Optimal internal stabilization of the linear system of elasticity

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Coimbra, December, 2007

Damped linear system of elasticity

$$\begin{cases} \mathbf{u}'' - \nabla_x \cdot \boldsymbol{\sigma} + a(x)\,\mathcal{X}_\omega(x)\,\mathbf{u}' = 0 & \text{in} \quad (0,T)\times\Omega, \\ \mathbf{u} = 0 & \text{on} \quad (0,T)\times\Gamma_0, \\ \boldsymbol{\sigma}\cdot\mathbf{n} = 0 & \text{on} \quad (0,T)\times\Gamma_1, \\ \mathbf{u}(0,\cdot) = \mathbf{u_0}, \quad \mathbf{u}'(0,\cdot) = \mathbf{u_1} & \text{in} \quad \Omega, \end{cases}$$

- $\Omega \subset \mathbf{R}^N$: bounded, Lipschitz boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$.
- $\mathbf{u} : [0,T]\times\Omega \to \mathbf{R}^N$ with derivatives $\mathbf{u}'$, $\mathbf{u}''$, $\nabla_x\mathbf{u}$.
- ε, $\boldsymbol{\sigma}$: strain and stress tensors

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla_x\mathbf{u} + (\nabla_x\mathbf{u})^T\right), \quad \boldsymbol{\sigma}(\mathbf{u}) = \left(\sigma_{ij} = a_{ijkl}\varepsilon_{kl}\right).$$

- $\omega \subset \Omega$, $a = a(x)$: damping subset and potential

$$a(x) \geq a_0 > 0.$$

- $\mathbf{n}$: outer normal to Γ_1.

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Interpretation.

Dissipative term $a(x) \mathcal{X}_{\omega}(x) \mathbf{u}' \mapsto$ feedback control mechanism

$\mathcal{X}_{\omega}$ place and shape of sensors and actuators.

Optimization problem

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$$\inf_{\omega\in\Omega_L} J(\mathcal{X}_\omega) = \frac{1}{2}\int_0^T \int_\Omega \left(\left|\mathbf{u}'\right|^2 + \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{u})\right) dx\, dt,$$

$$\Omega_L = \{\omega \subset \Omega : |\omega| = L\,|\Omega|\}, \quad 0 < L < 1.$$

$\mathbf{u}$, solution of

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How do we understand the nature of (some) minimizing sequences of designs?

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Magic term

RELAXATION.

New but related problem

$$\inf_{s\in L^\infty(\Omega)} J(s) = \frac{1}{2}\int_0^T \int_\Omega \left(\left|\mathbf{u}'\right|^2 + \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{u}) \right) dx\, dt,$$

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Theorem

Regularity on initial data: $(\mathbf{u_0}, \mathbf{u_1}) \in H^2(\Omega) \times H^1(\Omega)$. *The second problem (RP) is a full relaxation of the first (P).*

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Underlying reason.

Theorem

$(\mathbf{u_0}, \mathbf{u_1})$ *have the appropriate regularity.* $\mathcal{X}_{\omega_j}$, *minimizing for* (P) *with* $\mathbf{u}_j$ *the associated displacement fields.* $\mathcal{X}_{\omega_j} \rightharpoonup s$ *weak-*$\star$ *in* $L^\infty(\Omega)$ *implies* $\mathbf{u}_j \to \mathbf{u}$ *strong in* $\left(H^1\left((0,T)\times\Omega\right)\right)^N$.

Algorithm and data

Typical descent algorithm

$\eta \in \mathbb{R}^+$, $\eta << 1$, $s_1 \in L^\infty(\Omega)$, perturbation $s^\eta = s + \eta s_1$ of s. Derivative of J with respect to s in the direction s_1:

$$\frac{\partial J(s)}{\partial s} \cdot s_1 = \lim_{\eta \to 0} \frac{J(s + \eta s_1) - J(s)}{\eta}.$$

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- $N = 2$, $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \partial\Omega$,

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda tr(\nabla_x \cdot \mathbf{u})\mathbf{I}_{\mathbf{N}\times\mathbf{N}} + 2\mu\varepsilon(\mathbf{u}),$$

  $\lambda, \mu > 0$, Lamé coefficients.
- $a(x) = a\mathcal{X}_\Omega(x)$: a, constant in Ω.
- Initial conditions:

$$\mathbf{u_0} = (\sin(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\sin(\pi x_2)), \quad \mathbf{u_1} = (0,0).$$

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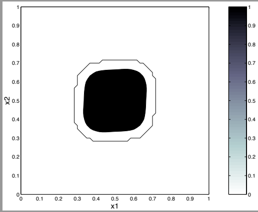

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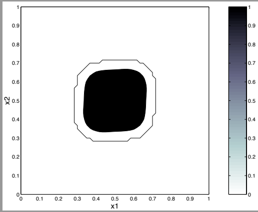

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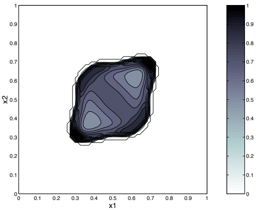

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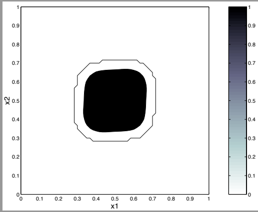

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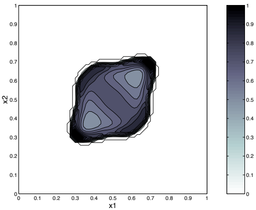

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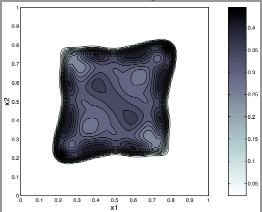

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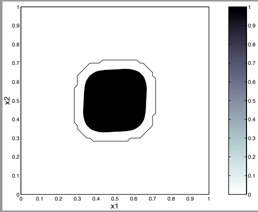

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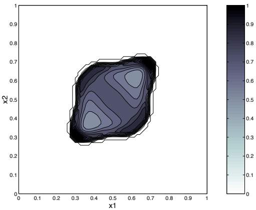

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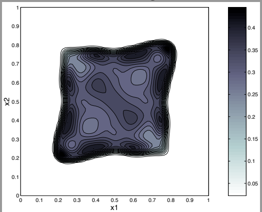

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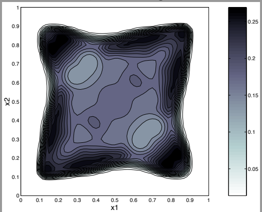

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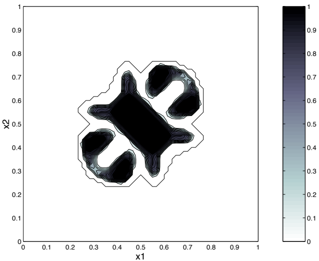

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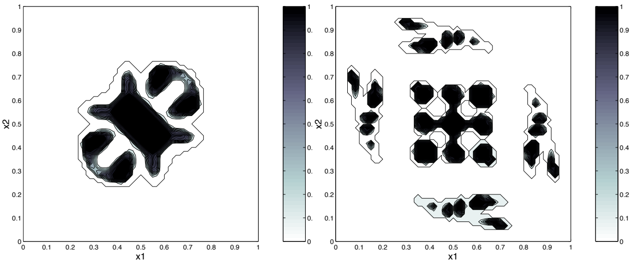

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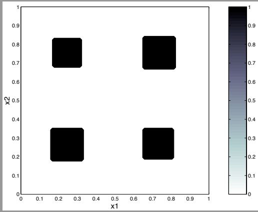

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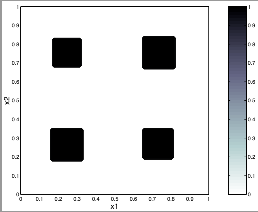

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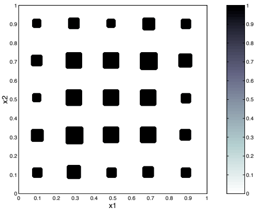

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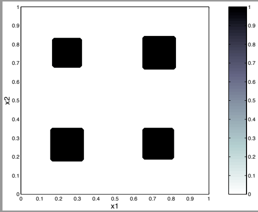

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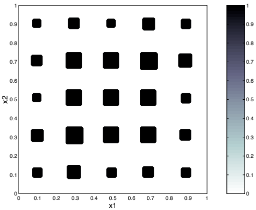

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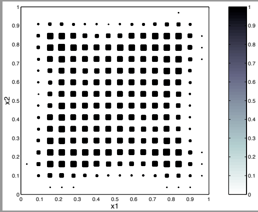

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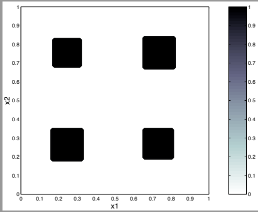

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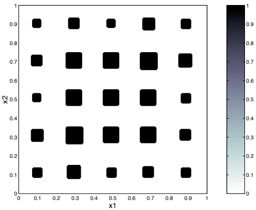

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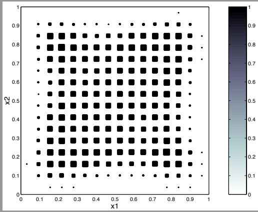

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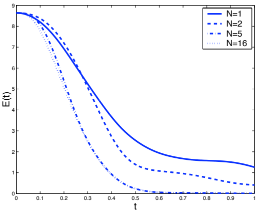

Perspective and main ingredients

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- $(\mathbf{F}_j, \mathbf{G}_j) \in \Lambda_0 \cup \Lambda_{1,C}$ ($C = a(x)\,\mathcal{X}_{\omega_j}(x)\,\mathbf{u}_j$), $A = (A_1, \overline{A})$:

$$\Lambda_{1,C} = \left\{(A, B) : A_1 = B_1 + C,\ \overline{A} = -\boldsymbol{\sigma}\left(\overline{B}\right)\right\}$$

$$\Lambda_0 = \left\{(A, B) : A_1 = B_1,\ \overline{A} = -\boldsymbol{\sigma}\left(\overline{B}\right)\right\}.$$

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- Main property. If $\mathcal{X}_{\omega_j}$ is minimizing, the projection of ν_x onto the second (gradient) component is trivial. This implies the strong convergence of $\mathbf{G}_j$.
- This main property depends in a fundamental way in the form of the cost (energy of the system) under the state law. It would not be true if we were to consider a different cost functional.