Variational Methods in Materials and Imaging Problems

Irene Fonseca

Department of Mathematical Sciences Center for Nonlinear Analysis Carnegie Mellon University

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equilibria \sim minima of energies

• bulk and interfacial energies

- multiple scales
- higher order derivatives
- discontinuities of underlying fields

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several settings ...

Imaging

- Thin Structures
- Micromagnetic Materials
- Foams
- Quantum Dots
- etc.

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Mumford-Shah model Staircasing in Imaging Inpainting

Mumford-Shah model

$$E(u) = \int_{\Omega} \left(|\nabla u|^{p} + |u - f|^{2} \right) dx + \int_{S(u)} \gamma(\nu) d\mathcal{H}^{N-1}$$

 $p \ge 1$, $p = 1 \dots TV$ model $u \in BV$ (bounded variation

$$Du = \nabla u \mathcal{L}^{N} \lfloor \Omega + [u] \otimes \nu \mathcal{H}^{N-1} \lfloor S(u) + C(u)$$

[De Giorgi, Ambrosio, Carriero, Leaci, Chan, Osher, et. al.]



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second order models in imaging

Blake-Zisserman Model [Leaci and Tomarelli, et.al.]

$$E(u) = \int_{\Omega} W(\nabla u, \nabla^2 u) \, dx + |u - f|^2 dx + \int_{\mathcal{S}(\nabla u)} \gamma(\nu) dH^{N-1}$$

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staircasing

T. Chan, A. Marquina and P. Mulet, SIAM J. Sci. Comput. 22 (2000), 503-516



FIG. 8.2. (a) Results of TV restoration; (b) results of our model.

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more on staircasing

[With G. Dal Maso, G. Leoni, M. Morini]

$$E(u) = \int_{\Omega} \left(|\nabla u| + |u - f|^2 \right) \, dx + \int_{\Omega} \psi(|\nabla u|) |\nabla^2 u|^p \, dx$$

$$\begin{array}{ll} p \geq 1, & \psi \sim 0 \text{ at } \infty \\ \text{e.g. } \psi(t) \leq \frac{c}{t^{\alpha}} \text{ for } t \geq 1, \ \alpha > 1 \\ & \int_{\infty}^{\infty} \psi(t) \, dt < +\infty, \qquad \inf_{t \in K} \psi(t) > 0 \end{array}$$
for every compact $K \subset \mathbb{R}$

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 $p \geq 1$, $\psi \sim 0$ at ∞ e.g. $\psi(t) \leq rac{c}{t^{lpha}}$ for $t \geq 1$, $\alpha > 1$

$$\int_{-\infty}^{\infty}\psi(t)\,dt<+\infty,\qquad \inf_{t\in K}\psi(t)>0$$

for every compact $K \subset \mathbb{R}$

(a)

Mumford-Shah model Staircasing in Imaging Inpainting

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more on staircasing

ocmpactness

• relaxed functional:

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{F}(u_k) : u_k \to u \text{ in } L^1(]a, b[) \right\}$$

• $f_k := f + h_k$, f smooth, $h_k \stackrel{*}{\rightharpoonup} 0$ Is u_k smooth for k >> 1?

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more on staircasing

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more on staircasing

- compactness
- relaxed functional:

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Mumford-Shah model Staircasing in Imaging Inpainting

inpainting

[With G. Leoni, F. Maggi, M. Morini] <u>Restoration</u> of color images by vector-valued *BV* functions Recovery is obtained from few, sparse *complete* samples and from

a significant *incomplete* information



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recolorization





Restoration of frescoes of Andrea Mantegna, Ovetari Chapel of the Eremitani's Church, Padova (Italy); destroyed by bombing in WWII

[courtesy of Massimo Fornasier et. al]

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inpainting

$D \subset \Omega \subset \mathbb{R}^2$... inpainting region

Minimize

$$\int_{\Omega} \phi(\nabla u) \, dx + \lambda \int_{D} |\mathcal{L}(u) - \bar{v}|^p \, dx + \mu \int_{\Omega \setminus D} |u - \bar{u}|^p \, dx$$

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Mumford-Shah model Staircasing in Imaging Inpainting

more on inpainting

RGB ... $u : \Omega \to [0, \infty)^3$ channels u(x) = (r(x), g(x), b(x))observed (\bar{u}, \bar{v}) \bar{u} ... correct information \bar{v} ... distorted information ... only gray level is known on D $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}$... e.g. $\mathcal{L}(u) := \frac{1}{3}(r+g+b)$

Goal

to produce a new color image that extends colors of the fragments to the gray region, constrained to match the known gray level

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- "optimal design" : what is the "best" *D*? How much color do we need to provide?
- are we creating spurious edges? $\operatorname{spt} Du_i \subset \operatorname{spt} D\overline{v}$?

(a)

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-



Hyperelastic material with reference configuration

$$\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$$



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Bending: First Order Bending: Second Order Order

more on thin structures

 $W: \Omega(\varepsilon) \times \mathbb{R}^{3 \times 3} \to [0, \infty) \dots$ stored energy density $g: \mathbb{R}^3 \times S^2 \to [0, +\infty) \dots$ surface energy density

 $x_{\alpha} := (x_1, x_2) \dots$ in-plane variables

 $l(\varepsilon)$... dead loads

 $au(arepsilon)\ldots$ surface traction on $\Sigma_arepsilon:=\omega imes\{-arepsilon\}\cup\omega imes\{arepsilon\}$

v(x) = x on $\partial \omega \times (-\varepsilon, \varepsilon)$... pinned on lateral bdry

(a)

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[Acerbi, Buttazzo Percivale, Anzellotti, Baldo, Le Dret, Raoult, Braides, Francfort, Müller, Bhattacharya, ...]

 $\Omega(\varepsilon) \to \Omega := \omega \times (-1, 1)$ Σ^+ Ω 1 ω 1 Σ $(x_{\alpha}, x_3) \mapsto (x_{\alpha}, \frac{x_3}{\varepsilon}), \ u(x) = v(x_{\alpha}, \varepsilon x_3),$ $x \in \Omega$

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membrane regime if

 $I(\varepsilon)(x_{\alpha}, \varepsilon x_{3}) = I(x_{\alpha}, x_{3}) \quad x \in \Omega$ $\tau(\varepsilon)(x_{\alpha}, \varepsilon x_{3}) = \varepsilon \tau(x_{\alpha}, x_{3}) \quad x \in \omega \times \{-1, 1\}$

 $u \in W^{1,p}(\Omega; \mathbb{R}^3), \ u(x) = v(x_{\alpha}, \varepsilon x_3)$

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membrane regime if

$$\begin{split} &l(\varepsilon)(x_{\alpha},\varepsilon x_{3})=l(x_{\alpha},x_{3}) \quad x\in \Omega \\ &\tau(\varepsilon)(x_{\alpha},\varepsilon x_{3})=\varepsilon\tau(x_{\alpha},x_{3}) \quad x\in\omega\times\{-1,1\} \end{split}$$

$$\begin{split} &\frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} W\left(Dv\right) dy + \int_{S(v)} g([v], \nu(v)) d\mathcal{H}^{2} = \int_{\Omega} W\left(D_{\alpha}u, \frac{1}{\varepsilon}D_{3}u\right) dx \\ &+ \int_{S(u)} g\left([u], \frac{\left(\nu_{\alpha}(u), \frac{1}{\varepsilon}\nu_{3}(u)\right)}{\left|\left(\nu_{\alpha}(u), \frac{1}{\varepsilon}\nu_{3}(u)\right)\right|}\right) \left|\left(\nu_{\alpha}(u), \frac{1}{\varepsilon}\nu_{3}(u)\right)\right| d\mathcal{H}^{2} \end{split}$$

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Bending: First Order Bending: Second Order Order

nonlocal?

[With G. Bouchitté and L. Mascarenhas] $W \sim |\xi|^p \qquad 1$

$$E(u,b) := \inf \left\{ \liminf \int_{\Omega} W\left(D_{\alpha} u_n | \frac{1}{\varepsilon_n} D_3 u_n \right) dx | \\ u_n \rightharpoonup u \quad W^{1,p} \quad \text{and} \quad \frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \quad L^p \right\}$$

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singular perturbations

[With F., Francfort and Leoni]

$$\begin{split} &\frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} \left[W\left(Dv_{\varepsilon} \right) + \varepsilon^{\gamma} \left| D^{2} v_{\varepsilon} \right|^{2} \right] dx \\ &= \int_{\Omega} \left[W\left(D_{\alpha} u_{\varepsilon} \left| \frac{1}{\varepsilon} D_{3} u_{\varepsilon} \right) \right. \\ &\left. + \varepsilon^{\gamma} \left(|D_{\alpha\beta}^{2} u_{\varepsilon}|^{2} + \frac{1}{\varepsilon^{2}} |D_{\alpha3}^{2} u_{\varepsilon}|^{2} + \frac{1}{\varepsilon^{4}} |D_{33}^{2} u_{\varepsilon}|^{2} \right) \right] dx \end{split}$$

 $u_{\varepsilon}(x) := v_{\varepsilon}(x_{\alpha}, \varepsilon x_3), \qquad \frac{1}{\varepsilon}D_3u_{\varepsilon} \rightharpoonup b$

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$\gamma = 2$

[Bhattacharya and James, Shu]

$$\begin{split} \int_{\Omega} \left[W\left(D_{\alpha} u_{\varepsilon} \Big| \frac{1}{\varepsilon} D_{3} u_{\varepsilon} \right) + \varepsilon^{2} |D_{\alpha\beta}^{2} u_{\varepsilon}|^{2} + |D_{\alpha3}^{2} u_{\varepsilon}|^{2} + \frac{1}{\varepsilon^{2}} |D_{33}^{2} u_{\varepsilon}|^{2} \right] dx \\ u_{\varepsilon}(x) &:= v_{\varepsilon}(x_{\alpha}, \varepsilon x_{3}), \qquad \frac{1}{\varepsilon} D_{3} u_{\varepsilon} \rightharpoonup b \end{split}$$

F-limit recently obtained!!!

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nonlocal?

WE STRONGLY SUSPECT THAT IT IS **NON LOCAL** in the *ORIGIANL* model without second order penalization)!

i.e. can't write

$$E(u,b) = \int_{\Omega} \tilde{W}(D_{\alpha}u(x_{\alpha}), b(x), D_{3}b(x)) dx$$

it is **NON LOCAL** with second order penalization (Dal Maso, F., Leoni)

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(a)

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$$\begin{cases} \operatorname{div} (h + \chi_{\Omega} m) = 0 & \operatorname{in} \mathcal{D}'(\mathbb{R}^3) \\ \operatorname{curl} h = 0 & \operatorname{in} \mathcal{D}'(\mathbb{R}^3) \\ |m| = 1 & \operatorname{a.e. in} \Omega \end{cases}$$

$$\int_{\Omega} \varepsilon^{\alpha} |\nabla m_{\varepsilon}|^2 + \frac{1}{\varepsilon^{\beta}} \varphi(m_{\varepsilon}) + \int_{\mathbb{R}^3} \frac{1}{\varepsilon^{\gamma}} |h_{\varepsilon}|^2$$

$$\dots \nabla m_{\varepsilon} \sim \nabla^2 \dots \alpha = \beta = \gamma = 1$$

[DeSimone, Kohn, Müller, Otto, Rivière, Serfaty, etc.]

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$$\int_{\Omega} \varepsilon^{\alpha} |\nabla m_{\varepsilon}|^2 + \frac{1}{\varepsilon^{\beta}} \varphi(m_{\varepsilon}) + \int_{\mathbb{R}^3} \frac{1}{\varepsilon^{\gamma}} |h_{\varepsilon}|^2$$

 $\dots \nabla m_{\varepsilon} \sim \nabla^2 \dots \alpha = \beta = \gamma = 1$

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

surfactants and foams: a singular perturbation problem



Irene Fonseca

Variational Methods in Materials and Imaging Problems

	The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ	
foams			

aequous foams, polymeric foams, metallic foams (AL_2O_3)

- metal foams: cellular structure of solid material (e.g. Aluminium) w/ large (\geq 80 % volume fraction of gas-filled pores); size of pores: 1mm–8mm
- open-cell foams (interconnected network): lightweight optics, advanced aerospace technology
- closed-cell foams (sealed pores): high impact loads

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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	The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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	The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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	The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
what for		

• detergents, oil recovery, snowboard wax, hair conditioner

- scaffolds for bone tissue engineering, biotechnology
- lightweight structural materials
- . . .

little liquid in thin film that separates adjacent bubbles; most liquid in *Plateau Borders/Struts* (region between 3 touching bubbles)

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

Plateau borders/struts; nodes/joints



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 The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

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surfactants

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organic compounds - wetting agents

Amphiphiles: hydrophobic tail and hydrophilic head \rightsquigarrow decrease surface tension; allow easier spreading; lower interfacial tension between two liquids

→→ stabilizing effect of multiple interfaces configurations – *Gibbs-Marangoni effect* : mass transfer due to gradients in surface tension

- water-oil ~> micelles of surfactants
- water-air \rightsquigarrow monolayers of surfactants at the interface, r_{\pm} , r_{\pm}

 The Issues
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

 Foams
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 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

 Foams
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The Issues The Context The Phase Field Model Imaging The Theorem Thin Structures The Surface Energy Density Micromagnetics The Convexity of Φ Foams

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
micelle	



A micelle of surfactant molecules surrounding a drop of oil in water

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Monolayers of surfactants molecules separated by a thin film of liquid

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The Issues The Context The Phase Field Model Imaging The Theorem Thin Structures The Surface Energy Density Micromagnetics The Convexity of Φ Foams

surfactants effects

• determine macroscopic properties

- segregate to interfaces
- facilitate formation of interfaces (wetness)

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 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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surfactants effects

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the phase field model

[R. Perkins, R. Sekerka, J. Warren and S. Langer]

Modification of the Cahn-Hilliard model for fluid-fluid phase transitions

$$F_{\varepsilon}^{CH}(u) := \int_{\Omega} \left(\frac{1}{\varepsilon} f(u) + \varepsilon |\nabla u|^2 \right) dx$$

f ... double well potential $\ldots \{f = 0\} = \{0, 1\}$



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 $\begin{array}{l} \mbox{The Phase Field Model} \\ \mbox{The Theorem} \\ \mbox{The Surface Energy Density} \\ \mbox{The Convexity of } \Phi \end{array}$

the phase field model

To this we add

$$F^{\rm surf}_{\alpha(\varepsilon)}(u,\rho_s) := \alpha(\varepsilon) \int_{\Omega} (\rho_s - |\nabla u|)^2 + \beta(\varepsilon) \int_{\Omega} |\nabla \rho_s|^2$$

$$F_{\varepsilon}(u,\rho) := \int_{\Omega} \left(\frac{1}{\varepsilon} f(u) + \varepsilon |\nabla u|^2 + \varepsilon \left(\rho - |\nabla u| \right)^2 \right) \, dx$$

 $\rho_s \geq 0 \dots$ surfactant density

 $\int_{\Omega} \rho_s = b$

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

asymptotic behavior

<u>Goal</u>: Study the asymptotics as $\varepsilon \to 0^+$ ($\beta(\varepsilon) = 0$)

$$F_{\varepsilon}(u, \rho_{s}) := F_{\varepsilon}^{CH}(u) + F_{\alpha(\varepsilon)}^{\mathrm{surf}}(u, \rho_{s})$$

minima of F_{ε} converge to minima of F ...

Theorem

 $\Gamma\text{-limit } F_{\varepsilon}(u,\mu) = F(u,\mu) \quad \text{ in } BV(\Omega; \{0,1\}) \times \mathcal{M}(\Omega)$

Different regimes:

- $\alpha(\varepsilon) \sim \varepsilon$
- arepsilon << lpha(arepsilon) << 1 . . . surfactant plays no role
- $\alpha(\varepsilon) << \varepsilon$... creation of interfaces needs ∞ energy

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 The Issues
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

 Foams
 The Convexity of Φ

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The Issues
The Context
Imaging
Thin Structures
Micromagnetics
Foams
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$$\alpha(\varepsilon) = \varepsilon$$

$$F(u,\mu) := \begin{cases} \int_{S(u)} \Phi\left(\frac{d\mu}{d\mathcal{H}^{N-1} \lfloor S(u)}(x)\right) d\mathcal{H}^{N-1} \\ & \text{if } u \in BV(\Omega; \{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

- $\bullet~\Phi$ described by optimal profile problem
- Φ nonincreasing, convex
- Φ is constant on $[1, +\infty)$... saturation

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Imaging
Thin Structures
Micromagnetics
Foams
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The Context
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Thin Structures
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Imaging
Thin Structures
Micromagnetics
Foams
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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

some conclusions

• surfactant segregates to interface

- prescribed distribution of surfactant dictates location of interfaces
- macroscopic energy *F* unchanged is density of surfactant on the interfaces exceeds 1 . . . *saturation threshold*
- stability implies uniform distribution of surfactant ... Marangoni effect

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surface energy density

$$\Phi(\gamma) := \inf_{(u,\lambda)\in\mathcal{A}(\gamma)} \left\{ \int_{-\infty}^{+\infty} \left[f(u) + \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \right] \right\}$$
$$\min\{\lambda^2 + |u'|^2, 2|u'|^2\} = |u'|^2 + (\max\{\lambda + |u'|, 0\} - |u'|)^2 = |u'|^2 + \min\{\lambda^2, |u'|^2\}$$

$$\rho \sim \max\{\lambda + |u'|, 0\}$$

$$\begin{split} \mathcal{A}(\gamma) &:= \left\{ (u, \lambda) \in H^1_{\text{loc}}(\mathbb{R}) \times (-\infty, 0] : \lim_{t \to -\infty} u(t) = 0, \\ \lim_{t \to +\infty} u(t) &= 1, \int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\} \leq \min\{\gamma, 1\} \right\} \end{split}$$

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

the optimal profile



The solution u to the optimal profile problem and the corresponding ρ

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

properties of the surface energy density

Theorem

 $\Phi \in C^1$, nonincreasing and is convex.

This is not a simple curiosity! It is needed in a crucial way to prove Γ – lim inf in the vectorial case!

For all $\gamma > 0$ the optimal profile $\Phi(\gamma)$ admits a unique solution $(u, \lambda) \in \mathcal{A}(\gamma)$

u nondecreasing and strictly increasing in $\{0 < u < 1\}$

 $\Phi(0)=2\sqrt{2}\int_0^1 f(s)\,ds$ $\Phi(1)=\sqrt{2}\int_0^1 f(s)\,dx$ (Modica-Mortola)

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The Issues The Context Imaging Thin Structures Micromagnetics Foams The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ Stability

Theorem

[Stability "a la" Kohn and Sternberg] $(u_0, \mu_0) \in BV(\Omega; \{0, 1\}) \times \mathcal{M}$ $u_0 \dots$ isolated local minimum of $F(\cdot, \mu_0)$, i.e.

$$F(u_0,\mu_0) < F(v,\mu_0)$$

if $0 < ||u_0 - v||_{L^1} \le \delta$, $\int_{\Omega} u_0 = \int_{\Omega} v$

Then there exists $\{u_{\varepsilon}, \rho_{\varepsilon}\}$ such that $u_{\varepsilon} \to u_0$ in L^1 , $\rho_{\varepsilon} \stackrel{*}{\rightharpoonup} \mu_0$

 $\mu_arepsilon$ local minimizer of $F_arepsilon(\cdot,
ho_arepsilon)$

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 The Issues
 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

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Then there exists $\{u_{\varepsilon}, \rho_{\varepsilon}\}$ such that $u_{\varepsilon} \to u_0$ in L^1 , $\rho_{\varepsilon} \stackrel{*}{\rightharpoonup} \mu_0$

 u_{ε} local minimizer of $F_{\varepsilon}(\cdot, \rho_{\varepsilon})$

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 The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

stability

Theorem

[Stability "a la" Kohn and Sternberg] $(u_0, \mu_0) \in BV(\Omega; \{0, 1\}) \times \mathcal{M}$ $u_0 \dots$ isolated local minimum of $F(\cdot, \mu_0)$, i.e.

$$F(u_0,\mu_0) < F(v,\mu_0)$$

if $0 < ||u_0 - v||_{L^1} \le \delta$, $\int_\Omega u_0 = \int_\Omega v$

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
next	

• 1D metastability of multiple interfaces configurations

- the impact of surfactants on the dynamics of phase separation
- vectorial models . . . more complicated interaction between u and ρ_s , e.g.

$$F_{\varepsilon}(u,\rho_s) = \frac{1}{\varepsilon} \int_{\Omega} f(x,u,\varepsilon \nabla u,\varepsilon \rho_s) \, dx$$

[Acerbi and Bouchitté]

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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ
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	The Issues The Context Imaging Thin Structures Micromagnetics Foams	The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ	
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The Issues
The Context
Imaging
Thin Structures
Micromagnetics
Foams
The Convexity of
$$\Phi$$

$$\Phi(\gamma) := \inf_{(u,\lambda)\in\mathcal{A}(\gamma)} \left\{ \int_{-\infty}^{+\infty} \left[f(u) + |u'|^2 + (\max\{\lambda + |u'|, 0\} - |u'|)^2 \right] \right\}$$

If $\gamma > 1$ then the unique minimizing pair is (u, 0) $(\rho = |u'|)$ with u the solution of the Modica-Mortola (Cahn-Hilliard) profile

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The Issues
The Context
Imaging
Thin Structures
Micromagnetics
Foams
The Phase Field Model
The Theorem
The Surface Energy Density
The Convexity of
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$$\Phi(\gamma) = \lim_{T \to \infty} \inf \left\{ E(u, \lambda; (-T, T)) : (u, \lambda) \in \mathcal{A}_T(\gamma) \right\}$$

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Irene Fonseca Variational Methods in Materials and Imaging Problems

 The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

the convexity of Φ

Theorem

 $\Phi \in C^1$, nonincreasing and is convex.

Proof Step 1: auxilliary energy density

$$\Phi(M, a, b; \gamma) := \inf_{t>0} \inf \left\{ Mt + \int_0^t \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \right\}$$

$$(u,\lambda) \in \mathcal{A}(t,a,b;\gamma)$$
$$\mathcal{A}(t,a,b;\gamma) := \left\{ (u,\lambda) \in H^1(0,t) \times (-\infty,0] : u(0) = a, u(t) = b, \\ \int_0^t \max\{\lambda + |u'|,0\} \le (=) \min\{\gamma, b-a\} \right\}_{z \in \mathbb{Z}}$$

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Variational Methods in Materials and Imaging Problems

the convexity of Φ

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Variational Methods in Materials and Imaging Problems

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Variational Methods in Materials and Imaging Problems



<u>Claim</u>: If $(u, \lambda) \in \mathcal{A}(t, a, b; \gamma)$ minimizes

$$Mt + \int_0^t \min\{\lambda^2 + |u'|^2, 2|u'|^2\}$$

then u is increasing, $(u, \max\{u' + \lambda, 0\})$ minimizes

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among (v, ρ) , $v \in H^1(0, t)$, v(0) = a, v(t) = b, $\rho \ge 0$, $\int_0^t \rho = \min\{\gamma, b - a\}$

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The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of
$$\Phi$$

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The Issues The Context	The Phase Field Model
Imaging	The Theorem
Thin Structures	The Surface Energy Density
Micromagnetics	The Convexity of Φ
Foams	•

Need

Lemma

 (Y,μ) measure space, μ non-atomic positive measure, $g: Y \to [0,+\infty)$ (non-zero) $\in L^1(Y,\mu) \cap L^2(Y,\mu)$, $0 < \gamma \leq \int_Y g \ d\mu$. The problem

$$\min\left\{\int_{Y} (v-g)^2 d\mu : v \ge 0, \int_{Y} v d\mu = \gamma\right\}$$

admits a (μ a.e.) unique solution

$$v := \max\{\lambda + g, 0\}$$

 λ is the unique (\leq 0) constant such that $\int_Y \max\{\lambda + g, 0\} d\mu = \gamma$

The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

the convexity of Φ ; Step 1, cont.

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

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The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

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use Lemma ...

$$\begin{split} \int_0^t |v'|^2 + \int_0^t (\rho - |v'|)^2 &\geq \int_0^t |v'|^2 + \int_0^t (\max\{|v'| + \bar{\lambda}, 0\} - |v'|^2) \\ &= \int_0^t \min\{\bar{\lambda}^2 + |v'|^2, 2|v'|^2\} \\ &\geq \int_0^t \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \\ &= \int_0^t |u'|^2 + \int_0^t (\max\{u' + \lambda, 0\} - |u'|^2) \end{split}$$

with $\overline{\lambda}$ s.t. $\int_0^t \max\{|v'| + \overline{\lambda}, 0\} = \min\{\gamma, b - a\}$

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variations $(u + \varepsilon \varphi, \max\{u' + \lambda, 0\})$ in

$$Mt + \int_0^t |v'|^2 + \int_0^t (
ho - |v'|)^2$$

yield Euler-Lagrange equation

$$2u' - \max\{u' + \lambda, 0\} = C \quad \text{a.e. in } (0, t)$$

for some constant *C* . . . *u*′ is constant . . .

$$u(x) = a + \frac{b-a}{t}x, \quad \lambda = \frac{\min\{\gamma - (b-a), 0\}}{t}$$

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$$\Phi(M, a, b; \gamma) = \min_{t>0} \left\{ Mt + \frac{(b-a)^2}{t} + \frac{[\min\{\gamma - (b-a), 0\}]^2}{t} \right\}$$
$$= \left\{ \begin{array}{c} 2\sqrt{M}\sqrt{(b-a)^2 + (\gamma - (b-a))^2} & \text{if } \gamma \le b-a \\ 2\sqrt{M}(b-a) & \text{otherwise} \end{array} \right\}$$

So, $\Phi(M, a, b; \cdot)$ is convex!

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 The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

the convexity of Φ ; Step 2

 $\frac{\text{Step 2: } f \text{ lsc, piecewise constant on } [0,1]}{\overline{f = M_i} \text{ in } (a_{i-1}, a_i), M_i > 0, 0 = a_0 < a_1 < \ldots < a_m = 1}$

Claim

$$\Phi(\gamma) = \min\left\{\sum_{i=1}^{m} \Phi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \ge 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\}\right\}$$

infimal convolution of $\Phi(M_i, a_{i-1}, a_i; \cdot)$

The infimal convolution of convex, nondecreasing functions is convex, nondecreasing!

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 The Issues

 The Context
 The Phase Field Model

 Imaging
 The Theorem

 Thin Structures
 The Surface Energy Density

 Micromagnetics
 The Convexity of Φ

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 The Issues

 The Context

 Imaging

 The The Orem

 Thin Structures

 Micromagnetics

 Foams

the convexity of Φ ; Step 2

Claim

 $\frac{\text{Step 2: } f \text{ lsc, piecewise constant on } [0,1]}{\overline{f = M_i} \text{ in } (a_{i-1}, a_i), \ M_i > 0, \ 0 = a_0 < a_1 < \ldots < a_m = 1$

 $\Phi(\gamma) = \min\left\{\sum_{i=1}^{m} \Phi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \ge 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\}\right\}$

infimal convolution of $\Phi(M_i, a_{i-1}, a_i; \cdot)$

The infimal convolution of convex, nondecreasing functions is convex, nondecreasing!

The Phase Field Model The Theorem The Surface Energy Density The Convexity of Φ

the convexity of Φ ; Step 3

f any continuous double-well potential, f_n as in Step 2, $f_n \rightarrow f$ decreasingly, $f_n = f$ outside (0, 1)

 Φ_n are convex... DONE if $\Phi_n \rightarrow \Phi$ pointwise

Clearly lim inf $\Phi_n \ge \Phi$. Fix $\varepsilon > 0$; choose $(u, \lambda) \in \mathcal{A}(\gamma)$ s.t. u(-T) = 0, u(T) = 1, some T > 0,

 $E(u,\lambda;(-T,T)) \leq \Phi(\gamma) + \varepsilon$

Then

 $\limsup \Phi_n(\gamma) \le \lim E_n(u,\lambda;(-T,T)) = E(u,\lambda;(-T,T)) \le \Phi(\gamma) + \varepsilon$



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