frac{(k-1)\lambda\, si\, sr}{s}\,.$$

k is the fixed degree of the RRG

Bifurcation diagram
k=4



The effect of correlations
analytical description

Pair approximation for stochastic SIRS

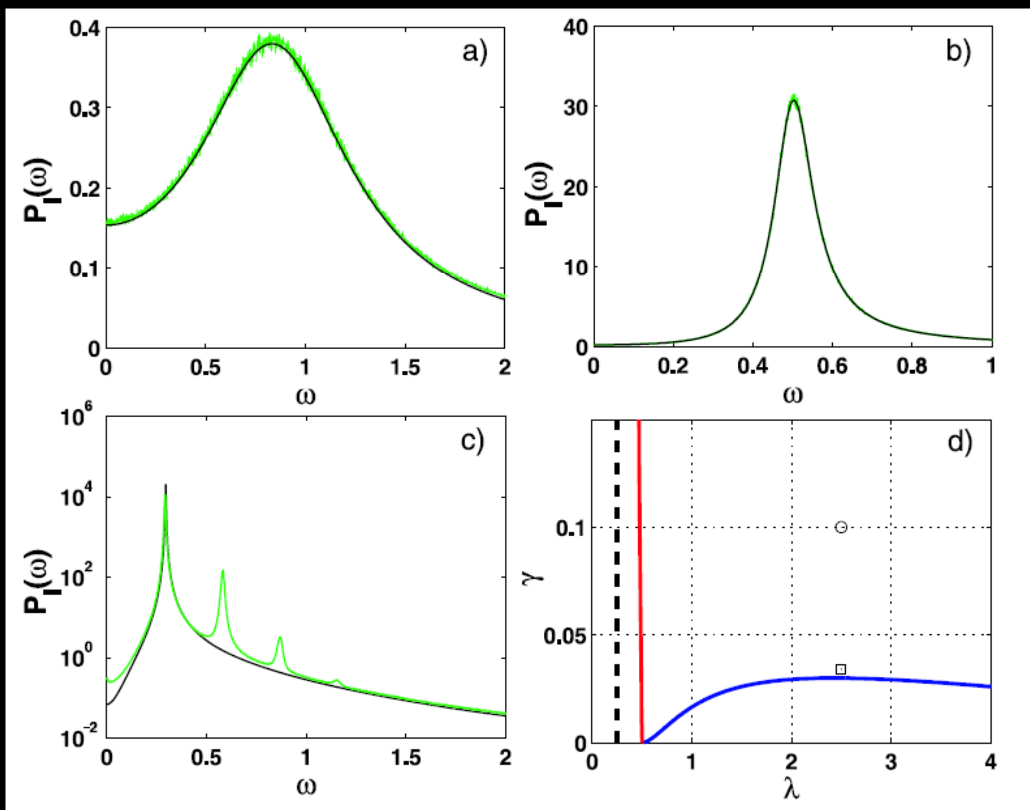
N individuals, kN/2 pairs

	Process	Rate
Infection	$SI \rightarrow II$	λ SI
Recovery	$I \rightarrow R$	d I
Loss of immunity	$R \rightarrow S$	γ R

Fluctuation power spectra around the equilibrium can be computed using van Kampen's expansion as before

The effect of correlations
analytical description

Pair approximation for stochastic SIRS



Amplitude and coherence of the resonant fluctuations are much enhanced in the PA.

Harmonic peaks show up close to the boundary of the oscillatory phase.

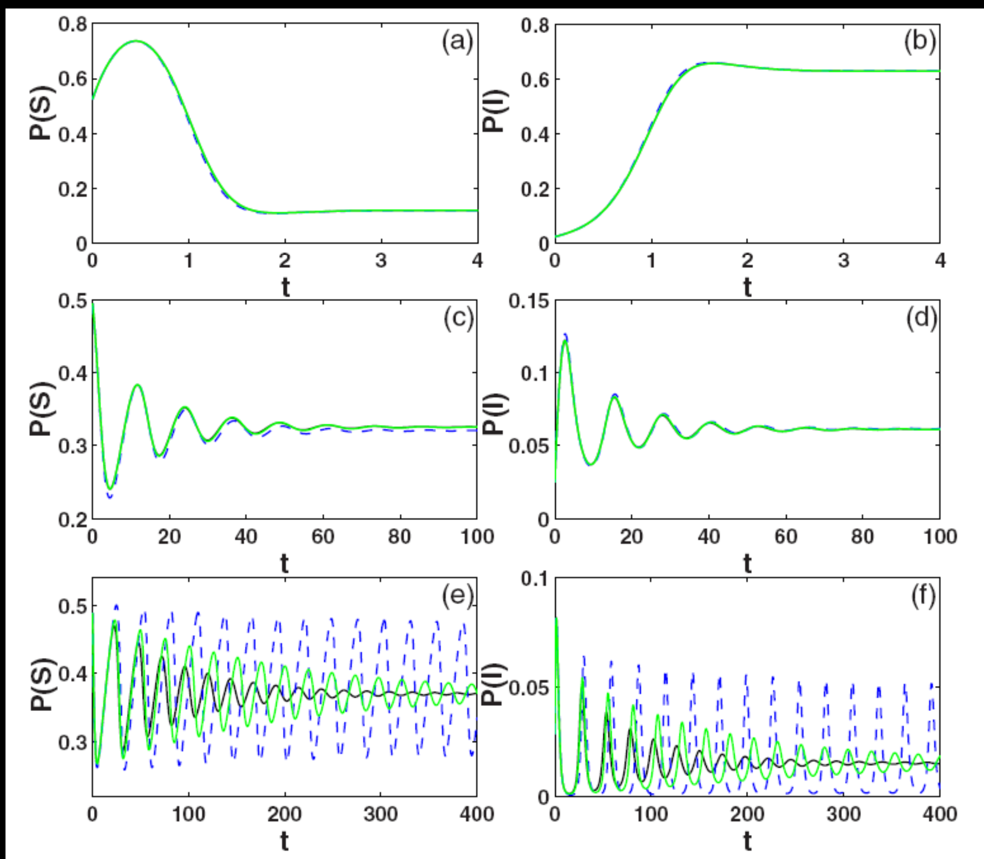
analytic

simulations $N=10^6$

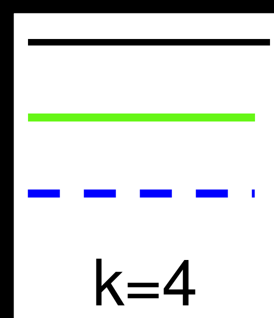
Rozhnova & Nunes, Phys. Rev. E 79, 041922 (2009)

The effect of correlations
analytical description

Beyond the pair approximation: SIRS on a k-RRG



Higher order cluster approximations are needed to describe some regimes of the dynamics on networks



simulations $N=10^6$

analytic TA

analytic PA

Rozhnova & Nunes,
Phys. Rev. E 80, 051915 (2009)

The effect of correlations
analytical description

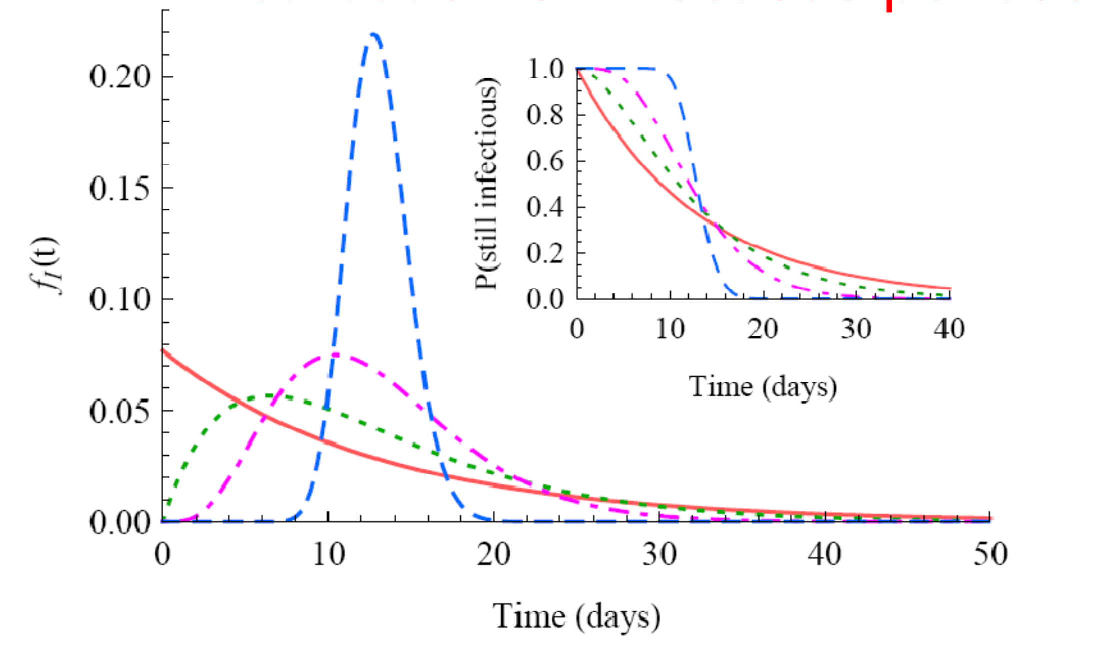
Stochastic d-SIR with distributed infectious periods

$$
\begin{array}{llcll}
\text{Infection} & S + I_j & \xrightarrow{\beta} & I_j + I_1 & j = 1, \ldots, L \\
\text{Birth/death} & \{I_j, R\} & \xrightarrow{\mu} & S & j = 1, \ldots, L \\
\text{Recovery} & I_j & \xrightarrow{L\gamma} & I_{j+1} & j = 1, \ldots, L-1 \\
 & I_L & \xrightarrow{L\gamma} & R &
\end{array}
$$

Gamma distributed recovery profiles that interpolate between exponentially distributed (L=1) and constant recovery period as L $\longrightarrow$ ¶

The effect of correlations
analytical description

Distribution of infectious periods

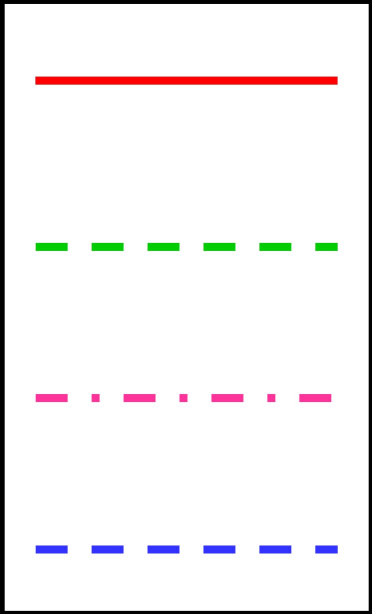


L=1

L=2

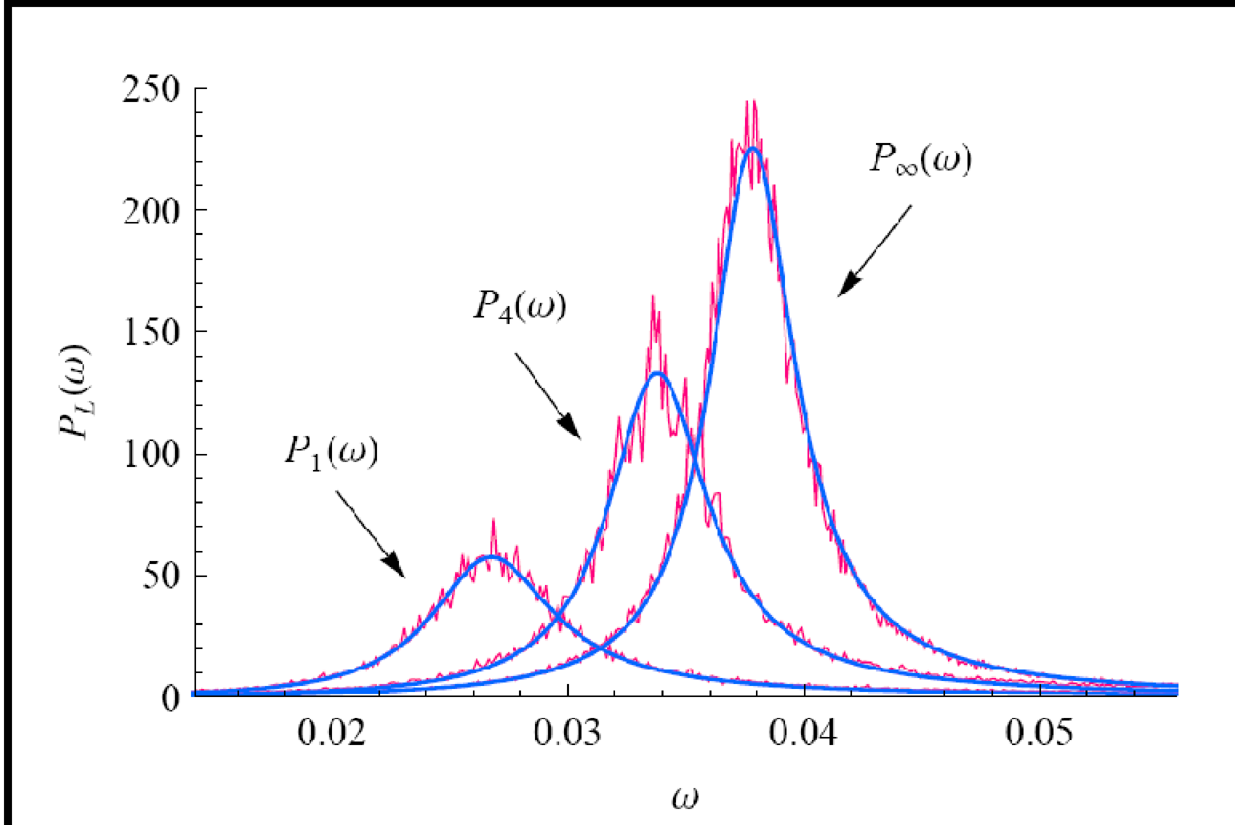
L=5

L=50



For each L, fluctuation power spectra around the equilibrium can be computed using van Kampen’s expansion as before

The effect of correlations
analytical description



The power spectrum can be obtained in closed form in the limit

L ⟶ ¶

The peak frequency is very sensitive to the recovery profile

analytic

simulations N=10^6

Black et al.,
Phys. Rev. E 80, 021922 (2009)

Conclusions & Perspectives

In infection dynamics, intrinsic stochasticity combined with correlations can give rise in finite systems to sustained noisy oscillations compatible with realistic patterns of recurrent epidemics

The power spectrum of the fluctuations around the endemic equilibrium can be computed analytically

The interplay between fluctuations and forcing is not yet well understood

Contact network structure and stochasticity can be treated analytically with cluster approximations and system size expansions

References:

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[3] A.J. McKane, T.J. Newman,
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[4] D. Alonso, A. J. McKane, M. Pascual,
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Cluster approximations for infection dynamics on random networks
Phys. Rev. E 80, 051915 (2009

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