Observability of time-discrete conservative systems

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Outline of the talk

An abstract problem

Problem

Let $(X, \|\cdot\|_X)$ be a Hilbert space.
Consider the conservative system

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 \in X, \end{cases} \tag{1}$$

observed through

$$y(t) = Bz(t). \tag{2}$$

Can we observe the time semi-discrete system ?

Conservative system

- $A : \mathcal{D}(A) \to X$ is a skew-adjoint operator.
  $\Longrightarrow$ The energy $\|z(t)\|_X$ is constant.
- A has a compact resolvent.
  $\Longrightarrow$ Its spectrum is discrete.
- Spectrum of A:

$$\sigma(A) = \{i\mu_j : \ j \in \mathbb{N}\}$$

with $(\mu_j)_{j\in\mathbb{N}}$ real numbers, associated to an orthonormal basis Ψ_j

$$A\Psi_j = i\mu_j\Psi_j$$

Examples

- Schrödinger equation in a bounded domain: $A = -\Delta$ + BC.
- Linearized KdV equation in a bounded domain: $A = \partial_{xxx}$ + BC.
- Wave equation in a bounded domain:

$$A = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}.$$

- Maxwell's equation
- …

The observation

- $B : \mathcal{D}(A) \to Y$, $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.

Definition

B is admissible if

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \le K_T \|z_0\|_X^2 \qquad \forall z_0 \in \mathcal{D}(A). \tag{3}$$

Definition

B is exactly observable in time $T > 0$ if

$$k_T \|z_0\|_X^2 \le \int_0^T \|Bz(t)\|_Y^2 \, dt \qquad \forall z_0 \in \mathcal{D}(A). \tag{4}$$

A natural time-discretization: *implicit midpoint*

Consider the following time-discretization

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\triangle t} = A\Big(\dfrac{z^{k+1} + z^k}{2}\Big), & \text{in } X, \quad k \in \mathbb{Z} \\ z^0 \text{ given}, \end{cases} \tag{5}$$

with the output function

$$y^k = Bz^k, \qquad k \in \mathbb{Z}.$$

$\rightsquigarrow$ The discrete system is conservative.

Time semi-discrete observability

Problem

To get a uniform in $\triangle t$ discrete observability estimate for the solutions of the time semi-discrete equation

$$\tilde{k}_T \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B z^k \right\|_Y^2 . \tag{6}$$

- Observability and controllability are dual notions*.
- The inequality (6) shall be derived uniformly with respect to $\triangle t$ for practical purpose in controllability theory, namely to guarantee the convergence of discrete controls†.

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*Lions, 1988, SIAM Review
†Zuazua, 2005, SIAM Review

The main tool : Resolvent estimate

Theorem (Burq & Zworski, 2004, J. AMS, and Miller, 2004, JFA)

Assume A is skew-adjoint with compact resolvent, and B is admissible.
Then the following assertions are equivalent:

1. The continuous system (1)–(2) is exactly observable in some time $T > 0$;
2. There exist constants $M, m > 0$ such that

$$M^2 \left\|(i\omega I - A)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \tag{7}$$

for all $\omega \in \mathbb{R},\ z \in \mathcal{D}(A)$.

Besides, (7) *implies observability in any time* $T > \pi M$.

Related works

- Controllability of *space* semi-discrete system: Zuazua, 2005, SIAM Review.
- Controllability of *time* semi-discrete wave equations:
  - Münch, 2005, M2AN: The 1-d wave equation.
  - Zhang, Zheng & Zuazua, 2007, Preprint: Exact controllability of the time discrete wave equation.

$\longrightarrow$ Spurious high frequency waves
$\Longrightarrow$ Filtering Technics.

$$\mathcal{C}_s = \text{span}\,\{\Psi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}.$$

Statement of the result

Theorem

Assume that B is an admissible operator for A, (A, B) satisfies the resolvent estimate (7) and that

$$\|Bz\|_Y \leq C_B \|Az\| .$$

Then,
$\forall \delta > 0, \exists T_\delta, \exists \triangle t_0 > 0, \forall T > T_\delta, \exists k_{T,\delta} > 0, \forall \triangle t \in (0, \triangle t_0)$, the solution z^k of (5) satisfies

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| Bz^k \right\|_Y^2, \qquad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t}.$$

Remarks

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- T_δ can be chosen as

$$T_\delta = \pi \left[M^2 \left(1 + \frac{\delta^2}{4}\right)^2 + m^2 C_B^2 \frac{\delta^4}{16} \right]^{1/2},$$

which yields πM when $\delta \to 0$.

- Order $1/\triangle t$ is the right scale!
  - We cannot go beyond this scale [a] !
  - No smallness condition of the filtering parameter δ !

[a] Zhang, Zheng & Zuazua, 2007

Resolvent estimate⇒Observability: Continuous case

Given $z^0 \in X$, set $z(t)$ the solution of the continuous system, and define, for $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$g(t) = \chi(t) z(t), \quad f(t) = g'(t) - Ag(t) = \chi'(t) z(t).$$

Then $\hat{f}(\omega) = (i\omega - A)\hat{g}(\omega)$. Apply the resolvent estimate to $\hat{g}(\omega)$:

$$\|\hat{g}(\omega)\|_X^2 \leq m^2 \left\| \widehat{Bg}(\omega) \right\|_Y^2 + M^2 \left\| \hat{f}(\omega) \right\|_X^2 .$$

After integration in ω and Parseval's identity

$$\left(\int \chi(t)^2 \, dt - M^2 \int \chi'(t)^2 \, dt \right) \left\| z^0 \right\|_X^2 \leq m^2 \int \chi(t)^2 \, \|Bz(t)\|_Y^2 \, dt$$

A slightly more general statement

Consider a family of vector spaces $X_{\delta,\triangle t} \subset X$ and a sequence of unbounded operators $(A_{\triangle t}, B_{\triangle t})$ s.t.

(H1) $\forall \triangle t > 0,\ A_{\triangle t}$ is skew-adjoint on $X_{\delta,\triangle t}$, and $A_{\triangle t} X_{\delta,\triangle t} \subset X_{\delta,\triangle t}$ is such that

$$\|A_{\triangle t} z\|_X \leq \frac{\delta}{\triangle t} \|z\|_X, \quad \forall z \in X_{\delta,\triangle t},\ \forall \triangle t > 0.$$

(H2) $\exists C_B$ such that

$$\|B_{\triangle t} z\|_Y \leq C_B \|A_{\triangle t} z\|_X, \qquad \forall z \in X_{\delta,\triangle t},\ \forall \triangle t > 0.$$

(H3) $\exists M, m > 0, \forall z \in X_{\delta,\triangle t} \cap \mathcal{D}(A_{\triangle t}), \forall \omega \in \mathbb{R},\ \forall \triangle t > 0,$

$$M^2 \|(A_{\triangle t} - i\omega I) z\|_X^2 + m^2 \|B_{\triangle t} z\|_Y^2 \geq \|z\|_X^2.$$

General statement

Theorem

Under (H1)–(H3), $\exists T_\delta, \forall\ T > T_\delta, \exists k_{T,\delta}, \forall \triangle t$ small enough, the solution of

$$\frac{z^{k+1} - z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^{k+1} + z^k}{2}\Big), \qquad \text{in } \ X_{\delta,\triangle t}, \qquad k \in \mathbb{Z}, .$$

with $z^0 \in X_{\delta,\triangle t}$ satisfies

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B_{\triangle t} z^k \right\|_Y^2, \qquad \forall\ z^0 \in X_{\delta,\triangle t}.$$

Moreover, T_δ can be taken as before.

General conservative schemes

Abstract conservative scheme given by

$$z^{k+1} = \mathbb{T}_{\triangle t} z^k, \qquad y^k = B z^k, \tag{8}$$

where $\mathbb{T}_{\triangle t}$ is a linear operator such that :

1. $\exists \lambda_{j,\triangle t}, \mathbb{T}_{\triangle t} \Psi_j = \exp(i\lambda_{j,\triangle t} \triangle t)\Psi_j$.
2. There is an explicit relation between $\lambda_{j,\triangle t}$ and μ_j:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t}\, h(\mu_j \triangle t),$$

where $h : \mathbb{R} \to (-\pi, \pi)$ is an increasing smooth function satisfying

$$\lim_{\eta \to 0} \frac{h(\eta)}{\eta} = 1.$$

Theorem

Assume that (A, B) is admissible and observable in the continuous setting, and $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.
Again, discrete observability holds uniformly in $\triangle t$ for any $z^0 \in \mathcal{C}_{\delta/\triangle t}$:

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B\left(\frac{z^k + z^{k+1}}{2} \right) \right\|_Y^2 .$$

Besides, we have the estimate on T_δ:

$$T_\delta \leq \pi \left[M^2 \left(1 + \tan^2\left(\frac{h(\delta)}{2} \right) \right)^2 \sup_{|\eta| \leq \delta} \left\{ \frac{\cos^4(h(\eta)/2)}{h'(\eta)^2} \right\} + m^2 C_B^2 \sup_{|\eta| \leq \delta} \left\{ \frac{2}{\eta} \tan\left(\frac{h(\eta)}{2} \right) \right\}^2 \tan^4\left(\frac{h(\delta)}{2} \right) \right]^{1/2} .$$

Sketch of the proof

Idea: Put the discrete system (8) into the form

$$\frac{z^{k+1}-z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^{k+1}+z^k}{2}\Big), \qquad \text{in } X_{\delta,\triangle t}, \qquad k \in \mathbb{Z},$$

and apply the previous theorem.

$\longrightarrow$ Need to prove a uniform resolvent estimate for $A_{\triangle t}$.

Application 1: The 4th order Gauss Method

$$\begin{cases} \kappa_i = A\Big(z^k + \triangle t \sum_{j=1}^{2} \alpha_{ij}\kappa_i\Big), \qquad i = 1, 2, \\ z^{k+1} = z^k + \frac{\triangle t}{2}(\kappa_1 + \kappa_2), \\ z^0 \in \mathcal{C}_{\delta/\triangle t} \text{ given}, \end{cases} \quad (\alpha_{ij}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{pmatrix}.$$

$\Longrightarrow$ Uniform admissibility holds for $\delta < 2\sqrt{3}$: Here,

$$h(\eta) = 2\arctan\left(\frac{\eta}{2 - \eta^2/6}\right).$$

Application 2: The Newmark method

Consider

$$\begin{cases} \ddot{u} + A_0 u = 0, \\ (u(0), \dot{u}(0)) = v_0, \end{cases} \qquad y(t) = B\dot{u}(t),$$

where A_0 is selfadjoint.
Newmark method with parameter $\beta \geq 1/4$:

$$\begin{cases} \dfrac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} + A_0\Big(\beta u^{k+1} + (1-2\beta)u^k + \beta u^{k-1}\Big) = 0, \\ \Big(\dfrac{u^0 + u^1}{2}, \dfrac{u^1 - u^0}{\triangle t}\Big) = (u_0, v_0), \quad y^{k+1/2} = B\Big(\dfrac{u^{k+1} - u^k}{\triangle t}\Big). \end{cases}$$

Observability holds uniformly in $\triangle t$!

Other applications

- Boundary observation of the Schrödinger equation.
- Boundary observation of the linearized KdV equation.
- Boundary observation of the wave equation.
- And a lot more . . .

Fully discrete schemes

Due to the explicit estimate, we can deal with fully discrete schemes.

1. First, study the space semi-discrete equations:

$$\dot{z} = A_h z, \quad y(t) = B_h z(t)$$

and prove that admissibility and observability hold uniformly in $h > 0$[‡].

2. Second, use the previous theorem to obtain uniform observability in $h, \triangle t > 0$ for the fully discrete scheme, for instance

$$\frac{z^{k+1} - z^k}{\triangle t} = A_h \left(\frac{z^k + z^{k+1}}{2} \right), \quad y^k = B_h z^k.$$

[‡] A lot of results exists ! See Zuazua, 2005, SIAM Rev.

On the time estimate

Consider the wave equation in a domain Ω observed in a subdomain ω.
In this case, the optimal time T^* of control is given by the Geometric Control Condition: Roughly, all the rays of Geometric Optics shall encounter ω in a time smaller than T.

For the time-discrete wave equation, we can use the formalism of the bicharacteristic rays to derive the rays. The relevant quantity is then the group velocity[§]

Group velocity

Our method accurately measures the group velocity !

[§] Trefethen, 1982, SIAM Rev.

Open problems

- Improve the time estimate.
- Weak observability ? Spectral characterization ?
- Spectral characterization of the observability for non conservative system ?
- ...

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Thanks

Thank you for your attention !

Related paper:
On the observability of time-discrete conservative linear systems[¶], S.E., Chuang Zheng & Enrique Zuazua, 2007.

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[¶] available on Enrique's webpage