## **Stability of Leaves**

### Rui Loja Fernandes

Departamento de Matemática Instituto Superior Técnico

### Seminars of the CIM Scientific Council Meeting 2008

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- Establish stability results for symplectic leaves of Poisson manifolds;
- Understand the relationship between (apparently) distinct stability results in different geometric settings;

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# Outline

### 1 Classical Results

- Flows
- Foliations
- Group actions
- 2 Stability of symplectic leaves
  - Poisson geometry
  - Symplectic leaves
  - Stability of symplectic leaves
  - Poisson cohomology
- 3 Universal Stability Theorem
  - Basic problem
  - Geometric Lie theory
  - Universal Stability Theorem

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Flows Foliations Group actions

# Flows: Stability of periodic orbits

#### Definition

A periodic orbit of a vector field  $X \in \mathfrak{X}(M)$  is called stable if every nearby vector field also has a nearby periodic orbit.

- Basic Fact: Stability is controlled by the Poincaré return map  $h: T \rightarrow T$ .
- Assumptions on  $d_{x_0}h$  can also guarantee stability.

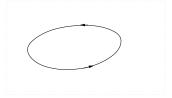
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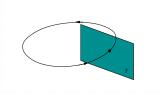
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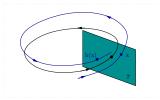
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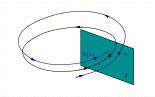
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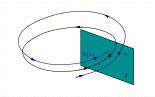
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# Foliations: stability of leaves

Fix a manifold *M* (dim M = n) and a foliation  $\mathcal{F}$  (codim( $\mathcal{F}$ ) = q).  $\mathcal{F}$  is given by a foliation atlas  $(U_i, \varphi_i)_{i \in I}$ 

### $\varphi_i: U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q, \quad \varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$

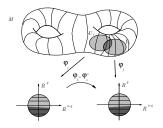
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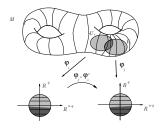
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# Foliations: stability of leaves

#### Theorem (Frobenius)

 $\operatorname{Fol}_q(M) \longleftrightarrow \{D: M \to \operatorname{Gr}_q(TM) | D \text{ is involutive} \}$ 

$$\mathcal{F} \longmapsto D := T\mathcal{F}$$

 $\implies$  Fol<sub>q</sub>(M) has a natural C<sup>r</sup> compact-open topology

#### Definition

A leaf *L* of a foliation  $\mathcal{F} \in Fol_k(M)$  is called stable if every nearby foliation in  $Fol_k(M)$  has a nearby leaf diffeomorphic to *L*.

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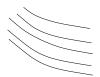
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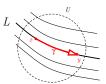


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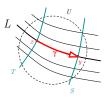


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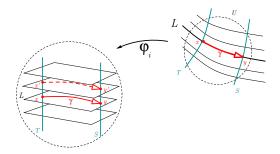


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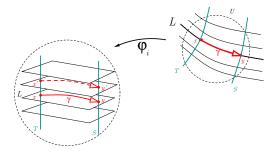


Construct diffeomorphism  $f : T \to S$  that satisfies f(x) = y and y' = f(x') iff x' and y' are in same plaque. Then:

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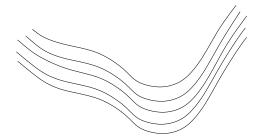


Construct diffeomorphism  $f : T \to S$  that satisfies f(x) = y and y' = f(x') iff x' and y' are in same plaque. Then: Hol<sup>*T*,*S*</sup>( $\gamma$ ) := germ<sub>*x*</sub>(f) : (*T*, x)  $\to$  (*S*, y).

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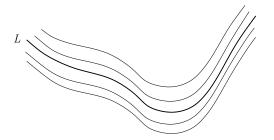


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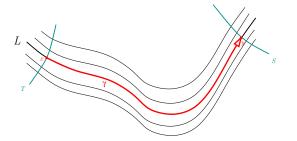


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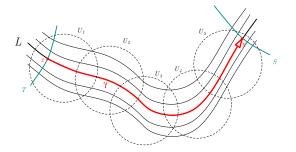
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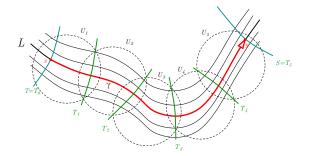
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 $\mathsf{Hol}^{\mathcal{T},\mathcal{S}}(\gamma):=\mathsf{Hol}^{\mathcal{T}_k,\mathcal{T}_{k-1}}(\gamma)\circ\cdots\circ\mathsf{Hol}^{\mathcal{T}_2,\mathcal{T}_1}(\gamma)\circ\mathsf{Hol}^{\mathcal{T}_1,\mathcal{T}_0}(\gamma)$ 

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# Foliations: stability of leaves

### Facts:

- Taking germs makes construction independent of choices;
- If  $\gamma, \eta$  are curves in *L* with  $\gamma(0) = \eta(1)$  then: Hol<sup>*T*,*S*</sup> $(\gamma \cdot \eta) = \text{Hol}^{$ *T*,*S* $}(\gamma) \circ \text{Hol}^{$ *S*,*R* $}(\eta);$
- If  $\gamma$  and  $\gamma'$  are homotopic curves in *L*, then: Hol<sup>*T*,*S*</sup>( $\gamma$ ) = Hol<sup>*T*,*S*</sup>( $\gamma'$ );

Hence, if we fix  $x \in L$  we obtain the holonomy homomorphism:

$$\operatorname{Hol} := \operatorname{Hol}^{T,T} : \pi_1(L, X) \to \operatorname{Diff}_X(T).$$

**Note:** The Poincaré return map is a special case of this construction.

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## Foliations: stability of leaves

Differentiating gives the linear holonomy representation:

$$\rho: \pi_1(L, \mathbf{X}) \to GL(\nu(L)_{\mathbf{X}}), \quad \rho:= \mathrm{d}_{\mathbf{X}} \circ \mathrm{Hol}$$

Denote by  $H^1(\pi_1(L, x), \nu(L)_x)$  the first group cohomology.

Theorem (Reeb, Thurston, Langevin & Rosenberg)

Let L be a compact leaf and assume that

$$H^1(\pi_1(L, x), \nu(L)_x) = 0.$$

Then L is stable.

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# Group actions: stability of orbits

Fix a manifold *M* and a Lie group *M* 

- $\alpha(g, x) := g \cdot x$  an action of G on M
- Action  $\alpha$  :  $G \times M \rightarrow M \Leftrightarrow$  homomorphism  $\alpha$  :  $G \rightarrow \text{Diff}(M)$

 $\operatorname{Act}(G; M) \subset \operatorname{Maps}(G; \operatorname{Diff}(M))$ 

 $\implies$  Act(G; M) has a natural C<sup>r</sup> compact-open topology

#### Definition

An orbit  $\mathcal{O}$  of  $\alpha \in Act(G; M)$  is called stable if every nearby action in Act(G; M) has a nearby orbit diffeomorphic to  $\mathcal{O}$ .

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#### The stability of an orbit $\mathcal{O}$ is controled by the isotropy of $\mathcal{O}$ :

- $G_x := \{g \in G : g \cdot x = x\}$  isotropy group at  $x \in O$ .
- $g \in G_x$  induces a map  $\alpha_g : M \to M, y \mapsto g \cdot y$  that fixes x.

#### $d_x \alpha_g : T_x M \to T_x M$ $\Rightarrow \rho(g) : \nu(\mathcal{O})_x \to \nu(\mathcal{O})_x \quad \text{where } \nu(\mathcal{O})_x = T_x M / T_x \mathcal{O}.$

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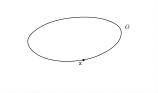
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Rui Loja Fernandes Stability of Leaves

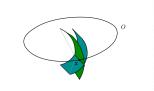
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Group actions

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Rui Loja Fernandes

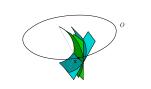
Stability of Leaves

Flows Foliations Group actions

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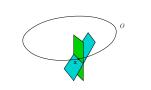
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Rui Loja Fernandes Stability of Leaves

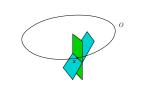
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## Group actions: stability of orbits

#### linear normal isotropy representation:

$$\rho: \mathbf{G}_{\mathbf{X}} \to \mathbf{GL}(\nu(\mathcal{O})_{\mathbf{X}}), \quad \rho(\mathbf{g}) := \mathrm{d}_{\mathbf{X}}\alpha_{\mathbf{g}}: \nu(\mathcal{O})_{\mathbf{X}} \to \nu(\mathcal{O})_{\mathbf{X}}$$

Denote by  $H^1(G_x, \nu(\mathcal{O})_x)$  the first group cohomology.

Theorem (Hirsch,Stowe)

Let O be a compact orbit and assume that

 $H^1(G_x,\nu(\mathcal{O})_x)=0.$ 

Then O is stable.

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Flows Foliations Group actions

#### Stability of leaves versus orbits

- In general, the two theorems are quite different (e.g., dimension of orbits can vary).
- If  $G_x$  is discrete, dimension of orbits is locally constant.
- If G<sub>x</sub> is discrete and G is 1-connected then π<sub>1</sub>(O, x) = G<sub>x</sub>.
  ⇒ the theorem for actions follows from the theorem for foliations.

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Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

## Hamilton's Equations

**R**<sup>2n</sup> with coordinates  $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ 

Classical Poisson bracket:

$$\{f_1, f_2\} = \sum_{i=1}^n \left( \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial f_1}{\partial p_i} \right)$$

Hamilton's equations:

$$\begin{cases} \dot{q}_i = \frac{\partial h}{\partial p_i} \\ \dot{p}_i = -\frac{\partial h}{\partial q_i} \end{cases} (i = 1, \dots, n) \qquad \Leftrightarrow \qquad \dot{x}_a = \{x_a, h\} \quad (a = 1, \dots, 2n)$$

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Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

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Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

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#### Poisson brackets

- Skew-symmetry:  $\{f, g\} = -\{g, f\};$
- Jacobi identity:  $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0;$
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A manifold *M* furnished with a Poisson bracket is called a Poisson manifold.

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#### **Basic examples**

Any symplectic manifold  $(M, \omega)$  is a Poisson manifold:

$$\{f,g\}=-\omega(X_f,X_g).$$

(X<sub>f</sub> is the unique vector field such that ι<sub>X<sub>f</sub></sub>ω = df.)
The dual of a Lie algebra M = g\* is a Poisson manifold:

$$\{f,g\}(\xi) = \langle \xi, [\mathrm{d}_{\xi}f, \mathrm{d}_{\xi}g] \rangle.$$

■ Any skew-symmetric matrix (*a<sub>ij</sub>*) defines a quadratic Poisson bracket on ℝ<sup>n</sup>:

$$\{x_i, x_j\} = a_{ij}x_ix_j.$$

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Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

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## Symplectic foliation

#### Definition

On  $(M, \{ , \})$ , the hamitonian vector field determined by  $h \in C^{\infty}(M)$  is the vector field  $X_h \in \mathfrak{X}(M)$  given by:

 $X_h(f) := \{f, h\}, \quad \forall f \in C^\infty(M).$ 

Write  $x \sim y$  if there exists a piecewise smooth curve joining x to y made of integral curves of hamiltonian vector fields:

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## Symplectic foliation

#### Theorem (Weinstein)

The decomposition of  $(M, \{, \})$  into equivalence classes of  $\sim$ :

 $M = \bigsqcup_{\alpha \in A} S_{\alpha}.$ 

#### satisfies:

(i) Each  $S_{\alpha}$  is a (immersed) submanifold of M;

(ii) Each  $S_{\alpha}$  carries a symplectic structure  $\omega_{\alpha}$ ;

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### Symplectic foliation Example

- $\blacksquare M = \mathfrak{sl}^*(2,\mathbb{R}) \simeq \mathbb{R}^3: \{x,z\} = y; \quad \{x,y\} = z; \quad \{z,y\} = x.$
- Symplectic foliation:  $\{(x, y, z)|x^2 + y^2 z^2 = c\}$ .
- Foliation is singular (dimension of leaves varies; e.g., cone x<sup>2</sup> + y<sup>2</sup> = z<sup>2</sup>)

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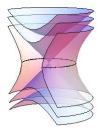
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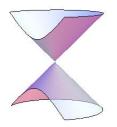
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### Poisson bivector

Given Poisson bracket  $\{, \}$  define the Poisson bivector:

 $\pi(\mathrm{d} f,\mathrm{d} g):=\{f,g\}.$ 

■  $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$  is a skew-symmetric contravariant tensor;

In local coordinates  $(x^1, \ldots, x^n)$ :

$$\pi = \sum_{i < j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

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Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

### Poisson bivector

Given Poisson bracket  $\{, \}$  define the Poisson bivector:

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### Stability of symplectic leaves

### $\mathsf{Poiss}(M) \quad \longleftrightarrow \quad \{\pi: M \to \wedge^2(TM) | \ [\pi,\pi] = 0\}.$

 $\Rightarrow$  Poiss(*M*) has a natural *C<sup>r</sup>* compact-open topology

#### Definition

A symplectic leaf *S* of  $\pi \in \text{Poiss}(M)$  is called stable if every nearby Poisson structure in Poiss(M) has a nearby leaf diffeomorphic to *S*.

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# Stability of symplectic leaves

#### Theorem (Crainic & RLF)

Let  $(M, \pi)$  be a Poisson structure and  $S \subset M$  a compact symplectic leaf such that:

$$H^2_{\pi}(M,S)=0.$$

Then S is stable.

Again, this result is quite **different** from the previous ones;

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# Stability of symplectic leaves

 $M = \mathfrak{su}^*(3) \simeq \mathfrak{su}(3)$  (via the Killing form) with linear Poisson structure:

Symplectic leaves are the conjugacy classes of *SU*(3):

$$A \sim \left(\begin{array}{rrrr} i\lambda_1 & 0 & 0\\ 0 & i\lambda_2 & 0\\ 0 & 0 & i\lambda_3 \end{array}\right)$$

$$(\lambda_1 + \lambda_2 + \lambda_3 = 0, \ 0 \le \lambda_1 \le \lambda_2)$$

Leaves have:

- (i) Dimension 6 (flag);
- (ii) Dimension 4 ( $\mathbb{C}P(2)$ );
- (iii) Dimension 0 (the origin);

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# Stability of symplectic leaves

■ All leaves of *su*<sup>\*</sup>(3) (including singular leaves) are stable;

- The same result applies for g\*, where g is any semi-simple Lie algebra of compact type;
- This is related to (and explains!) a famous linearization theorem of Conn;
- If g is semi-simple and non-compact stability, in general, does not hold (e.g., sl(2, ℝ)).

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## Poisson cohomology

Ordinary Geometry	Poisson Geometry
<ul> <li>Differential forms: <math display="block">\Omega^{k}(M) = \Gamma(\wedge^{k} T^{*}M);</math> </li> <li>DeRham differential: d: \Omega^{\u03c6}(M) \rightarrow \Omega^{\u03c6+1}(M), \u03c6 \u03c6 d\u03c6 \u03c6 \u03c6;</li> <li>DeRham cohomology: H^{\u03c6}_{DR}(M) := Ker d/ Im d;</li> </ul>	Multivector fields: $\mathfrak{X}^{k}(M) = \Gamma(\wedge^{k} TM);$ Lichnerowitz differential: $d_{\pi} : \mathfrak{X}^{\bullet}(M) \to \mathfrak{X}^{\bullet+1}(M),$ $d_{\pi}\theta := [\theta, \pi];$ Poisson cohomology: $H^{\bullet}_{\pi}(M) := \operatorname{Ker} d_{\pi}/\operatorname{Im} d_{\pi};$

Poisson geometry Symplectic leaves Stability of symplectic leaves Poisson cohomology

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Rui Loja Fernandes Stability of Leaves

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# Poisson cohomology

#### Geometric interpretations of $H^{\bullet}_{\pi}(M)$ in low degrees:

 $\blacksquare H^0_{\pi}(M) = Z(C^{\infty}(M)) - Casimirs;$ 

- H<sup>1</sup><sub>π</sub>(M) = {Poisson vect. fields}/{hamiltonian vect. fields} infinitesimal outer Poisson automorphisms;
- $H^2_{\pi}(M) = T_{\pi} \operatorname{Poiss}(M)$  infinitesimal (formal) deformations of  $\pi$ ;

$$\mathfrak{X}^{\bullet}(M, S) := \Gamma(\wedge^{\bullet} T_S M).$$

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### **Basic problem**

# Is there a general setup to deal with these kind of stability problems?

#### A positive answer to this question should lead to:

- (i) A universal stability theorem which would yield the stability theorems stated above.
- (ii) A way to handle with stronger notions of stability.

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### Lie algebroids

#### Definition

A Lie algebroid is a vector bundle  $A \rightarrow M$  with:

- (i) a Lie bracket  $[, ]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A);$
- (ii) a bundle map  $\rho : A \to TM$  (the anchor);

such that:

$$[\alpha, f\beta]_{\mathcal{A}} = f[\alpha\beta]_{\mathcal{A}} + \rho(\alpha)(f)\beta, \quad (f \in C^{\infty}(\mathcal{M}), \alpha, \beta \in \Gamma(\mathcal{A})).$$

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Lie algebroids have a characteristic foliation

Basic problem Geometric Lie theory Universal Stability Theorem

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## Lie algebroids Examples

Flows. For  $X \in \mathfrak{X}(M)$ , the associated Lie algebroid is:  $A = M \times \mathbb{R}, \quad [f, g]_A := fX(g) - gX(f), \quad \rho(f) = fX.$ Leaves of *A* are the orbits of *X*.

Foliations. For  $\mathcal{F} \in \operatorname{Fol}_k(M)$ , the associated Lie algebroid is:

 $A = T\mathcal{F}, \quad [X, Y]_A = [X, Y], \quad \rho = id.$ Leaves of A are the leaves of  $\mathcal{F}$ .

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#### Lie algebroids Examples

Actions. For α ∈ Act(G; M), the associated Lie algebroid is:
 A = M × g, ρ =infinitesimal action,
 [f,g]<sub>A</sub>(x) = [f(x), g(x)]<sub>g</sub> + L<sub>ρ(f(x))</sub>g(x) - L<sub>ρ(g(x))</sub>f(x).
 Leaves of A are the orbits of α (for G connected).

Poisson structures. For  $\pi \in \text{Poiss}(M)$ , the associated Lie algebroid is:

 $A = T^*M, \quad \rho = \pi^{\sharp},$ 

 $[\mathrm{d} f,\mathrm{d} g]_{\mathcal{A}}=\mathrm{d} \{f,g\}, \quad (f,g\in C^\infty(M)).$ 

Leaves of A are the symplectic leaves of  $\pi$ .

Basic problem Geometric Lie theory Universal Stability Theorem

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### Lie algebroids Examples

Actions. For α ∈ Act(G; M), the associated Lie algebroid is:
 A = M × g, ρ =infinitesimal action,
 [f,g]<sub>A</sub>(x) = [f(x), g(x)]<sub>g</sub> + L<sub>ρ(f(x))</sub>g(x) - L<sub>ρ(g(x))</sub>f(x).
 Leaves of A are the orbits of α (for G connected).

Poisson structures. For  $\pi \in \text{Poiss}(M)$ , the associated Lie algebroid is:

 $egin{aligned} & oldsymbol{A} = oldsymbol{T}^*oldsymbol{M}, & 
ho = \pi^{\sharp}, \ & [\mathrm{d}f,\mathrm{d}g]_{\mathcal{A}} = \mathrm{d}\{f,g\}, & (f,g\in C^{\infty}(\mathcal{M})). \end{aligned}$ 

Leaves of *A* are the symplectic leaves of  $\pi$ .

Basic problem Geometric Lie theory Universal Stability Theorem

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## Stability Theorem

For a fixed vector bundle A there is a natural compact-open topology on the set Algbrd(A) of Lie algebroid structures on A.

A leaf L of A is called stable if every nearby Lie algebroid structure in Algbrd(A) has a nearby leaf diffeomorphic to L.

■ There are natural A-cohomology theories. For a leaf L, one can define the relative A-cohomology with coefficients in the normal bundle v(L), denoted H<sup>●</sup>(A|<sub>L</sub>; v(L)).

Basic problem Geometric Lie theory Universal Stability Theorem

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Basic problem Geometric Lie theory Universal Stability Theorem

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# Stability Theorem

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Basic problem Geometric Lie theory Universal Stability Theorem

## Stability Theorem

#### Theorem (Crainic & RLF)

Let L be a compact leaf of a Lie algebroid A, and assume that  $H^1(A, L; \nu(L)) = 0$ . Then L is stable.

- The theorem says: infinitesimal stability ⇒ stability. Likewise, the proof is a "infinite dimensional transversality argument".
- All other stability theorems can be deduced from this one. This explains the appearence of different cohomologies.

The Lie algebroid approach allows the study of stronger notions of stability...

Basic problem Geometric Lie theory Universal Stability Theorem

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# **Moral:** There is a general framework to deal with stability of "leaf-type" problems.

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