A note on a Benney-type system

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Model for interaction between Long and Short waves

Short wave term $S(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ (Schrödinger-like equation)
Long wave term $L(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (Transport-like equation)

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Benney's equation is a rather universal model to describe LW-SW interactions:

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- General Theory of water waves interaction

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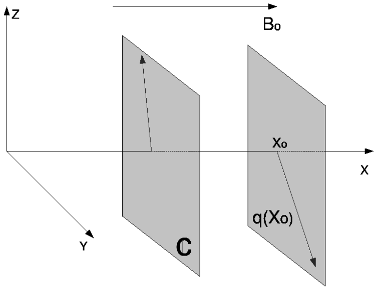
where **b** is the magnetic field, ρ the density of mass and **u** the fluid speed.

We present here a uni-dimensional asymptotic model for the evolution of wave trains of Alfvén waves with wave number k and frequency $\widetilde{\omega}$, in a frame travelling at the Alfvén-wave group velocity $v = 2\widetilde{\omega}^3 k^{-1}(k^2 + \widetilde{\omega}^2)^{-1}$.



(Champeaux & al, Nonlinear Processes in Geophysics, 1999)

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B is the transverse magnetic field, u is the ion speed in the (Ox) direction and ρ the density of mass.

We obtain here the Zakharov-Rubenchik equation, introduced as an (another) universal model for the interaction of long and short waves (1972).

First, a change of variables:

$$\begin{cases} iB_t + B_{xx} + \psi_1 B + \psi_2 B + |B|^2 B = 0 \\ \psi_{1t} - \psi_{1x} = |B|^2_x \\ \psi_{2tt} - \psi_{2xx} = |B|^2_{xx} \end{cases} \quad (1)$$

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The difficulty here is the derivative loss in the nonlinear terms.

Using Strichartz-type estimates for the free Schrödinger group, we can now obtain the existence of local (strong) solutions via a fixed-point in the Banach space

$$\begin{aligned}\|(F,\psi_1,\psi_2)\|_{X(T)} &= \|F\|_{L^\infty(0,T,L^2)} + \|F\|_{L^6(0,T,L^6)}\\ &+ \|\psi_1\|_{L^\infty(0,T,H^1)} + \|\psi_2\|_{L^\infty(0,T,H^1)}\\ &+ \|\psi_{1t}\|_{L^\infty(0,T,L^2)} + \|\psi_{2t}\|_{L^\infty(0,T,L^2)}.\end{aligned}$$

To obtain global solutions, we need to compute some invariants:

The following quantities are conserved by the Zakharov-Rubenchik flow:

$$
\begin{aligned}
I_1(t) &= \int_{\mathbb{R}} |B|^2 \\
I_2(t) &= \frac{\omega}{2}\int_{\mathbb{R}} |B_x|^2 + \frac{kq}{4}\int_{\mathbb{R}} |B|^4 + \frac{k}{2}\int_{\mathbb{R}} (u - \frac{v}{2}\rho)|B|^2 \\
&\quad + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 - \frac{v}{2}\int_{\mathbb{R}} u\rho, \\
I_3(t) &= \epsilon \int_{\mathbb{R}} u\rho + \frac{i}{2}\int_{\mathbb{R}} (B\overline{B_x} - B_x\overline{B}).
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Using these quantities, One can show the a priori estimation

$$\forall t \leq T, \ \|(F, \psi_1, \psi_2)\|_{X(T)} \leq D(T),$$

where D is a continuous function. This is enough to prove that the solutions are global (absence of blow-up)

Well posedness

$$\begin{cases} i\partial_T B + \omega\partial_{XX}B - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 & \text{(a)} \\ \epsilon\partial_T\rho + \partial_X(u - v\rho) = -k\partial_X|B|^2 & \text{(b)} \\ \epsilon\partial_T u + \partial_X(\beta\rho - vu) = \frac{k}{2}v\partial_X|B|^2 & \text{(c).} \end{cases}$$

Theorem

The Zakharov-Rubenchik system is globally well-posed in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

(FO, Physica D, 2003)

The adiabatic limit

In the adiabatic limit ($\epsilon \to 0$):

$$i\partial_T B + \omega \partial_{XX} B - k(u - \tfrac{v}{2}\rho + q|B|^2)B = 0 \quad \text{(a)}$$
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u and ρ become slaved to the magnetic field amplitude:

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$$B^{(\epsilon)} \to B \ ? \quad \text{If so, in what sense?}$$

Theorem

Assume $\tilde{\omega} < 0$, $\beta - v^2 > 0$ and $v < 0$.
Let $s > \frac{3}{2}$, $\epsilon < 1$ and

$$(B_o, \rho_o, u_o) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}).$$

Then there exists $T_o > 0$ independent of ϵ such that the Zakharov-Rubenchik possesses a unique solution

$$(B^{(\epsilon)}, \rho^{(\epsilon)}, u^{(\epsilon)}) \in \mathcal{C}^o([0; T_o]; H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R})).$$

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Furthermore, if $u_o - v\rho_o = -k|B_o|^2$ and $\beta\rho_o - vu_o = k\frac{v}{2}|B_o|^2$,

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(FO, Reports on Mathematical Physics, 2007)

Key: if $\tilde{\omega} < 0$, $\beta - v^2 > 0$ and $v < 0$,putting

$$(V, F, G) := (\epsilon \partial_x^{-1}(u + \frac{v}{2}\rho)_t, u - v\rho + k|B|^2, \beta\rho - vu + k\frac{v}{2}|B|^2)$$

and

$$(\alpha, \beta, \gamma, \delta) := \sqrt{2}(Re(B), Im(B), Re(B_x), Im(B_x)),$$

$$Y = (V, F, G, \alpha, \beta, \gamma, \delta)$$

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satisfies the perturbed symmetric hyperbolic system:

$$Y_t + \left(\frac{1}{\epsilon}M + N(Y)\right) Y_x + R(Y) + AY_{xx} = 0.$$

Here, M, $N(Y)$ are symmetric matrixes, A is antisymmetric and $R(Y)$ is a nonlinear term.

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Here, M, $N(Y)$ are symmetric matrixes, A is antisymmetric and $R(Y)$ is a nonlinear term. We the use Friedrich's general theory of hyperbolic systems, coupled with Klainerman and Maj'da ideas.

A similar result:
(**Kenig-Ponce-Vega, J. Functional Analysis, 1995**)

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$$\sup_{[0;T]} \|E^\epsilon - E\|_{H^s} \to 0,$$

for "large" s.

A previous result
(JP Dias & M Figueira, J. Hyperbolic Equations, 2007)

$$iu_t + u_{xx} = |u|^2 u + vu \qquad (2)$$

$$v_t + (f(v))_x = |u|^2_x \qquad (3)$$

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$$f(v) = av^2 - bv^3,$$

given initial data $u_o, v_o \in H^1$, there exists a weak solution u, v,

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This result was obtained by parabolic regularisation.

We were able to prove:
(JP Dias, M Figueira & FO,C.R. Acad. Sci. Paris, 2007)

Theorem

Let $f \in C^3$.
Given initial data $(u_o, v_o \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$, there exists $T > 0$ and a unique solution, with

$$(u, v) \in C^j([0, T]; H^{3-2j}(\mathbf{R})) \times C^j([0, T]; H^{2-j}(\mathbb{R})), \; j = 0, 1.$$

Here, the life-span $T > 0$ depends exclusively on f and on the initial data (u_0, v_0).

The proof relies on an algebraic "trick": we rewrite the system without derivative loss:
By setting $F = u_t$, we obtain

$$iF + u_{xx} - u = |u|^2 u + u(v-1).$$

Also, differentiating in time:

$$iF_t + F_{xx} = 2|u|^2 F + u^2 \overline{F} + Fv + u|u|_x^2 - uv_x f'(v).$$

Hence, we consider the following Cauchy problem:

$$\begin{aligned} iF_t + F_{xx} &= 2|u|^2 F + u^2\overline{F} + Fv + u|\tilde{u}|_x^2 - uv_x f'(v) \\ v_t + [f(v)]_x &= |\tilde{u}|_x^2 \end{aligned}$$

where u and $\tilde{u}$ are given in terms of F by

$$u(x,t) = u_0 + \int_0^t F(x,s)ds$$

and

$$\tilde{u}(x,t) = (\Delta - 1)^{-1}(|u|^2 u + u(v-1) - iF).$$

We then conclude by using Kato's general theory for quasilinear systems.

Thank you for your attention!