Superconvergence of projection methods for weakly singular integral operators

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A Weakly Singular Integral Equation

Given $\tau_* > 0$, $\omega_0 \in [0, 1[$, g such that

- $g(0^+) = +\infty$,
- g is continuous, positive and decreasing on $]0, \infty[$,
- $g \in L^1(\mathbb{R}) \cap W^{1,1}(\delta, +\infty)$ for all $\delta > 0$,
- $\|g\|_{L^1(\mathbb{R}^+)} \leq \frac{1}{2}$,

and a function f, find a function φ such that

$$\varphi(\tau) = \omega_0 \int_0^{\tau_*} g(|\tau - \tau'|)\varphi(\tau')\, d\tau' + f(\tau), \quad \tau \in [0, \tau_*].$$

An example: The Transfer Equation

$$\varphi(\tau) = \frac{\omega_0}{2} \int_0^{\tau_*} E_1(|\tau - \tau'|)\varphi(\tau')\,d\tau' + f(\tau), \quad \tau \in [0, \tau_*],$$

$$E_1(\tau) := \int_0^1 \mu^{-1} e^{-\tau/\mu}\,d\mu, \quad \tau > 0,$$

f, the source term, belongs to $L^1(0, \tau_*)$,
τ_*, the optical depth, is a very large number,
ω_0, the albedo, may be very close to 1.

Goal

For a class of four numerical solutions based on projections, find acurate error estimates which

- are independent of the grid regularity,
- are independent of τ_*,
- depend on ω_0 in an explicit way, and
- suggest global superconvergence phenomena.

Abstract framework

Set

$$(\Lambda\varphi)(\tau) := \int_0^{\tau_*} g(|\tau - \tau'|)\varphi(\tau')\, d\tau', \quad \tau \in [0, \tau_*].$$

For X and Y, suitable Banach spaces, the problem reads:

Given $f \in Y$, find $\varphi \in X$ such that $\quad \varphi = \omega_0 \Lambda\varphi + f.$

Remark:

$$\|(I - \omega_0\Lambda)^{-1}\| \leq \gamma_0 := \frac{1}{1 - \omega_0}.$$

The equality holds for $X = Y = L^2(0, \tau_*)$.

Projecting onto piecewise constant functions

Let be a grid of $n+1$ points in $[0, \tau_*]$:

$$
\begin{aligned}
0 =: \tau_0 < \tau_1 < \quad \dots \quad & < \tau_{n-1} < \tau_n := \tau_*, \\
h_i \quad &:= \quad \tau_i - \tau_{i-1}, \quad i \in [\![1, n]\!], \\
h \quad &:= \quad (h_1, h_2, \dots, h_n), \\
\mathcal{G}^h \quad &:= \quad (\tau_0, \tau_1, \dots, \tau_n), \\
\hat{h} \quad &:= \quad \max_{i \in [\![1,n]\!]} h_i, \\
\mathcal{I}_{i-\frac{1}{2}} \quad &:= \quad]\tau_{i-1}, \tau_i[, \quad i \in [\![1, n]\!], \\
\tau_{i-\frac{1}{2}} \quad &:= \quad (\tau_{i-1} + \tau_i)/2, \quad i \in [\![1, n]\!].
\end{aligned}
$$

The approximating space $\mathbb{P}_0^h(0, \tau_*)$

$$f \in \mathbb{P}_0^h(0, \tau_*) \iff \forall i \in [\![1, n]\!], \forall \tau \in \mathcal{I}_{i-\frac{1}{2}}, \quad f(\tau) = f(\tau_{i-\frac{1}{2}}).$$

The family of projections π_h:

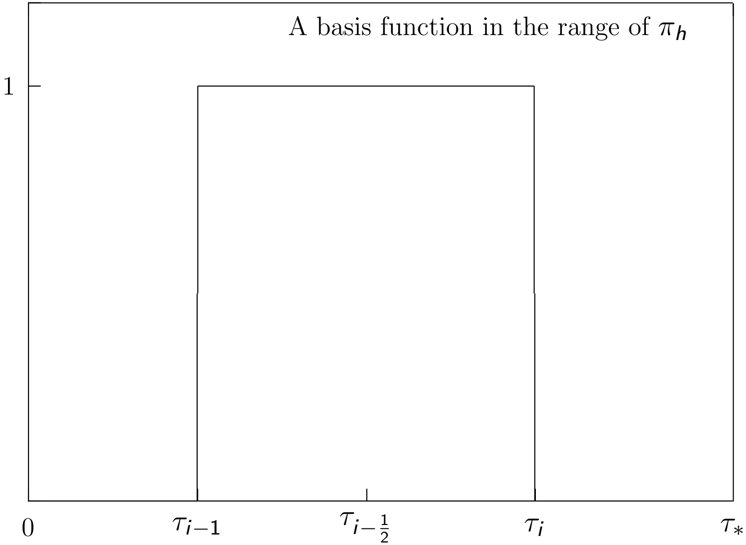
$$\pi_h : L^p(0, \tau_*) \to L^p(0, \tau_*)$$

$\forall i \in [\![1, n]\!], \forall \tau \in \mathcal{I}_{i-\frac{1}{2}}$,

$$(\pi_h \varphi)(\tau) := \frac{1}{h_i} \int_{\tau_{i-1}}^{\tau_i} \varphi(\tau')\, d\tau'.$$

Hence

$$\pi_h(L^p(0, \tau_*)) = \mathbb{P}_0^h(0, \tau_*).$$


A basis function in the range of π_h
1
0
τ_{i-1}
$\tau_{i-\frac{1}{2}}$
τ_i
τ_*

Four approximations based on π_h

The classical Galerkin approximation φ_h^{G} solves

$$\varphi_h^{\mathrm{G}} = \omega_0 \pi_h \Lambda \varphi_h^{\mathrm{G}} + \pi_h f.$$

The Sloan approximation φ_h^{S} (iterated Galerkin) solves

$$\varphi_h^{\mathrm{S}} = \omega_0 \Lambda \pi_h \varphi_h^{\mathrm{S}} + f.$$

The Kantorovich approximation φ_h^{K} solves

$$\varphi_h^{\mathrm{K}} = \omega_0 \pi_h \Lambda \varphi_h^{\mathrm{K}} + f.$$

The Authors' approximation φ_h^{A} (iterated Kantorovich) solves

$$\varphi_h^{\mathrm{A}} = \omega_0 \Lambda \pi_h \varphi_h^{\mathrm{A}} + f + \omega_0 \Lambda (I - \pi_h) f.$$

Useful remarks and relationships

- $\varphi_h^{\mathrm{G}} \in \mathbb{P}_0^h(0, \tau_*)$,
- φ_h^{G} is computed through an algebraic linear system,
- $\varphi_h^{\mathrm{S}} = \omega_0 \Lambda \varphi_h^{\mathrm{G}} + f$,
- $\psi_h = \omega_0 \pi_h \Lambda \psi_h + \pi_h \Lambda f \implies \varphi_h^{\mathrm{K}} = \omega_0 \psi_h + f$,
- $\phi_h = \omega_0 \Lambda \pi_h \phi_h + \Lambda f \implies \varphi_h^{\mathrm{A}} = \omega_0 \phi_h + f$.

The absolute errors and superconvergence

$$\varepsilon_h^{\mathrm{N}} := \varphi_h^{\mathrm{N}} - \varphi \qquad \text{for} \quad \mathrm{N} \in \{\mathrm{G}, \mathrm{K}, \mathrm{S}, \mathrm{A}\}.$$

Superconvergence is understood with respect to $\operatorname{dist}(\varphi, \mathbb{P}_0^h(0, \tau_*))$.

Theorem

In any Hilbert space setting, $\exists\ \alpha$ such that

$$\|\varphi - \pi_h\varphi\| \leq \|\varepsilon_h^{\mathrm{G}}\| \leq \alpha\|\varphi - \pi_h\varphi\|.$$

Main technical notions and results

Definitions

$$
\begin{aligned}
\Delta_\epsilon g(\tau) &:= g(\tau+\epsilon) - g(\tau), \\
\omega_1(g,\delta) &:= \sup_{0<\epsilon<\delta} \|\Delta_\epsilon g(|\cdot|)\|_{L^1(\mathbb{R})} \\
&= \sup_{0<\epsilon<\delta} \int_{\mathbb{R}} |g(|\tau+\epsilon|) - g(|\tau|)| \, d\tau.
\end{aligned}
$$

Main technical notions and results

The key for estimating errors

For $\alpha \in [0,1]$ and $i \in [\![1,n]\!]$:

$$\Delta_\alpha^h f(\tau) := \begin{cases} \Delta_{\alpha h_i} f(\tau) & \text{for } \tau \in [\tau_{i-1}, \tau_i - \alpha h_i], \\ 0 & \text{for } \tau \in]\tau_i - \alpha h_i, \tau_i], \end{cases}$$

$$\omega_p(f, \mathcal{G}^h) := 2^{1/p}\left[\int_0^1 \|\Delta_\alpha^h f\|_p^p \, d\alpha\right]^{1/p} \text{ for } p < +\infty,$$

$$\omega_\infty(f, \mathcal{G}^h) := \max_{i \in [\![1,n]\!]} \operatorname{essup}_{(\tau,\tau') \in [\tau_{i-1},\tau_i]^2} |f(\tau) - f(\tau')|,$$

$$\widehat{\omega}_1(g, \mathcal{G}^h) := \sup_{0 < \tau' < \tau_*} \omega_1(g(|\cdot - \tau'|, \mathcal{G}^h).$$

Global superconvergence

The main result

Theorem

If $0 \neq f \in L^p(0,\tau_*)$ *for some* $p \in [1,+\infty]$, *then* $0 \neq \varphi \in L^p(0,\tau_*)$ *and*

$$\frac{\|\varepsilon_h^{\mathrm{K}}\|_p}{\|\varphi\|_p} \leq 4 \cdot 3^{\frac{1}{p}} \cdot \omega_0 \cdot \gamma_0 \cdot \int_0^{\hat{h}} g(\tau)\,d\tau,$$

$$\frac{\|\varepsilon_h^{\mathrm{S}}\|_p}{\|\varphi\|_p} \leq 12 \cdot 3^{-\frac{1}{p}} \cdot \omega_0 \cdot \gamma_0 \cdot \int_0^{\hat{h}} g(\tau)\,d\tau,$$

$$\frac{\|\varepsilon_h^{\mathrm{A}}\|_p}{\|\varphi\|_p} \leq 48 \cdot \omega_0^2 \cdot \gamma_0 \cdot \left[\int_0^{\hat{h}} g(\tau)\,d\tau\right]^2.$$

The proof

First step

For all $f \in L^p(0, \tau_*)$,

$$\|(I - \pi_h)f\|_p \leq \omega_p(f, \mathcal{G}^h).$$

The proof

Second step

$$\frac{\|\varepsilon_h^{\mathrm{K}}\|_p}{\|\varphi\|_p} \leq \omega_0 \cdot \gamma_0 \cdot \|(I - \pi_h)\Lambda\|_p,$$

$$\frac{\|\varepsilon_h^{\mathrm{S}}\|_p}{\|\varphi\|_p} \leq \omega_0 \cdot \gamma_0 \cdot \|\Lambda(I - \pi_h)\|_p,$$

$$\frac{\|\varepsilon_h^{\mathrm{A}}\|_p}{\|\varphi\|_p} \leq \omega_0^2 \cdot \gamma_0 \cdot \|\Lambda(I - \pi_h)\Lambda\|_p.$$

The proof

Third step

$$
\begin{aligned}
\|(I - \pi_h)\Lambda\|_p &\leq \omega_1(g, \hat{h})^{1-\frac{1}{p}}\, \widehat{\omega}_1(g, \mathcal{G}^h)^{\frac{1}{p}}, \\
\|\Lambda(I - \pi_h)\|_p &\leq \omega_1(g, \hat{h})^{\frac{1}{p}}\, \widehat{\omega}_1(g, \mathcal{G}^h)^{1-\frac{1}{p}}, \\
\|\Lambda(I - \pi_h)\Lambda\|_p &\leq \omega_1(g, \hat{h})\, \widehat{\omega}_1(g, \mathcal{G}^h).
\end{aligned}
$$

The proof

Fourth step

$$\omega_1(g, \hat{h}) \leq 4 \int_0^{\hat{h}} g(\tau)\, d\tau,$$

$$\widehat{\omega}_1(g, \mathcal{G}^h) \leq 12 \int_0^{\hat{h}} g(\tau)\, d\tau.$$

The proof

Desired conclusion

$$\frac{\|\varepsilon_h^{\mathrm{K}}\|_p}{\|\varphi\|_p} \leq 4 \cdot 3^{\frac{1}{p}} \cdot \omega_0 \cdot \gamma_0 \cdot \int_0^{\hat{h}} g(\tau)\, d\tau,$$

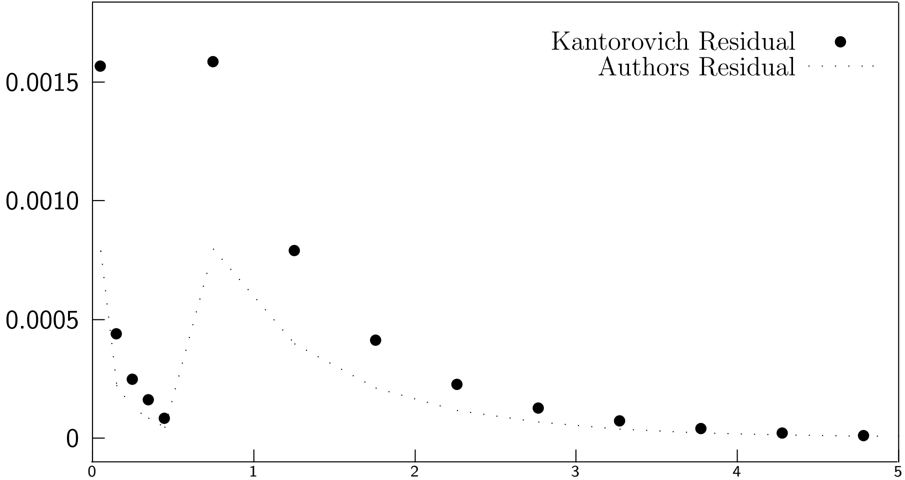
$$\frac{\|\varepsilon_h^{\mathrm{S}}\|_p}{\|\varphi\|_p} \leq 12 \cdot 3^{-\frac{1}{p}} \cdot \omega_0 \cdot \gamma_0 \cdot \int_0^{\hat{h}} g(\tau)\, d\tau,$$

$$\frac{\|\varepsilon_h^{\mathrm{A}}\|_p}{\|\varphi\|_p} \leq 48 \cdot \omega_0^2 \cdot \gamma_0 \cdot \left[\int_0^{\hat{h}} g(\tau)\, d\tau\right]^2.$$

Numerical evidence

- $g = \frac{1}{2}E_1 \implies \int_0^{\hat{h}} g(\tau)\, d\tau = O(\hat{h} \ln \hat{h})$
- Data: $\omega_0 = 0.5, \quad \tau_* = 500, \quad f(\tau) = 0.5$
- Number of grid points: $n = 1000$
- First 5 points equally spaced by 0.1
- Last 5 points equally spaced by 0.1
- Between 0.5 and 499.5: a uniform grid with 990 points
- $\hat{h} = 0.5\overline{04}$

Plotting residuals on $[0, 5]$



Looking for generalizations

A joint work with F. d'Almeida (Porto) and R. Fernandes (Braga)

X, a complex Banach space,

T, a bounded linear operator from X into itself,

$0 \neq \zeta \in \mathrm{re}(T)$, the resolvent set of T, and $f \in X$.

Problem:

Find $\varphi \in X$ such that $(T - \zeta I)\varphi = f$.

Looking for generalizations

A family of numerical schemes

Use an approximate operator T_n such that $\mathrm{re}(T) \subseteq \mathrm{re}(T_n)$ and an approximate second member f_n to solve exactly

$$(T_n - \zeta I)\varphi_n = f_n.$$

Remark that the resolvents

$$R(\zeta) := (T - \zeta I)^{-1}, \quad R_n(\zeta) := (T_n - \zeta I)^{-1}$$

satisfy

$$R_n(\zeta) - R(\zeta) = R(\zeta)(T - T_n)R_n(\zeta),$$

hence the error,

$$\varepsilon_n := \varphi_n - \varphi = R(\zeta)[f_n - f + (T - T_n)\varphi_n].$$

Looking for generalizations

Iterating

A fixed point approach leads to build the so-called

Iterated approximate solution $\widetilde{\varphi}_n := \frac{1}{\zeta}(T\varphi_n - f).$

The error of $\widetilde{\varphi}_n$ is related with the error φ_n by

$$\widetilde{\varepsilon}_n = \frac{1}{\zeta} T\varepsilon_n = \frac{1}{\zeta} R(\zeta) T[f_n - f + (T - T_n)\varphi_n].$$

Under some conditions on T and T_n we may expect that $\|\varphi_n\|$ is uniformly bounded in n, and that $T(T - T_n)$, $T(f_n - f)$ tend to 0 faster than $(T - T_n)\varphi_n$ and $f_n - f$, respectively.