Hardy's Uncertainty Principle, Convexity and Schrödinger Evolutions

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- What follows is joint work with C. Kenig, L. vega and G. Ponce.
- $$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-x\cdot\xi} f(x)\, dx.$$
-

Theorem (Hardy's Uncertainty Principle)

Assume $f(x) = O(e^{-x^2/\beta^2})$ and $\widehat{f}(\xi) = O(e^{-4\xi^2/\alpha^2})$. If $\alpha\beta < 4$, then $f \equiv 0$. If $\alpha\beta = 4$, f is a constant multiple of e^{-x^2/β^2}.

$$\begin{cases} \partial_t u = i\Delta u, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases}$$

- $$\int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}}\, u_0(y)\, dy = \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y/2t} e^{i|y|^2/4t} u_0(y)\, dy$$
- $$(4\pi it)^{\frac{n}{2}} e^{-i|x|^2/4t}\, u(x,t) = (e^{i|\cdot|^2/4t} u_0)^{\wedge}\left(\frac{x}{2t}\right)$$
- If $\alpha\beta < 4T$,

$$u(0) = O(e^{-|x|^2/\beta^2}) \text{ and } u(T) = O(e^{-|x|^2/\alpha^2})$$

then, $u \equiv 0$ in $\mathbb{R}^n \times [0, T]$. If $\alpha\beta = 4T$, u is the solution with initial data, $\omega e^{-\left(\frac{1}{\beta^2} + \frac{i}{4T}\right)|x|^2}$, for some complex number ω.

(Cowling M. and Price J. F.) If $p,\, q \in [1,\infty]$, with at least one of them finite, $\alpha\beta \leq 4$ and $e^{|x|^2/\beta^2} f \in L^p(\mathbb{R})$, $e^{4|\xi|^2/\alpha^2}\widehat{f} \in L^q(\mathbb{R})$, then $f \equiv 0$.
(Beurling) If f is in $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||\widehat{f}(\xi)|e^{|x\,\xi|}\,dx\,d\xi < \infty,$$

then $f \equiv 0$.

- If $e^{\frac{x^2}{\beta^2}} u_0 \in L^p(\mathbb{R})$, $e^{\frac{x^2}{\alpha^2}} e^{iT\partial_x^2} u_0 \in L^q(\mathbb{R})$, $p,\, q \in [1,\infty]$, with at least one of them finite and $\alpha\beta \leq 4T$, then $u_0 \equiv 0$.
- If $u_0 \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |u_0(x)||e^{iT\partial_x^2} u_0(y)|e^{|xy|/2T}\,dx\,dy < \infty,$$

then $u_0 \equiv 0$.

- There is $\epsilon > 0$ such that if $u \in C([0, T], L^2(\mathbb{R}^n))$ is a solution of

$$\begin{cases} \partial_t u = i(\triangle u + V(x,t)u + F(x,t)), \\ u(0) = u_0 \end{cases} , \quad \|V\|_{L^1_t L^\infty_x} \le \epsilon,$$

then

$$\sup_{[0,T]} \|e^{\lambda \cdot x} u(t)\| \le \|e^{\lambda \cdot x} u(0)\| + \|e^{\lambda \cdot x} u(T)\| + \|e^{\lambda \cdot x} F\|_{L^1([0,T],L^2(\mathbb{R}^n))}$$

- The identity $\int_{\mathbb{R}^n} e^{2\sqrt{\gamma}\lambda \cdot x - \frac{|\lambda|^2}{2}}\, d\lambda = (2\pi)^{n/2} e^{2\gamma|x|^2}$, gives

$$\sup_{[0,T]} \|e^{\gamma|x|^2} u(t)\| \le \|e^{\gamma|x|^2} u(0)\| + \|e^{\gamma|x|^2} u(T)\| + \|e^{\gamma|x|^2} F\|_{L^1([0,T],L^2(\mathbb{R}^n))}$$

- Assume that $u \in C([0,1], H^2(\mathbb{R}^n))$ is an strong solution of

$$\partial_t u = i\left(\triangle u + V(x,t)u\right),$$

$$V : \mathbb{R}^n \times [0,1] \to \mathbb{C},\ \ V \in L^\infty(\mathbb{R}^n \times [0,1]),$$

$$\nabla V \in L^1([0,1], L^\infty(\mathbb{R}^n)) \text{ and } \lim_{R\to+\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)}\, dt = 0.$$

There is $c = c(n, \|u\|_{L_t^\infty H_x^2}, \|V\|_{L_{t,x}^\infty}, \|\nabla_x V\|_{L_t^1 L_x^\infty})$ such that if $u(0)$ and $u(1)$ are in $H^1(e^{\gamma|x|^2}dx)$ and $\gamma \geq c$, then $u \equiv 0$.

- $u_1,\ u_2 \in C([0,1], H^k(\mathbb{R}^n)),\ k > n/2+1$, are solutions of

$$i\partial_t u + \triangle u + F(u,\overline{u}) = 0,$$

$$F : \mathbb{C}^2 \to \mathbb{C},\quad F \in C^k,\quad F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0.$$

There is $c = c(n, \|u_1\|_{L_t^\infty H_x^2}, \|u_2\|_{L_t^\infty H_x^2}, \|F\|_{C^k})$ such that if $u_1(0) - u_2(0)$ and $u_1(1) - u_2(1)$ are in $H^1(e^{\gamma|x|^2}dx)$ and $\gamma \geq c$, then $u_1 \equiv u_2$.

For u_0 in $\mathbb{S}'(\mathbb{R}^n)$ the following statements are equivalent:

(i) There are two different real numbers t_1 and t_2, such that $e^{a_j|x|^2}e^{it_j\triangle}u_0$ is in $L^2(\mathbb{R}^n)$, for some $a_j > 0$, $j = 1, 2$.

(ii) $e^{b_1|x|^2}u_0$ and $e^{b_2|x|^2}\widehat{u}_0$ are in $L^2(\mathbb{R}^n)$, for some $b_j > 0$, $j = 1, 2$.

(iii) There is $\nu : [0, +\infty) \longrightarrow (0, +\infty)$, such that $e^{\nu(t)|x|^2}e^{it\triangle}u_0$ is in $L^2(\mathbb{R}^n)$, for all $t \geq 0$.

(iv) $g(x) \equiv e^{i\tau|x|^2}u_0(x)$, $\tau \in \mathbb{R}$, verifies (ii) with possible different constants.

(v) $u_0(x + iy)$ is an entire function such that

$$|u_0(x + iy)| \leq Ne^{-a|x|^2+b|y|^2}, \text{ for some } N, a, b > 0.$$

(vi) $\widehat{u}_0(\xi + i\eta)$ verifies (v) with possible different constants.

(vii) There are δ and $\epsilon > 0$ and h in $L^2(e^{\epsilon|x|^2}dx)$ such that $u_0(x) = e^{\delta\Delta}h$.

Logarithmic Convexity

Assume that α, β, T are positive and λ is in $\mathbb{R}^n$. Then,

- $$\|e^{\frac{\lambda\cdot x}{\alpha t+\beta}}\, u(t)\| \le \|e^{\frac{\lambda\cdot x}{\beta}}\, u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\, \|e^{\frac{\lambda\cdot x}{\alpha T+\beta}}\, u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.
- $$\|e^{\frac{\lambda\cdot x}{\alpha t+\beta}}\, u(t)\| \le \|e^{\frac{\lambda\cdot x}{\beta}}\, u(0)\|^{\frac{\beta}{\alpha t+\beta}}\, \|e^{\frac{2\lambda\cdot\xi}{\alpha}}\, \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

- $f(x,t) = e^{\frac{\lambda\cdot x}{\alpha t+\beta}} u(x,t), \ \lambda \in \mathbb{R}^n$.
- $u(0)$ extends to the complex-space $\mathbb{C}^n$ as an analytic function and

$$|u(x+iy,0)| \le N e^{-a|x|^2+b|y|^2}, \text{ for all } x,y \in \mathbb{R}^n.$$

- $\sup_{0\le t\le T} \|e^{a|x|^2} u(t)\| < +\infty$.
- $\partial_t f = \mathcal{S} f + \mathcal{A} f$, $H(t) = \|f(t)\|^2$.
- $\mathcal{S}_t + [\mathcal{S},\mathcal{A}] \ge -\frac{2\alpha}{\alpha t+\beta}\mathcal{S}$.
- $\partial_t^2 \log H(t) \ge -\frac{2\alpha}{\alpha t+\beta}\partial_t \log H(t)$.
- $H(t)^{\alpha t+\beta}$ is logarithmically convex in $[0,T]$.
- $T^{\frac{n}{2}}|u(xT,T)| \to 2^{-\frac{n}{2}}|\widehat{u}(\xi/2,0)|$ and

$$\|e^{\frac{\lambda\cdot x}{\alpha T+\beta}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}} \to \|e^{\frac{2\lambda\cdot\xi}{\alpha}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}}$$

when $T \to +\infty$.

Logarithmic Convexity

- $$\|e^{\frac{|x|^2}{(\alpha t+\beta)^2}} u(t)\| \leq \|e^{\frac{|x|^2}{\beta^2}} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}} \|e^{\frac{|x|^2}{(\alpha T+\beta)^2}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$
  when $0 \leq t \leq T$.
- $$\|e^{\frac{|x|^2}{(\alpha t+\beta)^2}} u(t)\| \leq \|e^{\frac{|x|^2}{\beta^2}} u(0)\|^{\frac{\beta}{\alpha t+\beta}} \|e^{\frac{4|\xi|^2}{\alpha^2}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$
  when $t \geq 0$.
- $$\int_{\mathbb{R}^n} e^{2\sqrt{\gamma}\lambda\cdot\frac{x}{\alpha t+\beta}-\frac{|\lambda|^2}{2}}\, d\lambda = (2\pi)^{n/2} e^{2\gamma\frac{|x|^2}{(\alpha t+\beta)^2}}.$$

Other Convex Weights

Given $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ in $[0, \infty)^n$, the following holds:

- $$\|e^{\frac{\gamma_j x_j^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{\gamma_j x_j^2}{\beta^2}} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}} \|e^{\frac{\gamma_j x_j^2}{(\alpha T+\beta)^2}} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

when $0 \le t \le T$.

- $$\|e^{\frac{\gamma_j x_j^2}{(\alpha t+\beta)^2}} u(t)\| \le \|e^{\frac{\gamma_j x_j^2}{\beta^2}} u(0)\|^{\frac{\beta}{\alpha t+\beta}} \|e^{\frac{4\gamma_j \xi_j^2}{\alpha^2}} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

when $t \ge 0$.

Other Convex Weights

Given $p \in (1,2]$, there is $c = c(p,n) > 0$ such that,

- $$\|e^{\left|\frac{x}{\alpha t+\beta}\right|^p} u(t)\| \le c\, \|e^{\left|\frac{x}{\beta}\right|^p} u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\, \|e^{\left|\frac{x}{\alpha T+\beta}\right|^p} u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.
- $$\|e^{\left|\frac{x}{\alpha t+\beta}\right|^p} u(t)\| \le c\, \|e^{\left|\frac{x}{\beta}\right|^p} u(0)\|^{\frac{\beta}{\alpha t+\beta}}\, \|e^{\left|\frac{2\xi}{\alpha}\right|^p} \widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.
- There is $c = c(n,p)$ such that

$$c^{-1} e^{\frac{|x|^p}{p}} \le \int_{\mathbb{R}^n} e^{\lambda\cdot x - \frac{|\lambda|^{p'}}{p'}} |\lambda|^{\frac{n(p'-2)}{2}}\, d\lambda \le c\, e^{\frac{|x|^p}{p}},\ \text{when } x \in \mathbb{R}^n.$$

Interaction Inequalities

- $$\|e^{\frac{\lambda\cdot(x-y)}{\alpha t+\beta}}\, e^{it\Delta}u_0(x)\, e^{it\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}$$
  $$\leq \|e^{\frac{\lambda\cdot(x-y)}{\beta}}\, u_0(x)\, v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\, \|e^{\frac{\lambda\cdot(x-y)}{\alpha T+\beta}}\, e^{iT\Delta}u_0(x)\, e^{iT\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$
- $$\|e^{\frac{|x-y|^2}{(\alpha t+\beta)^2}}\, e^{it\Delta}u_0(x)\, e^{it\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}$$
  $$\leq \|e^{\frac{|x-y|^2}{\beta^2}}\, u_0(x)\, v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\, \|e^{\frac{|x-y|^2}{(\alpha T+\beta)^2}}\, e^{iT\Delta}u_0(x)\, e^{iT\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

when $0 \leq t \leq T$

Interaction Inequalities

- Letting $T \to +\infty$

$$\|e^{\frac{\lambda\cdot(x-y)}{\alpha t+\beta}}\, e^{it\Delta}u_0(x)\, e^{it\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}$$

$$\leq \|e^{\frac{\lambda\cdot(x-y)}{\beta}}\, u_0(x)\, v_0(y)\|^{\frac{\beta}{\alpha t+\beta}}_{L^2(\mathbb{R}^{2n}_{x,y})}\, \|e^{\frac{2\lambda\cdot(\xi-\eta)}{\alpha}}\, \widehat{u}_0(\xi)\, \widehat{v}_0(\eta)\|^{\frac{\alpha t}{\alpha t+\beta}}_{L^2(\mathbb{R}^{2n}_{\xi,\eta})},$$

-

$$\|e^{\frac{|x-y|^2}{(\alpha t+\beta)^2}}\, e^{it\Delta}u_0(x)\, e^{it\Delta}v_0(y)\|_{L^2(\mathbb{R}^{2n}_{x,y})}$$

$$\leq \|e^{\frac{|x-y|^2}{\beta^2}}\, u_0(x)\, v_0(y)\|^{\frac{\beta}{\alpha t+\beta}}_{L^2(\mathbb{R}^{2n}_{x,y})}\|e^{\frac{4|\xi-\eta|^2}{\alpha^2}}\, \widehat{u}_0(\xi)\, \widehat{v}_0(\eta)\|^{\frac{\alpha t}{\alpha t+\beta}}_{L^2(\mathbb{R}^{2n}_{\xi,\eta})},$$

when $t \geq 0$.

- The "logarithmic convexity" behind the previous interaction inequalities implies interaction Morawtez inequalities for the free particles in the same spirit of other recent interaction inequalities by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao:
-
$$\| |x-y| e^{it\triangle} u_0(x)\, e^{it\triangle} v_0(y) \|^2_{L^2(\mathbb{R}^{2n}_{x,y})}$$
is convex in $\mathbb{R}$.
-
$$\| |x-y|^{\frac{\alpha}{2}} e^{it\triangle} u_0(x)\, e^{it\triangle} v_0(y) \|^2_{L^2(\mathbb{R}^{2n}_{x,y})}$$
is convex, when $n \geq 3$ and $1 \leq \alpha \leq 2$
- The latter authors proved similar results, when $u_0 = v_0$.

Galilean Invariance

The Galilean invariant property of the free Schrödinger group,

$$e^{it\Delta}(e^{i\nu\cdot}\,u_0)(x) = e^{-i|\nu|^2 t} e^{i\nu\cdot x}(e^{it\Delta}u_0)(x-2t\nu)$$

describes the time evolution of the location of the "mass" of a Gaussian decaying solution:

- $$\|e^{\frac{|x+2t\nu|^2}{(\alpha t+\beta)^2}}u(t)\| \le \|e^{\frac{|x|^2}{\beta^2}}u(0)\|^{\frac{\beta(T-t)}{T(\alpha t+\beta)}}\|e^{\frac{|x+2T\nu|^2}{(\alpha T+\beta)^2}}u(T)\|^{\frac{(\alpha T+\beta)t}{(\alpha t+\beta)T}},$$

  when $0 \le t \le T$.

- $$\|e^{\frac{|x+t\nu|^2}{(\alpha t+\beta)^2}}u(t)\| \le \|e^{\frac{|x|^2}{\beta^2}}u(0)\|^{\frac{\beta}{\alpha t+\beta}}\|e^{\frac{4|\xi+\nu|^2}{\alpha^2}}\widehat{u(0)}\|^{\frac{\alpha t}{\alpha t+\beta}},$$

  when $t \ge 0$.

Variable Coefficients and the same Gaussian decay

Assume $\gamma > 0$ and that u in $C([0,1]), L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^n \times [0,1].$$

Then, there is a universal constant $N = N(\gamma)$ such that

$$sup_{[0,1]}\|e^{\gamma|x|^2}u(t)\| + \|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\le Ne^{N\left(M+M^2\right)}\left[\|e^{\gamma|x|^2}u(0)\| + \|e^{\gamma|x|^2}u(1)\|\right],$$

Moreover, $\|e^{\gamma|x|^2}u(t)\|$ is "logarithmically convex" in $[0,1]$:

$$\|e^{\gamma|x|^2}u(t)\| \le e^{N\left(M+M^2\right)}\|e^{\gamma|x|^2}u(0)\|^{1-t}\|e^{\gamma|x|^2}u(1)\|^{t}\ .$$

Conditions on the Potential

(i) $V(x,t) = V_0(x)$ is real-valued and $\|V_0\|_{L^\infty(\mathbb{R}^n)}$ is finite.

(ii) $V(x,t) = V_0(x) + V_1(x,t)$, V_0 as above and

$$\sup_{[0,1]} \|e^{\gamma|x|^2} V_1(t)\|_{L^\infty(\mathbb{R}^n)} \text{ is finite.}$$

(iii) $\|V\|_{L^\infty(\mathbb{R}^n \times [0,1])}$ is finite and

$$\lim_{R \to +\infty} \int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)}\, dt = 0.$$

The "logarithmic convexity" holds, when either (i) or (ii) holds.

Variable Coefficients and Different Gaussian Decay

There is a universal constant $N = N(\alpha\beta)$ such that

$$sup_{[0,1]}\|e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}u(s)\|+\|\sqrt{s(1-s)}e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\leq Ne^{N\left(M+M^2\right)}\left[\|e^{\frac{|y|^2}{\beta^2}}u(0)\|+\|e^{\frac{|y|^2}{\alpha^2}}u(1)\|\right],$$

Moreover, $\log\left(\|e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}u(s)\|^{\alpha s+\beta(1-s)}\right)$ is "convex" in $[0,1]$,

$$\|e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}u(s)\|\leq$$
$$e^{N\left(M+M^2\right)}\|e^{\frac{|y|^2}{\alpha^2}}u(0)\|^{\frac{\beta(1-s)}{\alpha s+\beta(1-s)}}\|e^{\frac{|y|^2}{\beta^2}}u(1)\|^{\frac{\alpha s}{\alpha s+\beta(1-s)}}$$

Conditions on the Potential for Different Gaussian Decay

(i) $V(y,s) = V_0(y)$ is real-valued and $\|V_0\|_{L^\infty(\mathbb{R}^n)}$ is finite.

(ii) $V(y,s) = V_0(y) + V_1(y,s),\ V_0$ as above and

$$\sup_{[0,1]} \|e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}} V_1(s)\|_{L^\infty(\mathbb{R}^n)} \text{ is finite.}$$

(iii) $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}$ is finite and

$$\lim_{R\to+\infty} \int_0^1 \|V(s)\|_{L^\infty(\mathbb{R}^n\setminus B_R)}\, ds = 0.$$

The logarithmic convexity holds, when either *(i)* or *(ii)* holds.

- Assume that u in $C([0,1]), L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^n \times [0,1],$$

$$\|e^{\frac{|y|^2}{\beta^2}} u(0)\| + \|e^{\frac{|y|^2}{\alpha^2}} u(1)\| < +\infty,$$

the potential $V(x,t)$ verifies one of the previous conditions and

$$\alpha\beta < 2.$$

Then, $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$.

- Reduction to case $\alpha = \beta$ using the Appell transform:

$$w(x,t) = t^{-\frac{n}{2}} u(x/t, 1/t) e^{\frac{|x|^2}{4(A+iB)t}}$$

verifies

$$\partial_t w = -(A+Bi)\left(\triangle w + t^{-\frac{n}{2}-2} F(x/t, 1/t) e^{\frac{|x|^2}{4(A+iB)t}}\right),$$

when

$$\partial_s u = (A+Bi)\left(\triangle u + F(y,s)\right).$$

- $$\widetilde{u}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} u\left(\frac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\beta-\alpha)i|x|^2}{4(\alpha(1-t)+\beta t)}},$$

verifies

$$\partial_t \widetilde{u} = i\left(\triangle_x \widetilde{u} + \widetilde{V}(x,t)\widetilde{u}\right), \text{ in } \mathbb{R}^n \times [0,1],$$

$$\widetilde{V}(x,t) = \tfrac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}\, V\left(\tfrac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right)$$

and

$$\|\widetilde{u}(t)e^{\gamma|x|^2}\| = \|u(s)e^{\frac{|y|^2}{(\alpha s+\beta(1-s))^2}}\|\ ,\ \text{when } \gamma = \frac{1}{\alpha\beta}\ ,$$

- $$\|\widetilde{V}\|_{L^\infty(\mathbb{R}^n\times[0,1])} \le \max\left\{\frac{\alpha}{\beta}\,,\,\frac{\beta}{\alpha}\right\}\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])},$$

- $$\int_0^1 \|\widetilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}\,dt = \int_0^1 \|V(s)\|_{L^\infty(\mathbb{R}^n)}\,ds\ ,\ \text{when } s = \tfrac{\beta t}{\alpha(1-t)+\beta t}\ .$$

Theorem

Assume $\gamma > 0$, u in $C([0,1]), L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i\left(\triangle u + V_0(x)u\right), \ \ in\ \mathbb{R}^n \times [0,1].$$

with V_0 real and bounded. Then, there is a universal constant $N = N(\gamma)$ such that

$$sup_{[0,1]}\|e^{\gamma|x|^2}u(t)\| + \|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} \\ \leq Ne^{N\left(M+M^2\right)}\left[\|e^{\gamma|x|^2}u(0)\| + \|e^{\gamma|x|^2}u(1)\|\right],$$

Gaussian decay for diffusions

- Assume that u in $L^\infty([0,1],L^2(\mathbb{R}^n))\cap L^2([0,1]),H^1(\mathbb{R}^n))$ satisfies

$$\partial_t u=(A+iB)\left(\triangle u+V(x,t)u+F(x,t)\right),\ \text{ in } \mathbb{R}^n\times(0,1],$$

$A>0$ and $B\in\mathbb{R}$. Then,

$$e^{-M_T}\|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)T}}u(T)\|$$
$$\le\|e^{\gamma|x|^2}u(0)\|+\sqrt{A^2+B^2}\|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)t}}F(t)\|_{L^1([0,T],L^2(\mathbb{R}^n))},$$

where

$$M_T=\|A\,Re\,V-B\,Im\,V\|_{L^1([0,T],L^\infty(\mathbb{R}^n))}$$

Convexity for Diffusions

- u in $L^\infty([0,1]), L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ verifies

$$\partial_t u = (A + iB)\,(\triangle u + V(x,t)u)\,, \text{ in } \mathbb{R}^n \times [0,1],$$

$A > 0,\ B \in \mathbb{R}$ and $\|V\|_{L^\infty(\mathbb{R}^n)} \le M.$ Then,

$$\|e^{\gamma|x|^2} u(t)\|$$

is "logarithmically convex" in $[0,1]$ and there is N such that

$$\|e^{\gamma|x|^2} u(t)\| \ \le e^{N\left((A^2+B^2)M^2+\sqrt{A^2+B^2}M\right)} \|e^{\gamma|x|^2} u(0)\|^{1-t} \|e^{\gamma|x|^2} u(1)\|^{t},$$

when $0 \le t \le 1$.

Moreover,

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])}\\&\quad\leq Me^{N\left((A^2+B^2)M^2+\sqrt{A^2+B^2}M\right)}\left(\|e^{\gamma|x|^2}u(0)\|+\|e^{\gamma|x|^2}u(1)\|\right).\end{aligned}$$

- When $u(t)=e^{itH}u_0$, $H=\triangle+V_0(x)$,

$$\begin{cases}\partial_t u=i\left(\triangle u+V_0(x)\right), \text{in } \mathbb{R}^n\times[0,1],\\ u(0)=u_0,\end{cases}$$

- $u_\epsilon(t) = e^{(\epsilon+i)tH}u_0$, solves

$$\begin{cases} \partial_t u = (\epsilon + i)\left(\triangle u + V_0(x)\right), \text{in } \mathbb{R}^n \times [0,1], \\ u(0) = u_0. \end{cases}$$

and if $\gamma_\epsilon = \frac{\gamma}{1+4\gamma\epsilon}$, $\|e^{\gamma_\epsilon|x|^2}u_\epsilon(t)\|$ is "logarithmically convex" and

$$\sup_{[0,1]} \|e^{\gamma_\epsilon|x|^2}u_\epsilon(t)\| + \|\sqrt{t(1-t)}e^{\gamma_\epsilon|x|^2}\nabla u_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\leq e^{N\left(M^2+M\right)}\left(\|e^{\gamma_\epsilon|x|^2}u_\epsilon(0)\| + \|e^{\gamma_\epsilon|x|^2}u_\epsilon(1)\|\right).$$

- $u_\epsilon(1) = e^{(\epsilon+i)H}u_0 = e^{\epsilon H}e^{iH}u_0 = e^{\epsilon H}u(1)$ and

$$\sup_{[0,1]} \|e^{\frac{\gamma|x|^2}{1+4\gamma\epsilon t}}e^{\epsilon tH}f\| \leq e^{\epsilon M}\|e^{\gamma|x|^2}f\|.$$

and let ϵ tend to zero in the above inequality.

A Carleman inequality

- The inequality

$$R\|e^{\gamma|x+Rt(1-t)e_1|^2-(1+\epsilon)\frac{R^2t(1-t)}{16\gamma}}g\|_{L^2(\mathbb{R}^{n+1})} \leq$$
$$N_{\gamma,\epsilon}\|e^{\gamma|x+Rt(1-t)e_1|^2-(1+\epsilon)\frac{R^2t(1-t)}{16\gamma}}(\partial_t - i\triangle)g\|_{L^2(\mathbb{R}^{n+1})}$$

holds, when $\gamma > 0$, $\epsilon > 0$, $R > 0$ and $g \in C_0^\infty(\mathbb{R}^{n+1})$.

- It is related to the logarithmic convexity of

$$H(t) = \|e^{\gamma|x+Rt(1-t)e_1|^2-\frac{R^2t(1-t)}{16\gamma}}u(t)\|,$$

when u is a solution to the free Schrödinger evolution in $\mathbb{R}^n \times [0,1]$.

General Framework

- u verifies $\partial_t u = (A + iB)(\triangle u + V(x,t)u)$, $f = e^{\gamma\varphi(x,t)}u$.
- Need to control $H(t) = \|f(t)\|^2$.
- f verifies $|\partial_t f - \mathcal{S}f - \mathcal{A}f| \leq M|f|$, $\mathcal{S}$ is symmetric and $\mathcal{A}$ is skew-symmetric.

$$\partial_t H(t) = 2\Re(\partial_t f, f) = 2\Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) + 2(\mathcal{S}f, f)$$

Is $\mathcal{S}$ a negative operator?

$$\begin{aligned}\partial_t^2 H = 2\partial_t Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) + 2(\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]f, f)\\ + \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2\end{aligned}$$

Is $\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]$ a non-negative operator?

- To get the log-convexity of $H(t)$

$$\partial_t \log H(t) = 2\Re(\partial_t f, f) = 2\Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f)/H + N(t).$$

- $N(t) = 2(\mathcal{S}f, f)/H$ and

$$\begin{aligned}\partial_t N(t) = {} & 2\left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]\, f, f\right)/H \\ & + \left[\|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2\|f\|^2 - \left(\Re\left(\partial_t f - \mathcal{A}f + \mathcal{S}f, f\right)\right)^2\right]/H^2 \\ & + \left[\left(Re\left(\partial_t f - \mathcal{A}f - \mathcal{S}f, f\right)\right)^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2\|f\|^2\right]/H^2\end{aligned}$$

Is $\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]$ a non-negative operator?

- To prove a Carleman inequality where the right hand side is

$$\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|$$

the standard argument is to write

$$\|\partial_t f-\mathcal{S}f-\mathcal{A}f\|^2 = \|\mathcal{S}f\|^2+\|\partial_t f-\mathcal{A}f\|^2-2\mathrm{Re}\iint \mathcal{S}f\overline{\partial_t f - \mathcal{A}f}\,dxdt,$$

but

$$-2\mathrm{Re}\iint \mathcal{S}f\overline{\partial_t f - \mathcal{A}f}\,dxdt = \int \left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]f, f\right)\,dt,$$

- It shows the relation between Carleman inequalities for evolutions and logarithmic convexity properties of solutions.

Parabolic analog

- Assume that

$$|\triangle u - \partial_t u| \le M\left(|u| + |\nabla u|\right)\ ,\ |u(x,t)| \le Me^{M|x|^2}$$

in $\mathbb{R}^n \times [0,1]$ and $|u(x,1)| \le C_k e^{-k|x|^2}$ in $\mathbb{R}^n$ for all $k \ge 1$. Then, $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$.

- If $e^{\frac{|x|^2}{\beta^2}} e^{\triangle} f \in L^2(\mathbb{R}^n)$, $\widehat{f} = e^{\frac{4|\xi|^2}{2^2}} \widehat{e^{\triangle} f} \in L^2(\mathbb{R}^n)$ and $2\beta \le 4$. Then, $f \equiv 0$.

- Let $V \in L^\infty(\mathbb{R}^n \times [0,1])$ and u verify

$$\partial_t u = \triangle u + V(x,t)u\ ,\ \text{in } \mathbb{R}^n \times [0,1],$$

$u(0) \in L^2(\mathbb{R}^n)$, $e^{\frac{|x|^2}{\beta^2}} u(1) \in L^2(\mathbb{R}^n)$ and $\beta < \sqrt{2}$. Then, $u \equiv 0$.