Variational Methods in Materials and Imaging Problems

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equilibria $\sim$ minima of energies

- bulk and interfacial energies
- multiple scales
- higher order derivatives
- discontinuities of underlying fields

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several settings . . .

- Imaging
- Thin Structures
- Micromagnetic Materials
- Foams
- Quantum Dots
- etc.

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Mumford-Shah model

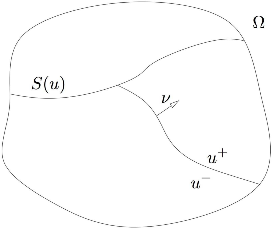
$$E(u) = \int_\Omega \left(|\nabla u|^p + |u - f|^2 \right) dx + \int_{S(u)} \gamma(\nu) d\mathcal{H}^{N-1}$$

$p \geq 1$, $p = 1$ ... **TV model**

$u \in BV$ (bounded variation)

$$Du = \nabla u\, \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu\, \mathcal{H}^{N-1} \lfloor S(u) + C(u)$$

[De Giorgi, Ambrosio, Carriero, Leaci, Chan, Osher, et. al.]



$[u] = u^+ - u^-$

Mumford-Shah model

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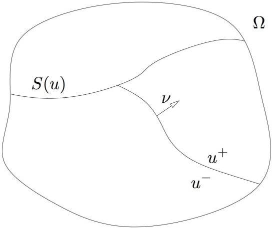

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second order models in imaging

Blake-Zisserman Model [Leaci and Tomarelli, et.al.]

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\, dx + |u - f|^2 dx + \int_{S(\nabla u)} \gamma(\nu) dH^{N-1}$$

staircasing

T. Chan, A. Marquina and P. Mulet, SIAM J. Sci. Comput. **22** (2000), 503–516

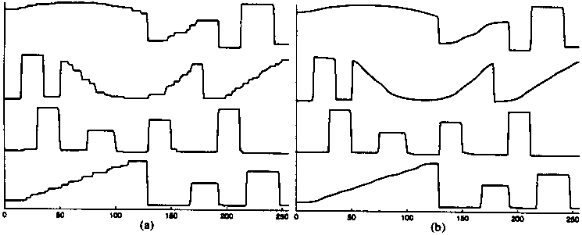

FIG. 8.2. (a) *Results of TV restoration;* (b) *results of our model.*

more on staircasing

[With G. Dal Maso, G. Leoni, M. Morini]

$$E(u) = \int_\Omega \left(|\nabla u| + |u - f|^2\right)\, dx + \int_\Omega \psi(|\nabla u|)|\nabla^2 u|^p\, dx$$

$p \geq 1, \quad \psi \sim 0$ at ∞
e.g. $\psi(t) \leq \frac{c}{t^\alpha}$ for $t \geq 1$, $\alpha > 1$

$$\int_\infty^\infty \psi(t)\, dt < +\infty, \qquad \inf_{t \in K} \psi(t) > 0$$

for every compact $K \subset \mathbb{R}$

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- compactness
- relaxed functional:

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{F}(u_k) : \ u_k \to u \text{ in } L^1(]a,b[) \right\}$$

- $f_k := f + h_k$, f smooth, $h_k \overset{*}{\rightharpoonup} 0$
  Is u_k smooth for $k >> 1$?

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inpainting

[With G. Leoni, F. Maggi, M. Morini]
Restoration of color images by vector-valued *BV* functions

Recovery is obtained from few, sparse *complete* samples and from a significant *incomplete* information

recolorization

Restoration of frescoes of Andrea Mantegna, Ovetari Chapel of the Eremitani's Church, Padova (Italy); destroyed by bombing in WWII

[courtesy of Massimo Fornasier *et. al*]

inpainting

$D \subset \Omega \subset \mathbb{R}^2$ ... *inpainting region*

Minimize

$$\int_\Omega \phi(\nabla u)\, dx + \lambda \int_D |\mathcal{L}(u) - \bar{v}|^p\, dx + \mu \int_{\Omega \setminus D} |u - \bar{u}|^p\, dx$$

more on inpainting

RGB . . . $u : \Omega \to [0, \infty)^3$ channels $u(x) = (r(x), g(x), b(x))$

observed $(\bar{u}, \bar{v})$

$\bar{u}$ . . . *correct* information

$\bar{v}$ . . . *distorted information* . . . only gray level is known on D

$\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}$ . . . e.g. $\mathcal{L}(u) := \frac{1}{3}(r + g + b)$

Goal

to produce a new color image that extends colors of the fragments to the gray region, constrained to match the known gray level

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questions

- "optimal design" : what is the "best" D? How much color do we need to provide?
- are we creating spurious edges? $\mathrm{spt}\, Du_i \subset \mathrm{spt}\, D\bar{v}$?

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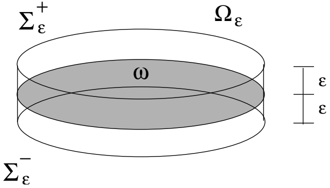
thin structures

Hyperelastic material with reference configuration

$$\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$$



$\omega \subset \mathbb{R}^2$ . . . open, bounded domain

more on thin structures

$W : \Omega(\varepsilon) \times \mathbb{R}^{3\times 3} \to [0, \infty)$ ... stored energy density
$g : \mathbb{R}^3 \times S^2 \to [0, +\infty)$ ... surface energy density

$x_\alpha := (x_1, x_2)$ ... in-plane variables

$l(\varepsilon)$ ... dead loads

$\tau(\varepsilon)$ ... surface traction on $\Sigma_\varepsilon := \omega \times \{-\varepsilon\} \cup \omega \times \{\varepsilon\}$

$v(x) = x$ on $\partial\omega \times (-\varepsilon, \varepsilon)$... pinned on lateral bdry

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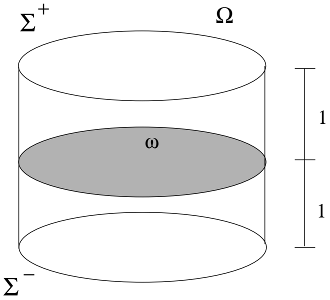
rescale

[Acerbi, Buttazzo Percivale, Anzellotti, Baldo, Le Dret, Raoult, Braides, Francfort, Müller, Bhattacharya, ...]

$$\Omega(\varepsilon) \to \Omega := \omega \times (-1, 1)$$



$$(x_\alpha, x_3) \mapsto \left(x_\alpha, \frac{x_3}{\varepsilon}\right), \; u(x) = v\left(x_\alpha, \varepsilon x_3\right), \quad x \in \Omega$$

membrane regime if

$$I(\varepsilon)(x_\alpha, \varepsilon x_3) = I(x_\alpha, x_3) \quad x \in \Omega$$

$$\tau(\varepsilon)(x_\alpha, \varepsilon x_3) = \varepsilon \tau(x_\alpha, x_3) \quad x \in \omega \times \{-1, 1\}$$

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(Dv)\,dy + \int_{S(v)} g([v], \nu(v))\,d\mathcal{H}^2 = \int_\Omega W\left(D_\alpha u, \frac{1}{\varepsilon}D_3 u\right)dx$$

$$+ \int_{S(u)} g\left([u], \frac{(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u))}{|(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u))|}\right)\left|\left(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\right)\right| d\mathcal{H}^2$$

$$u \in W^{1,p}(\Omega; \mathbb{R}^3),\ u(x) = v(x_\alpha, \varepsilon x_3)$$

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nonlocal?

[With G. Bouchitté and L. Mascarenhas]
$W \sim |\xi|^p \qquad 1 < p < +\infty$

$$E(u,b) := \inf \left\{ \liminf \int_\Omega W\left(D_\alpha u_n | \frac{1}{\varepsilon_n} D_3 u_n \right) \, dx | \right.$$

$$\left. u_n \rightharpoonup u \quad W^{1,p} \quad \text{and} \qquad \frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \quad L^p \right\}$$

singular perturbations

[With F., Francfort and Leoni]

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left[W\left(Dv_\varepsilon\right)+\varepsilon^\gamma\left|D^2v_\varepsilon\right|^2\right]\,dx$$
$$=\int_\Omega\left[W\left(D_\alpha u_\varepsilon\middle|\frac{1}{\varepsilon}D_3u_\varepsilon\right)\right.$$
$$\left.+\varepsilon^\gamma\left(|D^2_{\alpha\beta}u_\varepsilon|^2+\frac{1}{\varepsilon^2}|D^2_{\alpha 3}u_\varepsilon|^2+\frac{1}{\varepsilon^4}|D^2_{33}u_\varepsilon|^2\right)\right]\,dx$$

$u_\varepsilon(x) := v_\varepsilon(x_\alpha, \varepsilon x_3)$, $\quad \frac{1}{\varepsilon}D_3u_\varepsilon \rightharpoonup b$

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$$
\begin{aligned}
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&=\int_\Omega\left[W\left(D_\alpha u_\varepsilon\Big|\frac{1}{\varepsilon}D_3 u_\varepsilon\right)\right. \\
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$\gamma = 2$

[Bhattacharya and James, Shu]

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Γ-limit recently obtained!!!

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nonlocal?

WE STRONGLY SUSPECT THAT IT IS **NON LOCAL** in the *ORIGIANL* model without second order penalization)!

i.e. can't write

$$E(u,b) = \int_\Omega \tilde{W}\left(D_\alpha u(x_\alpha), b(x), D_3 b(x)\right)\, dx$$

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Maxwell's equations

[With G. Bouchitté, G. Leoni and V. Millot]

$$\begin{cases} \mathrm{div}\,(h + \chi_\Omega m) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3) \\ \mathrm{curl}\, h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3) \\ |m| = 1 & \text{a.e. in } \Omega \end{cases}$$

$$\int_\Omega \varepsilon^\alpha |\nabla m_\varepsilon|^2 + \frac{1}{\varepsilon^\beta}\varphi(m_\varepsilon) + \int_{\mathbb{R}^3} \frac{1}{\varepsilon^\gamma}|h_\varepsilon|^2$$

$\ldots \nabla m_\varepsilon \sim \nabla^2 \ \ldots \alpha = \beta = \gamma = 1$

[DeSimone, Kohn, Müller, Otto, Rivière, Serfaty, etc.]

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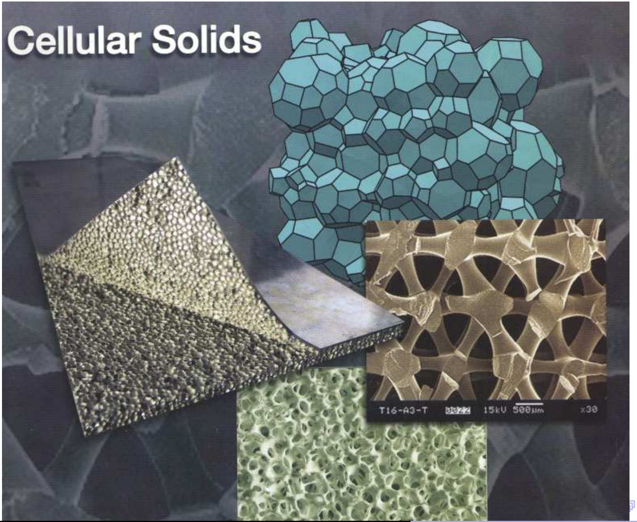
surfactants and foams: a singular perturbation problem



foams

[With M. Morini and V. Slastikov]

aequous foams, polymeric foams, metallic foams (AL_2O_3)

- metal foams: cellular structure of solid material (e.g. Aluminium) w/ large ($\geq$ 80 % volume fraction of gas-filled pores); size of pores: 1mm–8mm
- open-cell foams (interconnected network): lightweight optics, advanced aerospace technology
- closed-cell foams (sealed pores): high impact loads

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[With M. Morini and V. Slastikov]

aequous foams, polymeric foams, metallic foams (AL_2O_3)

- metal foams: cellular structure of solid material (e.g. Aluminium) w/ large (≥ 80 % volume fraction of gas-filled pores); size of pores: 1mm–8mm
- open-cell foams (interconnected network): lightweight optics, advanced aerospace technology
- closed-cell foams (sealed pores): high impact loads

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what for

- detergents, oil recovery, snowboard wax, hair conditioner
- scaffolds for bone tissue engineering, biotechnology
- lightweight structural materials
- ...

little liquid in thin film that separates adjacent bubbles; most liquid in *Plateau Borders/Struts* (region between 3 touching bubbles)

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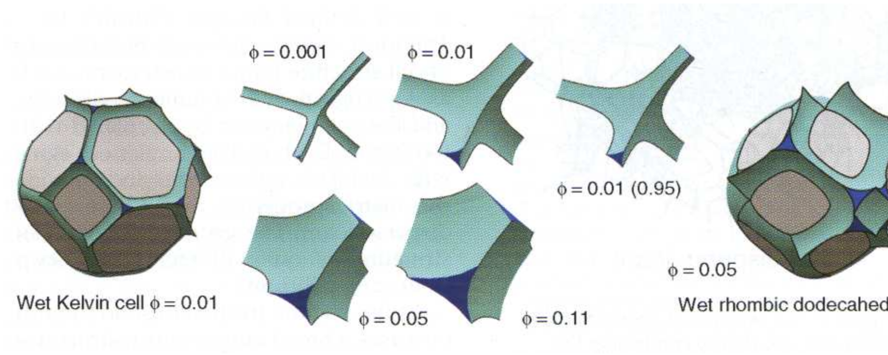
Plateau borders/struts; nodes/joints



surfactants

Surf(ace) **Act**(ive) **A**(ge)**nt**

organic compounds - wetting agents

Amphiphiles: hydrophobic tail and hydrophilic head $\rightsquigarrow$ decrease surface tension; allow easier spreading; lower interfacial tension between two liquids

$\rightsquigarrow$ stabilizing effect of multiple interfaces configurations – *Gibbs-Marangoni effect* : mass transfer due to gradients in surface tension

Typical mixtures:

– water-oil $\rightsquigarrow$ micelles of surfactants

– water-air $\rightsquigarrow$ monolayers of surfactants at the interface

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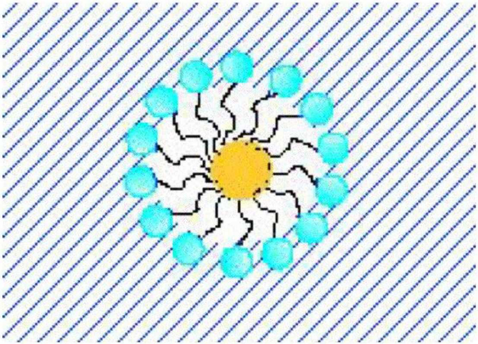
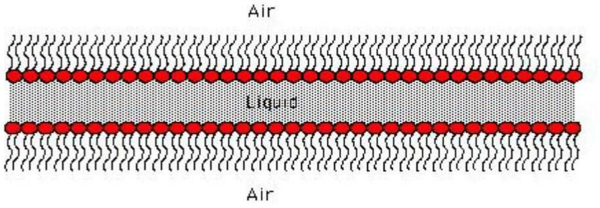
micelle



A micelle of surfactant molecules surrounding a drop of oil in water

monolayers of surfactant



Monolayers of surfactants molecules separated by a thin film of liquid

surfactants effects

- determine macroscopic properties
- segregate to interfaces
- facilitate formation of interfaces (wetness)

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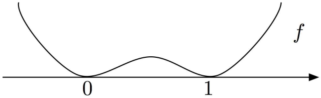
the phase field model

[R. Perkins, R. Sekerka, J. Warren and S. Langer]

Modification of the Cahn-Hilliard model for fluid-fluid phase transitions

$$F_\varepsilon^{CH}(u) := \int_\Omega \left(\frac{1}{\varepsilon} f(u) + \varepsilon |\nabla u|^2 \right) dx$$

f ... double well potential ... $\{f = 0\} = \{0, 1\}$



$u : \Omega \to [0, 1]$ order parameter, $\int_\Omega u = a$

the phase field model

To this we add

$$F^{\mathrm{surf}}_{\alpha(\varepsilon)}(u,\rho_s) := \alpha(\varepsilon)\int_\Omega (\rho_s - |\nabla u|)^2 + \beta(\varepsilon)\int_\Omega |\nabla \rho_s|^2$$

$$F_\varepsilon(u,\rho) := \int_\Omega \left(\frac{1}{\varepsilon} f(u) + \varepsilon|\nabla u|^2 + \varepsilon\,(\rho - |\nabla u|)^2\right)\,dx$$

$\rho_s \geq 0$ ... surfactant density

$\int_\Omega \rho_s = b$

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asymptotic behavior

Goal: Study the asymptotics as $\varepsilon \to 0^+$ $(\beta(\varepsilon) = 0)$

$$F_\varepsilon(u, \rho_s) := F_\varepsilon^{CH}(u) + F^{\mathrm{surf}}_{\alpha(\varepsilon)}(u, \rho_s)$$

minima of F_ε converge to minima of F ...

Theorem

Γ-limit $F_\varepsilon(u, \mu) = F(u, \mu)$ in $BV(\Omega; \{0, 1\}) \times \mathcal{M}(\Omega)$

Different regimes:

- $\alpha(\varepsilon) \sim \varepsilon$
- $\varepsilon << \alpha(\varepsilon) << 1$ ... surfactant plays no role
- $\alpha(\varepsilon) << \varepsilon$ ... creation of interfaces needs ∞ energy

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$\alpha(\varepsilon) = \varepsilon$

$$F(u,\mu) := \begin{cases} \int_{S(u)} \Phi\left(\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x)\right) d\mathcal{H}^{N-1} \\ \qquad \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty \quad \text{otherwise} \end{cases}$$

- Φ described by optimal profile problem
- Φ nonincreasing, convex
- Φ is constant on $[1,+\infty)$ ...saturation

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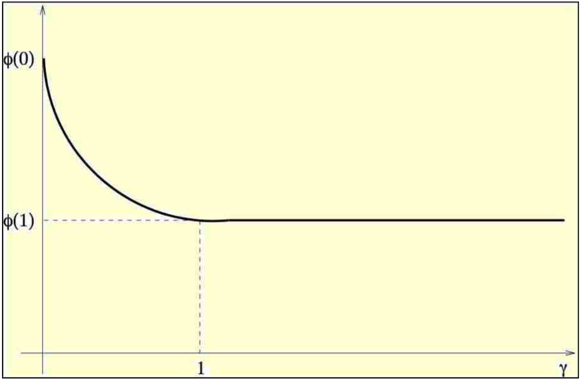
Theorem, still



The energy density ϕ

some conclusions

- surfactant segregates to interface
- prescribed distribution of surfactant dictates location of interfaces
- macroscopic energy F unchanged is density of surfactant on the interfaces exceeds 1 . . . *saturation threshold*
- stability implies uniform distribution of surfactant . . . Marangoni effect

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surface energy density

$$\Phi(\gamma) := \inf_{(u,\lambda)\in\mathcal{A}(\gamma)} \left\{ \int_{-\infty}^{+\infty} \left[f(u) + \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \right] \right\}$$

$$\min\{\lambda^2 + |u'|^2, 2|u'|^2\} = |u'|^2 + (\max\{\lambda + |u'|, 0\} - |u'|)^2 = |u'|^2 + \min\{\lambda^2, |u'|^2\}$$

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$$\left. \lim_{t\to+\infty} u(t) = 1, \int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\} \le \min\{\gamma, 1\} \right\}$$

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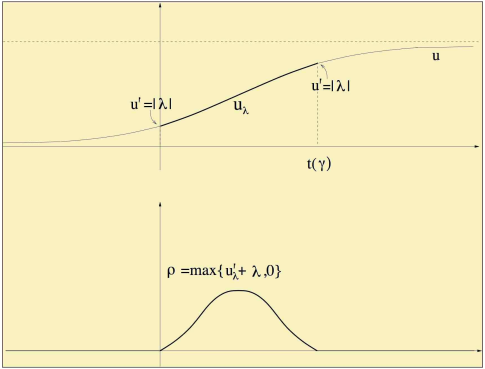
the optimal profile



The solution u to the optimal profile problem and the corresponding ρ

properties of the surface energy density

Theorem

$\Phi \in C^1$, nonincreasing and is convex.

This is not a simple curiosity! It is needed in a crucial way to prove $\Gamma - \liminf$ in the vectorial case!

For all $\gamma > 0$ the optimal profile $\Phi(\gamma)$ admits a unique solution $(u, \lambda) \in \mathcal{A}(\gamma)$

u nondecreasing and strictly increasing in $\{0 < u < 1\}$

$\Phi(0) = 2\sqrt{2}\int_0^1 f(s)\,ds \qquad \Phi(1) = \sqrt{2}\int_0^1 f(s)\,dx$ (Modica-Mortola)

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stability

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[Stability "a la" Kohn and Sternberg]
$(u_0, \mu_0) \in BV(\Omega; \{0,1\}) \times \mathcal{M}$
u_0 . . . isolated local minimum of $F(\cdot, \mu_0)$, i.e.

$$F(u_0, \mu_0) < F(v, \mu_0)$$

if $0 < ||u_0 - v||_{L^1} \leq \delta$, $\int_\Omega u_0 = \int_\Omega v$

Then there exists $\{u_\varepsilon, \rho_\varepsilon\}$ such that
$u_\varepsilon \longrightarrow u_0$ in L^1, $\quad \rho_\varepsilon \overset{*}{\rightharpoonup} \mu_0$

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next . . .

- 1D metastability of multiple interfaces configurations
- the impact of surfactants on the dynamics of phase separation
- vectorial models . . . more complicated interaction between u and ρ_s, e.g.

$$F_\varepsilon(u, \rho_s) = \frac{1}{\varepsilon} \int_\Omega f(x, u, \varepsilon \nabla u, \varepsilon \rho_s)\, dx$$

[Acerbi and Bouchitté]

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Recall ...

$$\Phi(\gamma) := \inf_{(u,\lambda)\in\mathcal{A}(\gamma)} \left\{ \int_{-\infty}^{+\infty} \left[f(u) + |u'|^2 + (\max\{\lambda + |u'|, 0\} - |u'|)^2 \right] \right\}$$

If $\gamma > 1$ then the unique minimizing pair is $(u, 0)$ $(\rho = |u'|)$ with u the solution of the Modica-Mortola (Cahn-Hilliard) profile

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the convexity of Φ

Theorem

$\Phi \in C^1$, nonincreasing and is convex.

Proof Step 1: auxilliary energy density

$$\Phi(M,a,b;\gamma) := \inf_{t>0} \inf \left\{ Mt + \int_0^t \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \right\}$$

$(u,\lambda) \in \mathcal{A}(t,a,b;\gamma)$

$$\mathcal{A}(t,a,b;\gamma) := \Big\{ (u,\lambda) \in H^1(0,t) \times (-\infty, 0] : u(0) = a, u(t) = b,$$
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the convexity of Φ

Theorem

$\Phi \in C^1$, nonincreasing and is convex.

Proof Step 1: auxilliary energy density

$$\Phi(M, a, b; \gamma) := \inf_{t>0} \inf \left\{ Mt + \int_0^t \min\{\lambda^2 + |u'|^2, 2|u'|^2\} \right\}$$

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Claim: If $(u, \lambda) \in \mathcal{A}(t, a, b; \gamma)$ minimizes

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then u is increasing, $(u, \max\{u' + \lambda, 0\})$ minimizes

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among (v, ρ), $v \in H^1(0, t)$, $v(0) = a$, $v(t) = b$, $\rho \geq 0$, $\int_0^t \rho = \min\{\gamma, b - a\}$

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(Y, μ) measure space, μ non-atomic positive measure, $g : Y \to [0, +\infty)$ (non-zero) $\in L^1(Y, \mu) \cap L^2(Y, \mu)$, $0 < \gamma \leq \int_Y g \, d\mu$. The problem

$$\min \left\{ \int_Y (v - g)^2 \, d\mu : v \geq 0, \int_Y v \, d\mu = \gamma \right\}$$

admits a (μ a.e.) unique solution

$$v := \max\{\lambda + g, 0\}$$

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$$\int_0^t |v'|^2 + \int_0^t (\rho - |v'|)^2 \geq \int_0^t |v'|^2 + \int_0^t (\max\{|v'| + \bar{\lambda}, 0\} - |v'|^2)$$
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yield *Euler-Lagrange equation*

$$2u' - \max\{u'+\lambda, 0\} = C \quad \text{a.e. in } (0,t)$$

for some constant C ... u' is constant ...

$$u(x) = a + \frac{b-a}{t}x, \quad \lambda = \frac{\min\{\gamma-(b-a), 0\}}{t}$$

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$$\Phi(M,a,b;\gamma) = \min_{t>0}\left\{ Mt + \frac{(b-a)^2}{t} + \frac{[\min\{\gamma-(b-a),0\}]^2}{t} \right\}$$
$$= \begin{cases} 2\sqrt{M}\sqrt{(b-a)^2+(\gamma-(b-a))^2} & \text{if } \gamma \le b-a \\ 2\sqrt{M}(b-a) & \text{otherwise} \end{cases}$$

So, $\Phi(M,a,b;\cdot)$ is convex!

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the convexity of Φ; Step 2

Step 2: f lsc, piecewise constant on $[0,1]$
$f = M_i$ in (a_{i-1}, a_i), $M_i > 0$, $0 = a_0 < a_1 < \ldots < a_m = 1$

Claim

$$\Phi(\gamma) = \min\left\{\sum_{i=1}^{m} \Phi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \geq 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\}\right\}$$

infimal convolution of $\Phi(M_i, a_{i-1}, a_i; \cdot)$

The infimal convolution of convex, nondecreasing functions is convex, nondecreasing!

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the convexity of Φ; Step 3

f any continuous double-well potential, f_n as in Step 2, $f_n \to f$ decreasingly, $f_n = f$ outside $(0,1)$

$$\Phi_n \text{ are convex... DONE if } \Phi_n \to \Phi \text{ pointwise}$$

Clearly $\liminf \Phi_n \geq \Phi$. Fix $\varepsilon > 0$; choose $(u, \lambda) \in \mathcal{A}(\gamma)$ s.t. $u(-T) = 0$, $u(T) = 1$, some $T > 0$,

$$E(u, \lambda; (-T, T)) \leq \Phi(\gamma) + \varepsilon$$

Then

$$\limsup \Phi_n(\gamma) \leq \lim E_n(u, \lambda; (-T, T)) = E(u, \lambda; (-T, T)) \leq \Phi(\gamma) + \varepsilon$$

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