The homology of amalgams of topological groups

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4 Kač-Moody groups
Definition of amalgam

\( \phi_i: A \rightarrow G_i \) group homomorphisms. The amalgam \( A \star G_i \) is the colimit of the diagram:

\[
\begin{array}{ccc}
A & \longrightarrow & G_1 \\
\uparrow \phi_1 & & \uparrow \\
A & \longrightarrow & : \\
\downarrow \phi_n & & \downarrow \\
G_n & \longrightarrow & : \\
\end{array}
\]
Definition of amalgam

\( \phi_i : A \rightarrow G_i \) group homomorphisms. The amalgam \( \ast_A G_i \) is the colimit of the diagram:

\[ A \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_i} A \]
Definition of amalgam

\( \phi_i : A \to G_i \) group homomorphisms. The amalgam \( *_{A} G_i \) is the colimit of the diagram:

\[
\begin{array}{c}
A \\
\downarrow \phi_1 \\
G_1 \\
\downarrow \phi_1 \\
\downarrow \phi_n \\
G_n \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow \quad *_{A} G_i \\
\rightarrow \quad \exists! \quad \rightarrow H
\end{array}
\]

Amalgams are familiar from Van Kampen's theorem: \( X = U \cup V \), \( U, V, U \cap V \) connected. Then \( \pi_1(X) = \pi_1(U) *_{A} \pi_1(U \cap V) *_{A} \pi_1(V) \).
Definition of amalgam

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\begin{array}{c}
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G_1 \\
\downarrow \phi_i \\
\vdots \\
A \\
\downarrow \phi_n \\
G_n \\
\end{array}
\]

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G_1 & \to & H
\end{array} \]

Amalgams are familiar from Van Kampen’s theorem: \( X = U \cup V, U, V, U \cap V \) connected. Then \( \pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \).

**Aim:** Compute the Pontryagin algebra \( H_* \left( *_{A} G_i \right) \) for \( \phi_i \) inclusions.
Examples of amalgams

1. $SL(2; \mathbb{Z}) = \mathbb{Z}/4 \ast \mathbb{Z}/6$

Figure: The tree of $SL(2; \mathbb{Z})$. 
Examples of amalgams

1. \( SL(2; \mathbb{Z}) = \mathbb{Z}/4 \ast_{\mathbb{Z}/2} \mathbb{Z}/6 \)

![Figure: The tree of SL(2; \mathbb{Z}).](image)

2. \( \text{Diff}(S^2 \times S^2, \omega) \cong \text{colim}(S^1 \times SO(3) \leftarrow SO(3) \xrightarrow{\triangle} SO(3) \times SO(3)) \)
   if \( \omega(S^2 \times 1) = 1, \omega(1 \times S^2) \in ]1, 2]. \)
More examples of amalgams

3. \( \text{Diff}(\mathbb{C}P^2 \# \mathbb{C}P^2, \omega) \cong \text{colim}(U(2) \xleftarrow{(1,0)} S^1 \xrightarrow{(2,1)} U(2)) \) if \( \omega(\mathbb{C}P^1) = 1, \omega(E) = \lambda \in [1, 2[. \)
More examples of amalgams

3. \( \text{Diff}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega) \cong \text{colim}(U(2) \xleftarrow{(1,0)} S^1 \xrightarrow{(2,1)} U(2)) \) if \( \omega(\mathbb{C}P^1) = 1, \omega(E) = \lambda \in [1, 2]. \)

4. \( K \) simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

\[
\begin{array}{ccc}
\star P_i & \pi & \rightarrow K \\
\downarrow & & \\
B & & \\
\end{array}
\]

where \( P_i \) are the minimal parabolics and \( B \) is the Borel subgroup [Kač-Peterson].
More examples of amalgams

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where \( P_i \) are the minimal parabolics and \( B \) is the Borel subgroup [Kač-Peterson].

The homomorphism \( \pi \) induces a surjection on homology [Kitchloo].
Main theorem

**Theorem:** Consider the diagram $F: I \to \text{TopGps}$ described by

\[
\begin{array}{ccc}
A & \xleftarrow{\phi_1} & G_1 \\
& \phi_n & \downarrow \\
& & \cdots \\
& & G_n
\end{array}
\]

Suppose the $\phi_i$ are *inclusions* and the projections $G_i \to G_i/A$ admit local sections. Then
**Main theorem**

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![Diagram](image)

Suppose the $\phi_i$ are *inclusions* and the projections $G_i \rightarrow G_i/A$ admit local sections. Then

1. The following canonical map is a weak equivalence

$$\text{hocolim}_{i \in I} \text{TopGps} F(i) \rightarrow \text{colim}_{i \in I} \text{TopGps} F(i) = \ast G_i$$
Main theorem

**Theorem:** Consider the diagram $F : I \to \text{TopGps}$ described by

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\]

2. There is a functor $G : \Pi_n \to \text{Spaces}$ and a spectral sequence of graded algebras

\[
E_{k,j}^2 = \colim_w H_k G(w) \Rightarrow H_{j+k}(\ast G_i)_A.
\]
Remarks on the main Theorem

- If $A, G_i$ are discrete this is a well known theorem of J.H.C. Whitehead in group cohomology.
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Remarks on the main Theorem

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In this case the $E_2$ term of the spectral sequence is concentrated on the 0-line which is given by

$$E_{k,0}^2 = \operatorname{colim}_{w \in \Pi} H_k G(w) = \left(\begin{array}{c} \ast \\ H_*(A) \end{array} \begin{array}{c} * \\ H_*(G_i) \end{array}\right)_{k}.$$
A functor $F: I \to C$ is called a *diagram in $C$ indexed by $I$*. 
A functor $F: I \to C$ is called a \textit{diagram in $C$ indexed by $I$}. The \textit{colimit} of a diagram $F$ is an object $C \in C$ together with morphisms $F(i) \xrightarrow{\phi_i} C$ satisfying, for all morphisms $\alpha: i \to j$ in $I$,

\[
\begin{array}{ccc}
F(i) & \xrightarrow{\phi_i} & C \\
F(\alpha) \downarrow & & \downarrow \\
F(j) & \xrightarrow{\phi_j} & C
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$$
\begin{array}{c}
F(i) \\
\downarrow F(\alpha) \\
F(j) \\
\downarrow \phi_j \\
\end{array}
\xrightarrow{\exists!} \begin{array}{c}
C \\
\downarrow \phi_i \\
D
\end{array}
$$

**Examples:** Let $C = \text{Sets}$.

1. $\colim \left( X \leftarrow A \xrightarrow{g} Y \right) = (X \bigsqcup Y) / f(a) \sim g(a)$.
2. For $I = G$ a (discrete) group,

$$\colim \left( \begin{array}{c} \circlearrowleft \\
X
\end{array} \xrightarrow{G} \right) = X / G.$$
The trouble with colimits

They are \textit{not} homotopy invariant.
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**Example:** The two diagrams

\[ S^{n-1} \subset D^n \quad \text{and} \quad S^{n-1} \to \ast \]

are naturally homotopy equivalent.
The trouble with colimits

They are *not* homotopy invariant.

**Example:** The two diagrams

\[ S^{n-1} \hookrightarrow D^n \quad \text{and} \quad S^{n-1} \rightarrow \ast \]

are *naturally* homotopy equivalent. Their colimits

\[ S^n \quad \ast \]

are not homotopy equivalent.
Let $\mathcal{C}$ be a category with a notion of homotopy equivalence (e.g. Spaces, TopGps, $\text{Ch}_R^+ =$ chain complexes of modules over a ring $R$).
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Let $C$ be a category with a notion of homotopy equivalence (e.g. Spaces, TopGps, $\text{Ch}_R^+$ = chain complexes of modules over a ring $R$).

Let $C^I$ = category of diagrams in $C$ indexed by $I$.

1st definition of homotopy colimit: hocolim is the terminal homotopy invariant functor mapping to colim
2nd definition of homotopy colimit: To give a map $\text{hocolim}_i F(i) \to C$ consists of giving

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- For each \( i \in I \), a map \( \phi_i : F(i) \to C \),
- For each \( \alpha : i \to j \), a homotopy \( F(i) \times [0, 1] \to C \) between \( \phi_i \) and \( \phi_j \circ F(\alpha) \),
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- For each $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$, homotopies $F(i) \times \Delta^2 \to C$ restricting to the previous homotopies on the edges, etc...
2nd definition of homotopy colimit: To give a map \( \text{hocolim} F(i) \to C \) \( \left(i \in I\right) \) consists of giving

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- For each \( i \xrightarrow{\alpha} j \xrightarrow{\beta} k \), homotopies \( F(i) \times \Delta^2 \to C \) restricting to the previous homotopies on the edges, etc...

This suggests a construction of the homotopy colimit.
Examples of homotopy colimits of spaces

\[ \text{hocolim} \left( \begin{array}{ccc}
X & \xleftarrow{f} & A \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & \end{array} \right) = \]
\[ (X \coprod A \times [0,1] \coprod Y) / ((a,0) \sim f(a), (a',1) \sim g(a')). \]
Examples of homotopy colimits of spaces

\[ \text{hocolim} \left( X \xleftarrow{f} A \xrightarrow{g} Y \right) = \left( \bigvee X \bigoplus A \times [0, 1] \bigoplus \bigvee Y \right) / \left( (a, 0) \sim f(a), (a', 1) \sim g(a') \right). \]

If \( f \) or \( g \) is a cofibration, the map \( \text{hocolim} \to \text{colim} \) is a weak equivalence.

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Examples of homotopy colimits of spaces

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  If \( f \) or \( g \) is a cofibration, the map \( \text{hocolim} \rightarrow \text{colim} \) is a weak equivalence.

- \( \text{hocolim} \left( X \right. \left( \begin{array}{c} \right) \\
  G \\
  X \\
  \end{array} \right) = E_G \times_G X, \) usually called the Borel construction, or the homotopy orbit space.
Examples of homotopy colimits of spaces

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X \xleftarrow{f} A \xrightarrow{g} Y \end{array} \right) = \rightleftharpoons \)
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- \( \text{hocolim} \left( \begin{array}{c}
X \overset{G}{\longrightarrow} \end{array} \right) = EG \times_G X \), usually called the Borel construction, or the homotopy orbit space.
If the action is free the map \( EG \times_G X \to X/G \) is a weak equivalence.
Examples of homotopy colimits of spaces

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  \]
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  If the action is free the map \( EG \times_G X \to X/G \) is a weak equivalence.

- Inclusions of topological groups are not cofibrations of topological groups!
**Homotopy colimits of topological groups**

**Theorem [Kan]:** There is an equivalence of homotopy theories

\[
\text{Ho}(\text{TopGps}) \leftrightarrow \text{Ho}(\text{ConnectedSpaces})
\]

given by the loop and classifying space functors.
Proof of theorem

Homotopy colimits of topological groups

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This is how one would usually think of homotopy colimit of topological groups.
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**New approach:** For \( n \geq 2 \), let \( \Pi_n \) be the category with

- Objects: finite ordered sets labeled with \( n \) colors
- Morphisms: order preserving maps preserving the colors.
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- **Objects:** finite ordered sets labeled with \( n \) colors
- **Morphisms:** order preserving maps preserving the colors.

\( \Pi_n \) is a monoidal category with product given by concatenation. The unit is the empty word.
Homotopy colimits of topological groups II

Given a diagram of topological groups

\[
\begin{array}{ccc}
A & \longrightarrow & A \\
\downarrow & & \downarrow \\
G_1 & \longrightarrow & \cdots & \longrightarrow & G_n
\end{array}
\]

and \( w = (a_1, \ldots, a_k) \in \Pi_n \) with \( a_i \in \{1, \ldots, n\} \) define

\[
G(w) = G_{a_1} \times_A G_{a_2} \times_A \cdots \times_A G_{a_n}
\]
Given a diagram of topological groups

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A & \\
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\]

Define \( G(w \to w') \) using multiplication and inclusions.
Given a diagram of topological groups

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G(w) = G_{a_1} \times_A G_{a_2} \times_A \cdots \times_A G_{a_n}
\]

Define \( G(w \to w') \) using multiplication and inclusions.

\[
\text{colim}_{w \in \Pi_n} G(w) = \ast G_i \\
\text{with} \quad G_i \quad \text{in} \quad \mathcal{A}
\]
Homotopy colimits of topological groups II

Given a diagram of topological groups

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\begin{array}{c}
A \\
\downarrow \\
G_1 & \cdots & G_n
\end{array}
\]

and \( w = (a_1, \ldots, a_k) \in \Pi_n \) with \( a_i \in \{1, \ldots, n\} \) define

\[
G(w) = G_{a_1} \times_A G_{a_2} \times_A \cdots \times_A G_{a_n}
\]

Define \( G(w \rightarrow w') \) using multiplication and inclusions.

- \( \operatorname{colim}_{w \in \Pi_n} G(w) = \ast G_i \)
- \( G : \Pi_n \rightarrow (A - \text{Spaces} - A) \) is a monoidal functor, hence
Homotopy colimits of topological groups II

Given a diagram of topological groups

\[
\begin{array}{c}
A \\
\downarrow \\
G_1 \quad \cdots \quad G_n \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
G_1 \quad \cdots \quad G_n
\end{array}
\]

and \( w = (a_1, \ldots, a_k) \in \Pi_n \) with \( a_i \in \{1, \ldots, n\} \) define

\[
G(w) = G_{a_1} \times_A G_{a_2} \times_A \cdots \times_A G_{a_n}
\]

Define \( G(w \to w') \) using multiplication and inclusions.

- \( \text{colim}_{w \in \Pi_n} G(w) = \ast_{G_i} \)
- \( G : \Pi_n \to (A - \text{Spaces} - A) \) is a monoidal functor, hence
- \( \text{hocolim}_{w \in \Pi_n} G(w) \) is a monoid and the canonical map \( \text{hocolim} \to \text{colim} \) is a map of monoids.
Prop: $\operatorname{hocolim}_{w \in \Pi_n} G(w)$ is homotopy equivalent to

$$\operatorname{hocolim}^\text{TopGps} A \leftarrow \cdots \rightarrow G_1 \leftarrow \cdots \rightarrow G_n$$

This implies the first part of the Theorem.
Proof of theorem

Homotopy colimits of topological groups III

Prop: \( \operatorname{hocolim}_{w \in \Pi_n} G(w) \) is homotopy equivalent to \( \operatorname{hocolim} \operatorname{TopGps} \)

\[
\begin{array}{ccc}
\text{hocolim} & A & \text{colim} \\
\downarrow & \downarrow & \downarrow \\
G_1 & \cdots & G_n
\end{array}
\]

Prop: If \( \phi_i : A \to G_i \) are inclusions, the map

\[
\operatorname{hocolim}_{w \in \Pi_n} G(w) \to \operatorname{colim}_{w \in \Pi_n} G(w) = \ast_{A} G_i
\]

is a weak equivalence.
Homotopy colimits of topological groups III

**Prop:** $\hocolim_{w \in \Pi} G(w)$ is homotopy equivalent to

$$\hocolim_{\text{TopGps}} A \rightarrow G_1 \rightarrow \cdots \rightarrow G_n$$

**Prop:** If $\phi_i : A \rightarrow G_i$ are inclusions, the map

$$\hocolim_{w \in \Pi} G(w) \rightarrow \colim_{w \in \Pi} G(w) = \ast G_i$$

is a weak equivalence.

This implies the first part of the Theorem.
The spectral sequence

Given $F : I \to \text{Spaces}$ there is a standard spectral sequence

$$E^2_{p,q} = \text{colim}_p H_q(F(i); R) \Rightarrow H_{p+q}(\text{hocolim}_i F(i); R)$$
The spectral sequence

Given $F : I \to \text{Spaces}$ there is a standard spectral sequence

$$E^2_{p,q} = \text{colim}_{i \in I} H_q(F(i); R) \Rightarrow H_{p+q}(\text{hocolim}_{i \in I} F(i); R)$$

**Example:** For $X \xrightarrow{G}$ this is the usual spectral sequence $H_p(G; H_q(X; R)) \Rightarrow H_{p+q}(EG \times_G X; R)$. 
Given $F : I \to \text{Spaces}$ there is a standard spectral sequence

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**Example:** For $X \downarrow \downarrow G$ this is the usual spectral sequence

$$H_p(G; H_q(X; R)) \Rightarrow H_{p+q}(EG \times_G X; R).$$

Applying this to $G : \Pi_n \to \text{Spaces}$ gives the spectral sequence in the second part of the Theorem.
The spectral sequence

Given $F : I \to \text{Spaces}$ there is a standard spectral sequence

$$E^2_{p,q} = \colim_{i \in I} H_q(F(i); R) \Rightarrow H_{p+q}(\hocolim_{i \in I} F(i); R)$$

**Example:** For $X \xrightarrow{G} \text{Spaces}$ this is the usual spectral sequence

$$H_p(G; H_q(X; R)) \Rightarrow H_{p+q}(EG \times_G X; R).$$

Applying this to $G : \Pi_n \to \text{Spaces}$ gives the spectral sequence in the second part of the Theorem.

$G$ monoidal $\Rightarrow$ the spectral sequence is multiplicative.
**Rank 1 parabolics**

**Recall:** $K$ simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

\[ \ast P_i \overset{\pi}{\longrightarrow} K \]

where $P_i$ are the *minimal parabolics* and $B$ is the *Borel* subgroup.
Recall: $K$ simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

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$B$ deformation retracts to the maximal torus $T^n$. 
Rank 1 parabolics

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$$\pi : \bigstar_{B} P_i \rightarrow K$$

where $P_i$ are the minimal parabolics and $B$ is the Borel subgroup.

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The minimal parabolics have semisimple rank 1. They deformation retract (in the simply connected case) to either

$$T^{n-1} \times SU(2) \quad \text{or} \quad T^{n-2} \times U(2).$$
Rank 1 parabolics

Recall: $K$ simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

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Want to prove the algebra $H_*(\bigast K_i)$ with $K_i$ one of the two groups above is finitely generated.
A cell decomposition of $SU(2)$

$$SU(2) = \left\{ \begin{bmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{bmatrix} : |z_1|^2 + |z_2|^2 = 1 \right\}.$$
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$$e^2 = \left\{ (z_1, z_2) : 0 \leq |z_1| < 1, z_2 = \sqrt{1 - |z_1|^2} \right\}.$$
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$e^2$ provides a transverse to both right and left actions of $S^1$ on $SU(2) \setminus S^1$.

$$e^3 = e^1 e^2$$
The homology DGA of $SU(2)$

The cellular chains form a differential graded algebra

$$C_*(SU(2); \mathbb{Z}) = \mathbb{Z}\langle x_1, z_2 \rangle / \langle x_1^2, z_2^2, x_1z_2 + z_2x_1 \rangle$$

with $\partial(x_1) = 0$, $\partial(z_2) = x_1$. 
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$T^n$ has an obvious multiplicative cell decomposition. The adapted cell decomposition for $SU(2)$ gives a multiplicative cell decomposition for $\ast$ $K_i$ when $K_i$ are of type $T^{n-1} \times SU(2)$. 
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This gives a simple formula for the differential graded algebra of cellular chains $C_\ast(^\ast K_i; \mathbb{Z})$ in this case.
A simple example

\[ C_\bullet(\text{SU}(2) \times S^1 \ast S^1 \times \text{SU}(2); \mathbb{Z}) = \mathbb{Z}[x_1, y_1, z_2, w_2]/J \]

with \( J \) the ideal

\[ J = \langle x_1^2, y_1^2, z_2^2, w_2^2, x_1 z_2 + z_2 x_1, y_1 w_2 + w_2 y_1, x_1 y_1 + y_1 x_1, z_2 y_1 - y_1 z_2, w_2 x_1 - x_1 w_2 \rangle \]
A simple example

\[ C_\ast(SU(2) \times S^1 \ast_{T^2} S^1 \times SU(2); \mathbb{Z}) = \mathbb{Z}[x_1, y_1, z_2, w_2]/J \]

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It follows that

\[ H_\ast(SU(2) \times S^1 \ast_{T^2} S^1 \times SU(2); \mathbb{Z}) = \mathbb{Z}(A_3, B_3) \otimes \mathbb{Z}[C_4]. \]
A simple example

$$C_*(\text{SU}(2) \times S^1 \ast_{T^2} S^1 \times \text{SU}(2); \mathbb{Z}) = \mathbb{Z}[x_1, y_1, z_2, w_2]/J$$

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More generally one can prove in this way that the homology is finitely generated when the factors are all of type $T^{n-1} \times \text{SU}(2)$. 