Mathematical Games

Jorge Nuno Silva

Coimbra, October 13, 2007

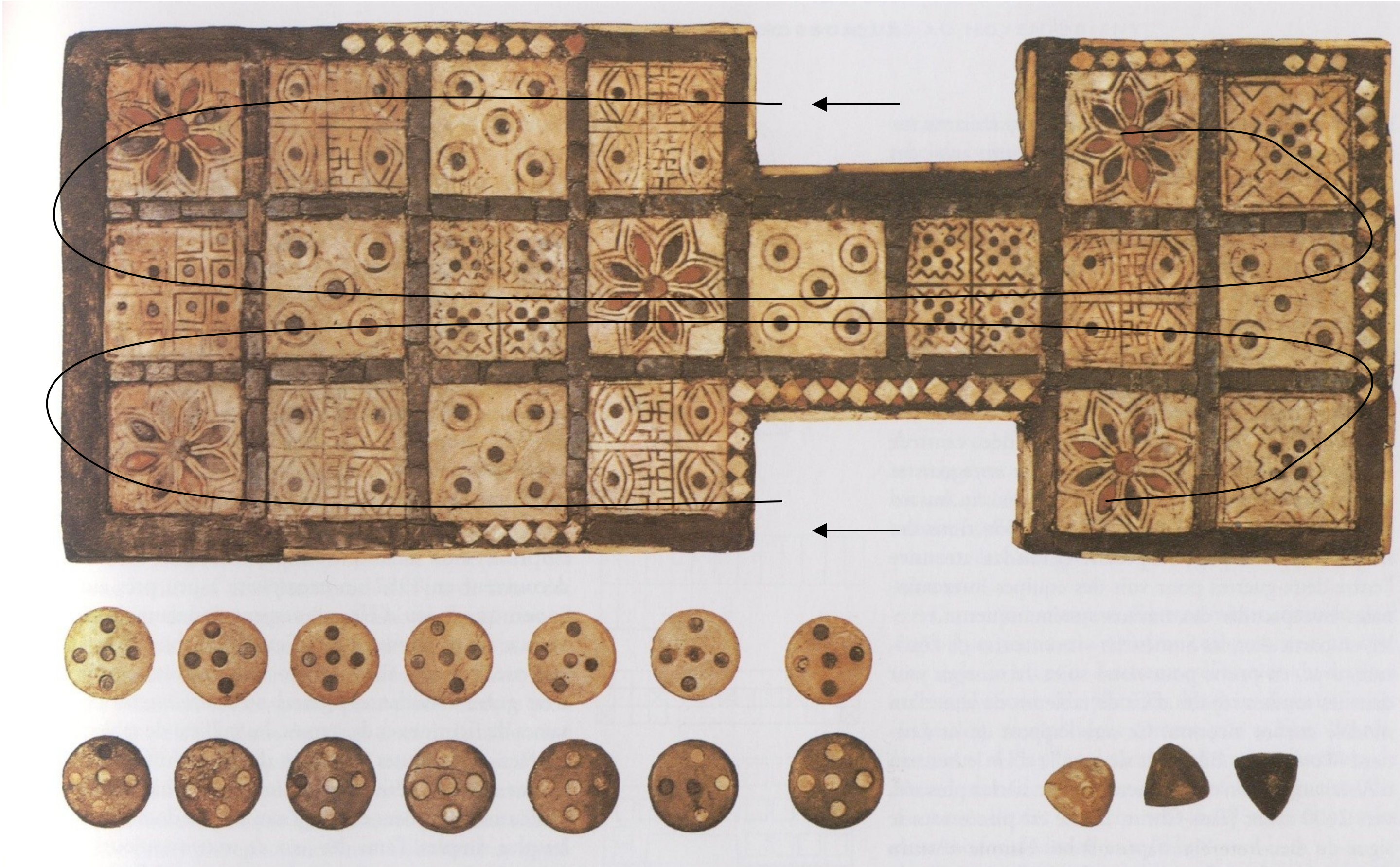
What are mathematical games?

Strategy games are mathematical in more than one way. Some in ways that are tough to identify...

Ur

The oldest game we know the rules of. Found in *Ur*, is maybe 5 000 years old.

It is a race game with some interaction between players. There are “good” houses and “bad” houses.



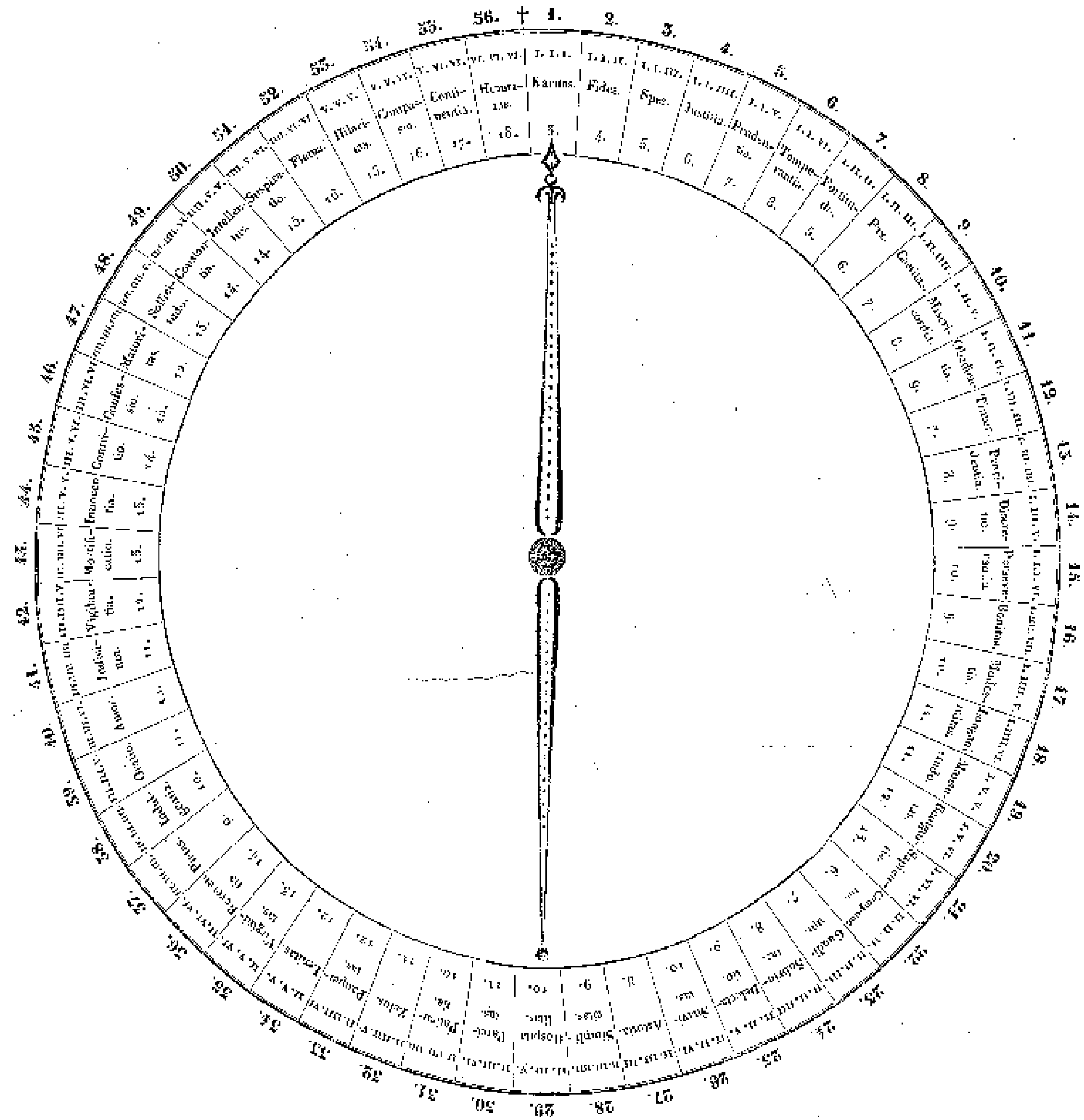
In the middle row pieces can be sent out (except when in a special house).

Ludus Regularis

Invented by a priest, in the 10th century, in France.

The main idea was to get a game religious people could play.

It uses three dice (first occurrence of 21 and 56 in a game).


Fides.
Spes.
Justitia.
Pax.

Boethius

- Anicius Manlius Severinus Boethius (480-524) was born in Rome in a powerful family, the Anicii. His father was consul, and he became consul himself.
- Boethius was raised by Symmachus, who taught him Greek and Philosophy.
- Boethius planned the translation of Plato and Aristotle. He did part of this.

- Wrote works on Arithmetic and Music.
- Was sentenced to death by the emperor Theodoric. While under house arrest, he wrote *Consolation of Philosophy,* where he

  turned to philosophy, instead of God, in those hard last times.

  This work was popular for several hundred years.

Quadrivium

- Arithmetic, Music, Geometry and Astronomy. (With Grammar, Rhetorics and Logic the seven liberal arts).
- “Quadrivium” was a word used by Boethius, meaning the four fold way leading to Philosophy (the main knowledge was Theology).

Boethius’ Arithmetic

- Was Pythagoric, a version of Nichomacus of Gerasa (fl -100) own *Arithmetic*.
- Without proofs or any deductive struture, was a collection of terminology and results, with mystic and religious commentaries.
- Even, odd, evenly even, prime, composite, perfect, abundant,...

- Evenly even: 4, 16, 32, ... (when divided by 2, we get an even number).
- Prime: 2, 3, 5, 7, 11, ... (can only be divided by themselves and by 1).
- Composite: 4=2x2, 6=2x3, 8=2x2x2, ...
- A perfect number: 6, because 6=1+2+3.
- A deficient number: 8, because 1+2+4=7, which is less than 8.
- An abundant number: 12, because 1+2+3+4+6=16, which is more than 12.

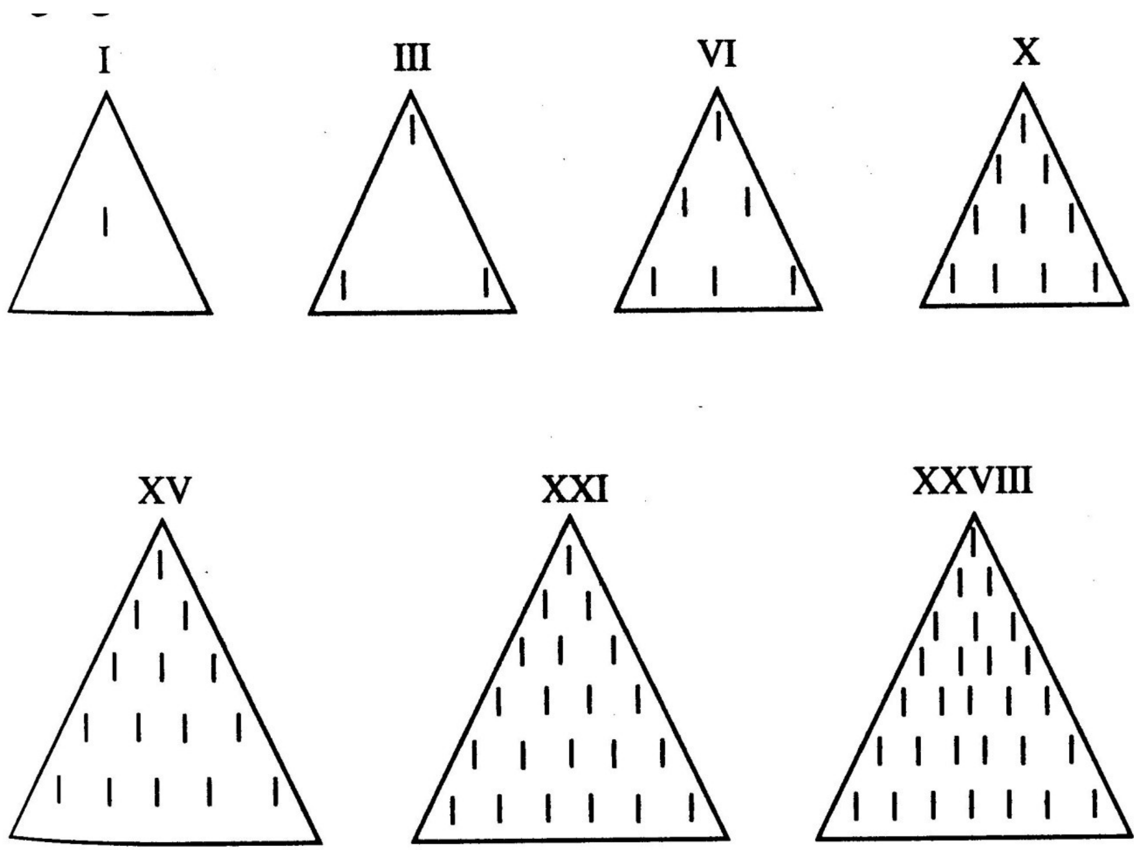
- Some results are also in Euclid’s Elements, but occur here without proof, like the Erathostenes sieve and the algorithm to find the GCD.
- Main topics are particular relations between integers, like *multiplex* (n:1), *superparticular* ((n+1): n), *superpartient* ((n+ m):n), and means.

Three main means

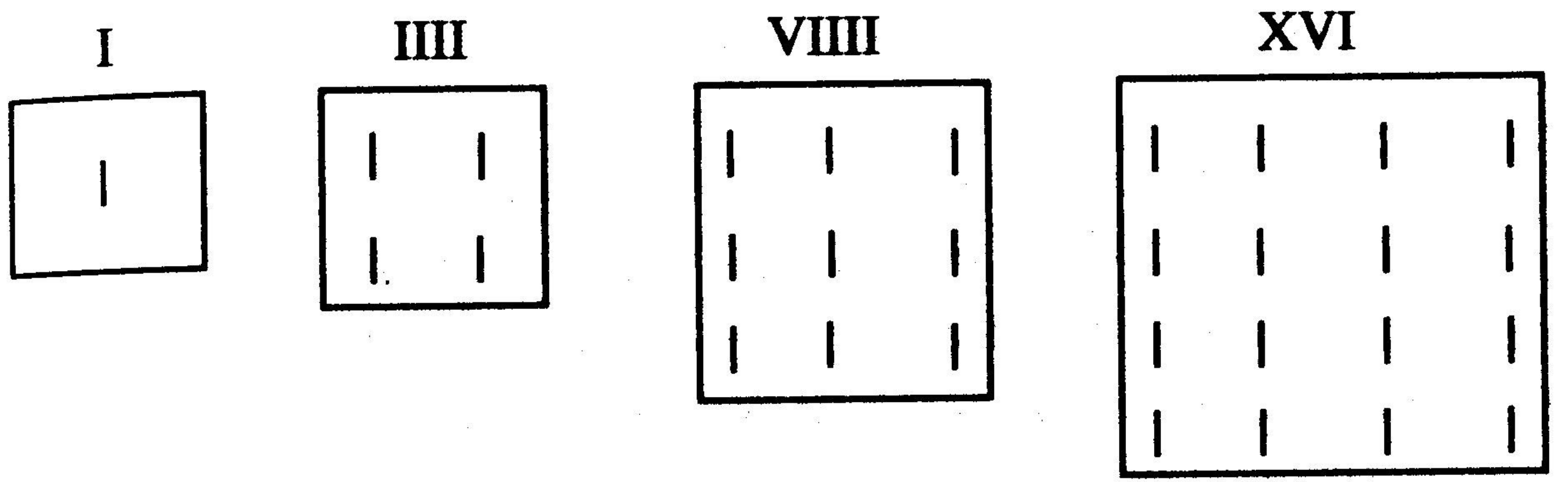
- For numbers $x < y < z$, they define a progression
  - Arithmetic if $(z-y)/(y-x)=x/x$
  - Geometric if $(z-y)/(y-x)=y/x$
  - Harmonic if $(z-y)/(y-x)=z/x$
  - (Boethius gives seven more...)

Boethius' Figurate Numbers

Triangular

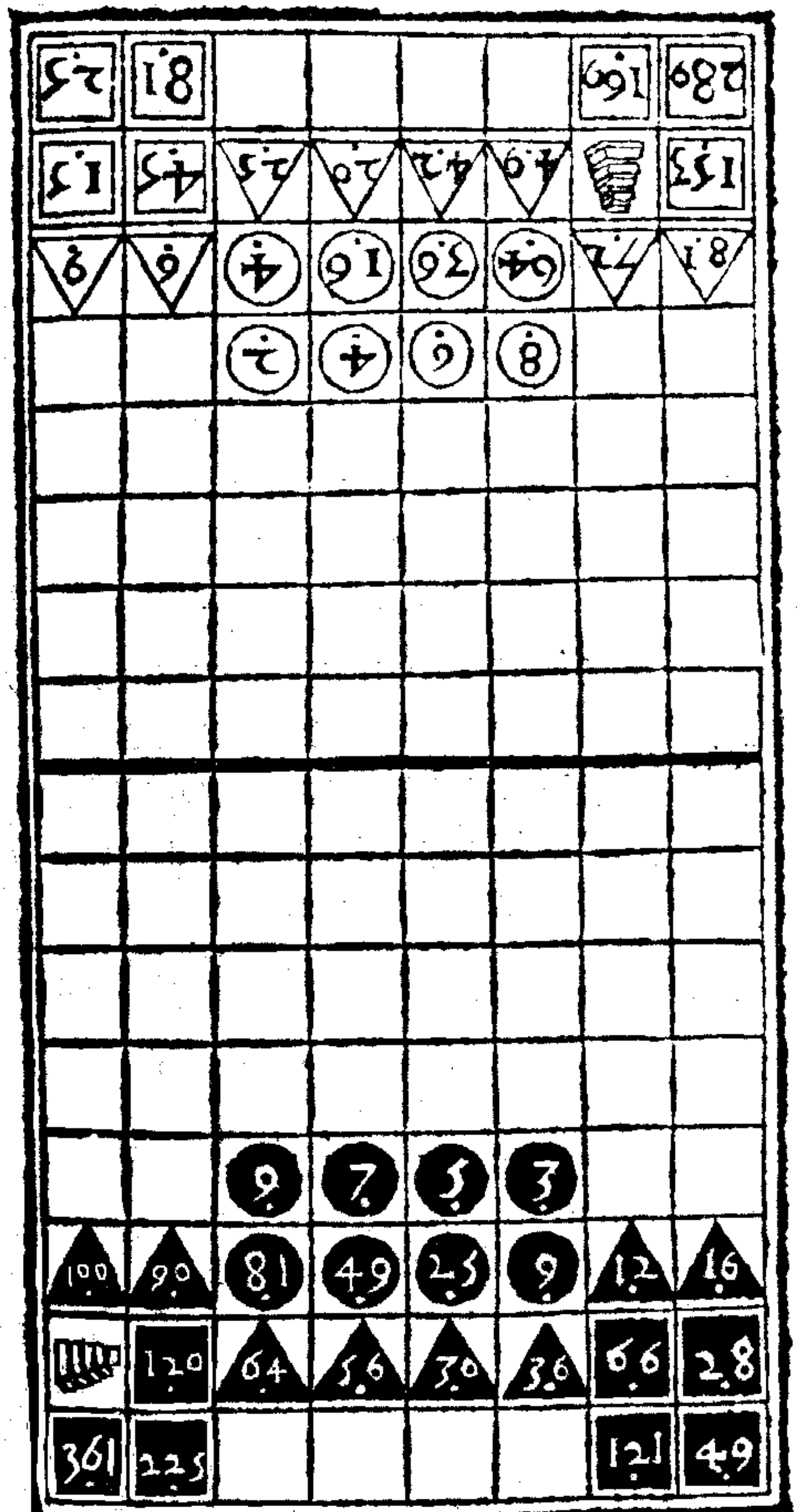


- Square

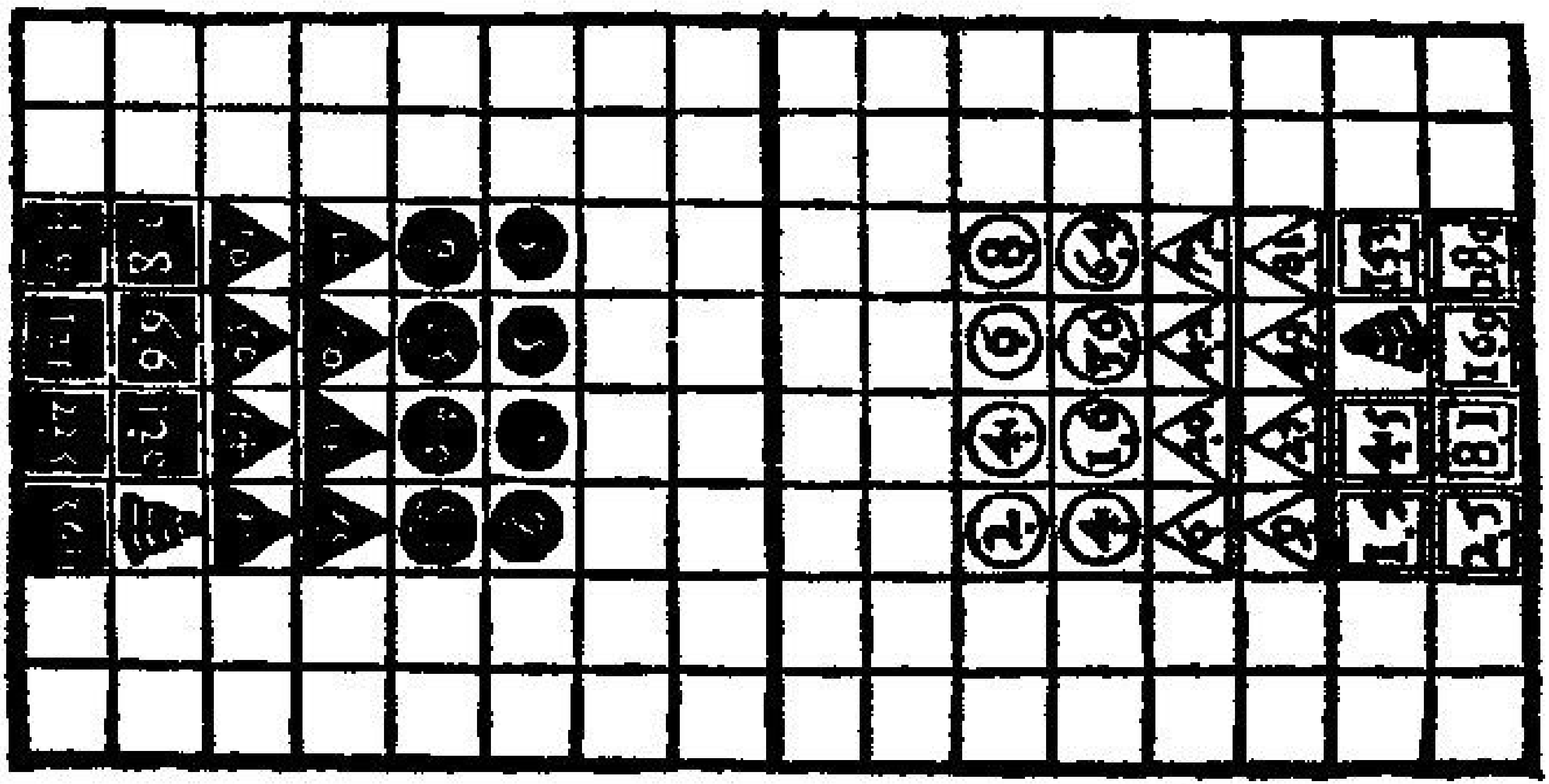


Rythmomachia

- Rhythmo=proportion, machia=fight.
- Invented as a pedagogical game, to help the teaching of Arithmetic, in the 11th century.
- Even the setup of the pieces on the board was an important experience...



25 81 169 289
15 45 25 20 42 49 153
9 6 4 16 36 64 72 81
2 4 6 8
9 7 5 3
100 90 81 49 25 9 12 16
120 64 56 30 36 66 28
361 225 121 49



- This game was very respected, Roger Bacon (1214-1294) (Oxford) included it in the list of sciences (after Pythagorean tables, astronomical tables, pratical geometry and commerce).
- Thomas Bradwardine (1290-1349) (Oxford) wrote a manual.
- Robert Burton (1577-1640), in *Anatomy of Melancholy*, lists some winter/loneliness pastimes, among them is the Philosopher's Game.

- Thomas More (1478-1535), in *Utopia*:

  “They know nothing about gambling with dice or other such foolish and ruinous games. They play two games not unlike our chess. One is a battle of numbers, in which one number plumbers another. The other is a game in which the vice battles against the virtues.”

BAROZZI (1537-1604)

- Learned Greek, physics, metaphysics, philisophy, arts, mathematics, ...
- Francesco Barozzi studied at Padua University (where he taught in 1559)
- Lived in Crete and in Venice.
- B. translated Proclus' version of Euclid's *Elements*. He was 22 by then.
- Played a major role in Paduan debates on the philosophy of mathematics

"The fulfilment of the soul is accomplished only by the study of philosophy, and mathematics, as the most important part of philosophy, should be studied above all".

Barozzi planned to translate several Greek authors, and he did some.

B. tried to show that the views on the role of mathematics in the study of nature, from Aristotle (tabula rasa, learn from exp.) and Plato (innate ideas obsc. by exp.), where consistent with each other.

- While for Aristotle observation and induction were the right way to go about studying nature, Plato believed in a universe made up of mathematical relations and numerical harmonies...
- Barozzi addressed also the problem of the meaning of mathematical demonstration, mathematical truth,... the epistemological discussions of his time.
- For Aristotle, mathematics has a small role in addressing nature, because its truths hold only to immaterial things...

- For Plato, number is the link between the intellect and the universe.

- Barozzi, as other astronomers of the renaissance, got tired of the old text of Sacrobosco (*On the sphere*), and wrote *Cosmographia (1585)* to replace it.

  He was also very interested in prophecies and necromancy, which got him in trouble with the Inquisition. He was condemned for causing a storm, for talking with dead people, for love charms,...

- Through his activities as restorer and patron of mathematics he helped to bring about that renaissance of mathematics, with humanist touch, which was to culminate in the scientific revolution.

- In 1572 he published a manual of Rythmomachia, based on an older one written by Boissière (1554).

IL NOBILISSIMO ET ANTIQVISSIMO GIVOCO PYTHAGOREO NOMINATO Rythmomachia CIOE BATTAGLIA DE CONSONANTIE DE NVMERI,

Ritrouato per vtilità, & solazzo delli Studiosi.

Et al presente per Francesco Barozzi Gentil'huomo Venetiano in lingua volgare in modo di Paraphrasi composto.

IN VENETIA.

Appresso Gratioso Perchacino. 1572.

The setup

- Each piece shows the same number on both sides, each side with one colour.
- On the first row the first four even (odd) numbers
- On the second, their squares.

$$\begin{array}{ccccc} 2 & 4 & 6 & 8 & n \\ 4 & 16 & 36 & 64 & n^2 \end{array}$$

- To get the third, we add the first two rows.

$$
\begin{array}{ccccc}
2 & 4 & 6 & 8 & n \\
4 & 16 & 36 & 64 & n^2 \\
6 & 20 & 42 & 72 & n(n+1)
\end{array}
$$

- To get the second row of white triangles, Barozzi adds the values of the first row of white triangles to the first row of black rounds.
- Equivalently, because

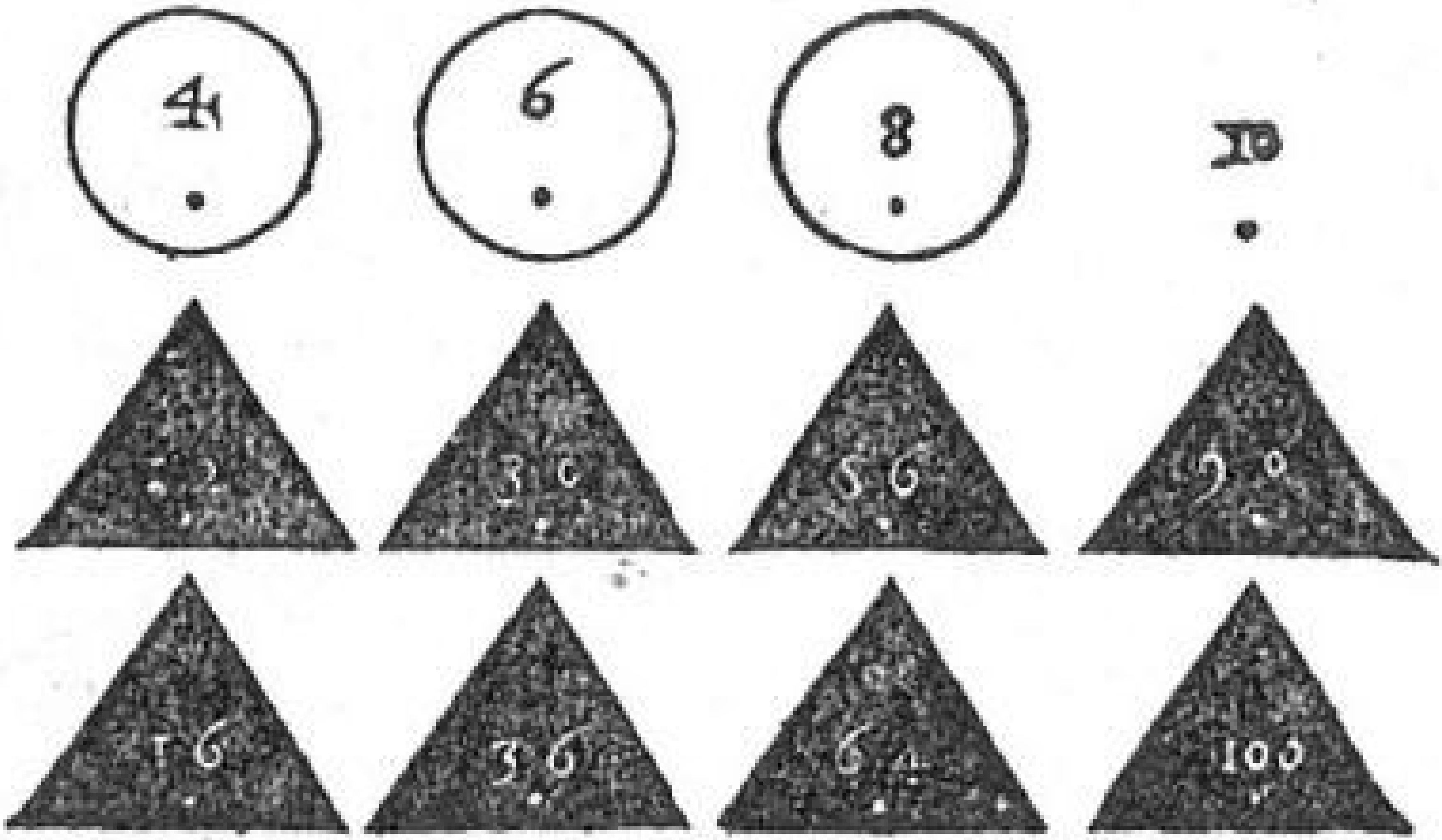
$n(n+1)+n+1=(n+1)(n+1)$

3	5	7	9
6	20	42	72
9	25	49	81

- On the fourth, the squares of the successors of the elements of the first row.

2	4	6	8	n
4	16	36	64	n^2
6	20	42	72	$n(n+1)$
9	25	49	81	$(n+1)^2$

- Each time Barozzi needs n+1 for the largest element of a black row, like 10 (9+1) he uses the number, in all the other cases he uses the values of white pieces.
- We present here the way we can get all the numbers withou leaving each colour.


4
6
8
10
30
56
90
16
36
64
100

- Add two more rows to get the fifth.

2	4	6	8	n
4	16	36	64	n^2
6	20	42	72	$n(n+1)$
9	25	49	81	$(n+1)^2$
15	45	91	153	$(n+1)(2n+1)$

- Double each original number. Add one and square what you got.

2	4	6	8	n
4	16	36	64	n^2
6	20	42	72	$n(n+1)$
9	25	49	81	$(n+1)^2$
15	45	91	153	$(n+1)(2n+1)$
25	81	69	289	$(2n+1)^2$

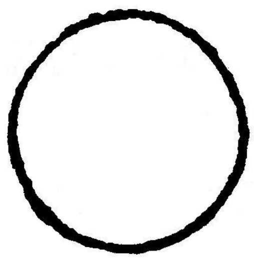
Kings or Pyramids

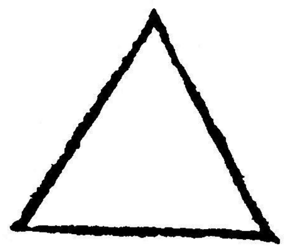
The pieces that correspond to the numbers 91 and 190 are special.

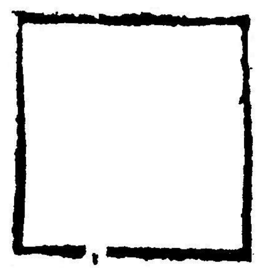
They are made out of stories, according to the decompositions:

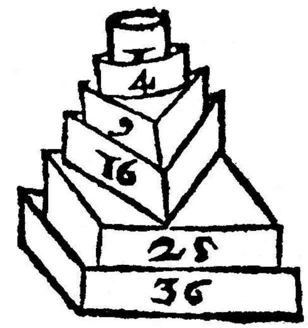
$$91=1+4+9+16+25+36$$

$$190=16+25+36+49+64$$

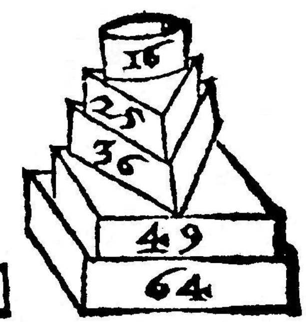




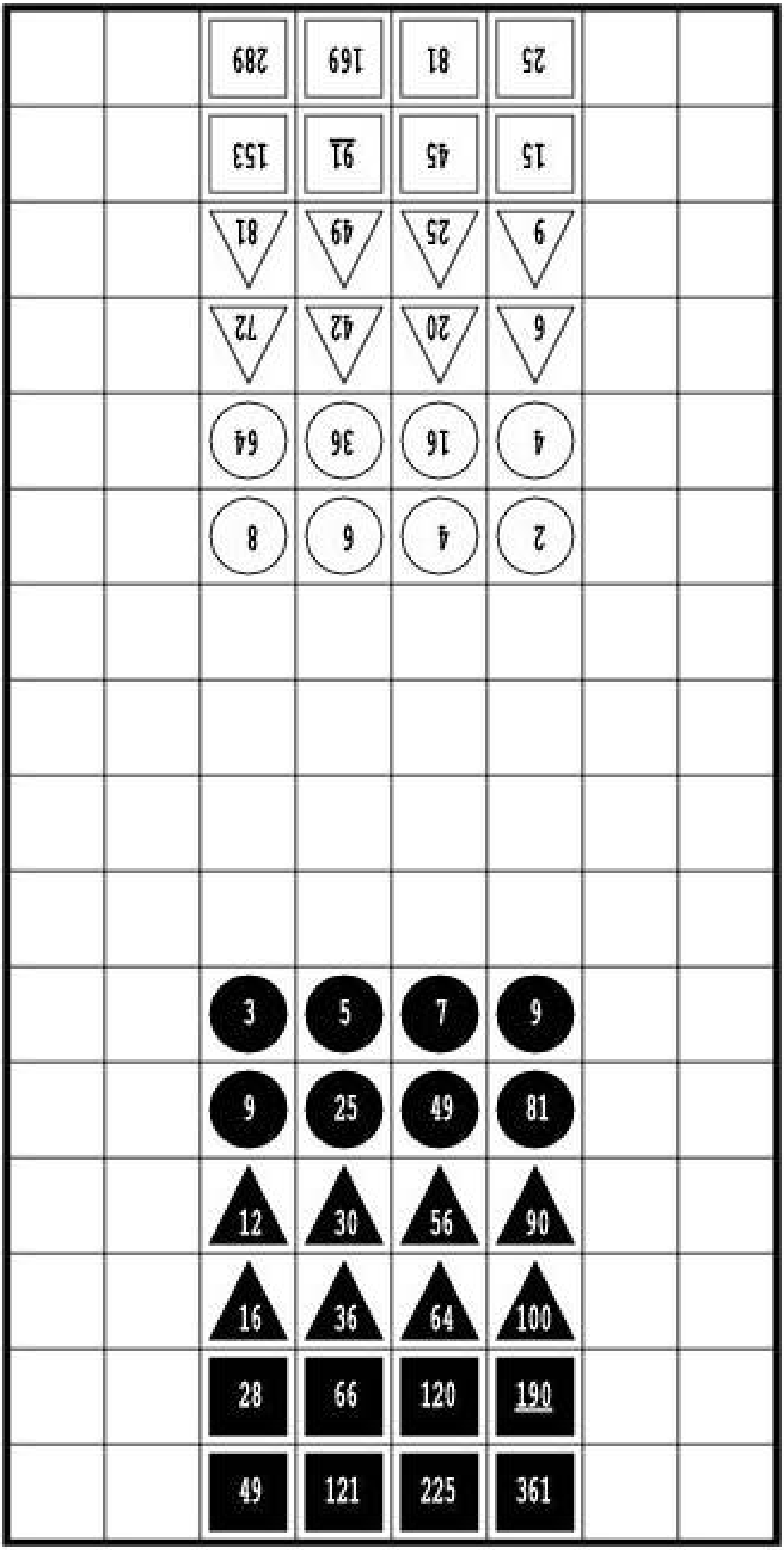


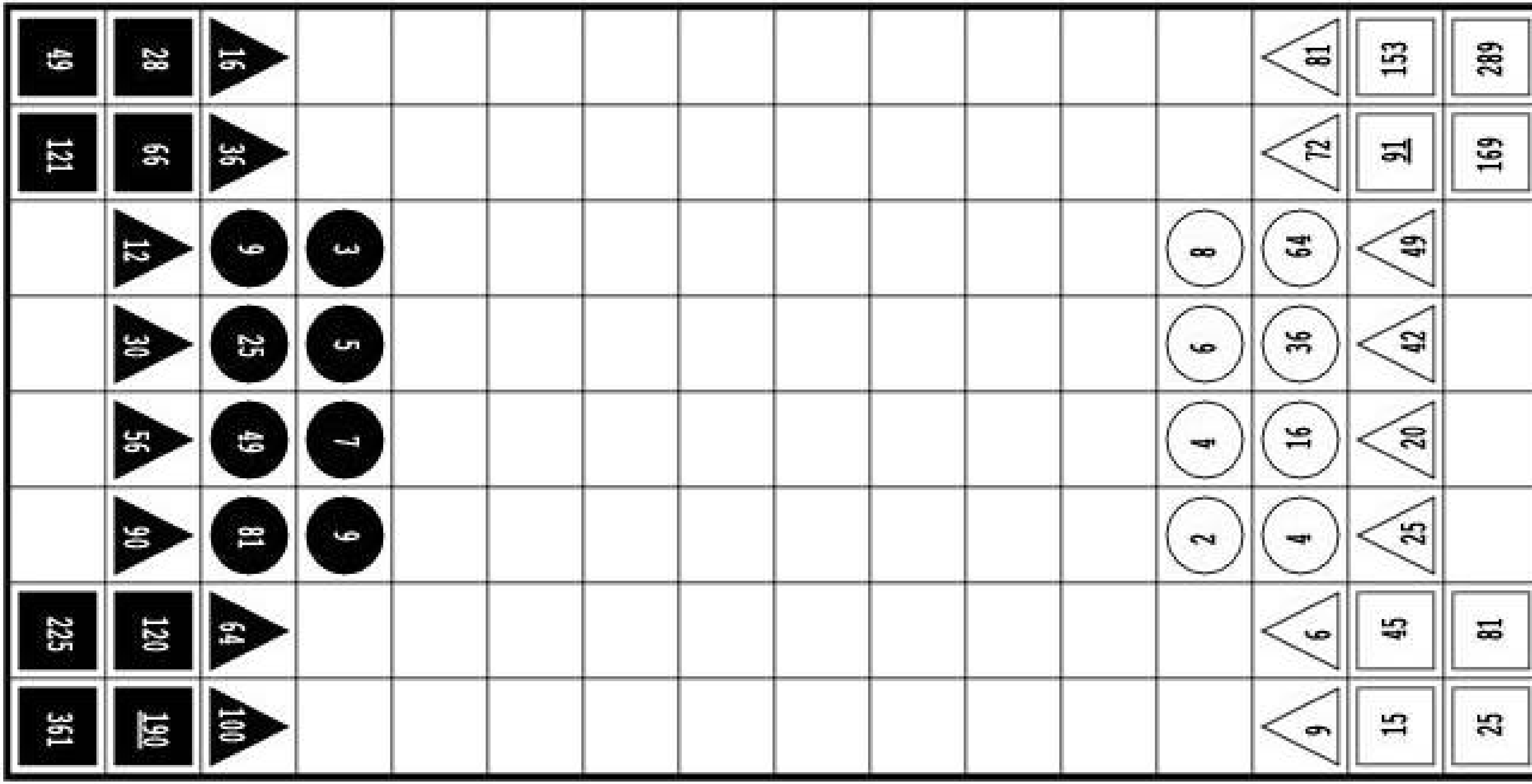


91



190

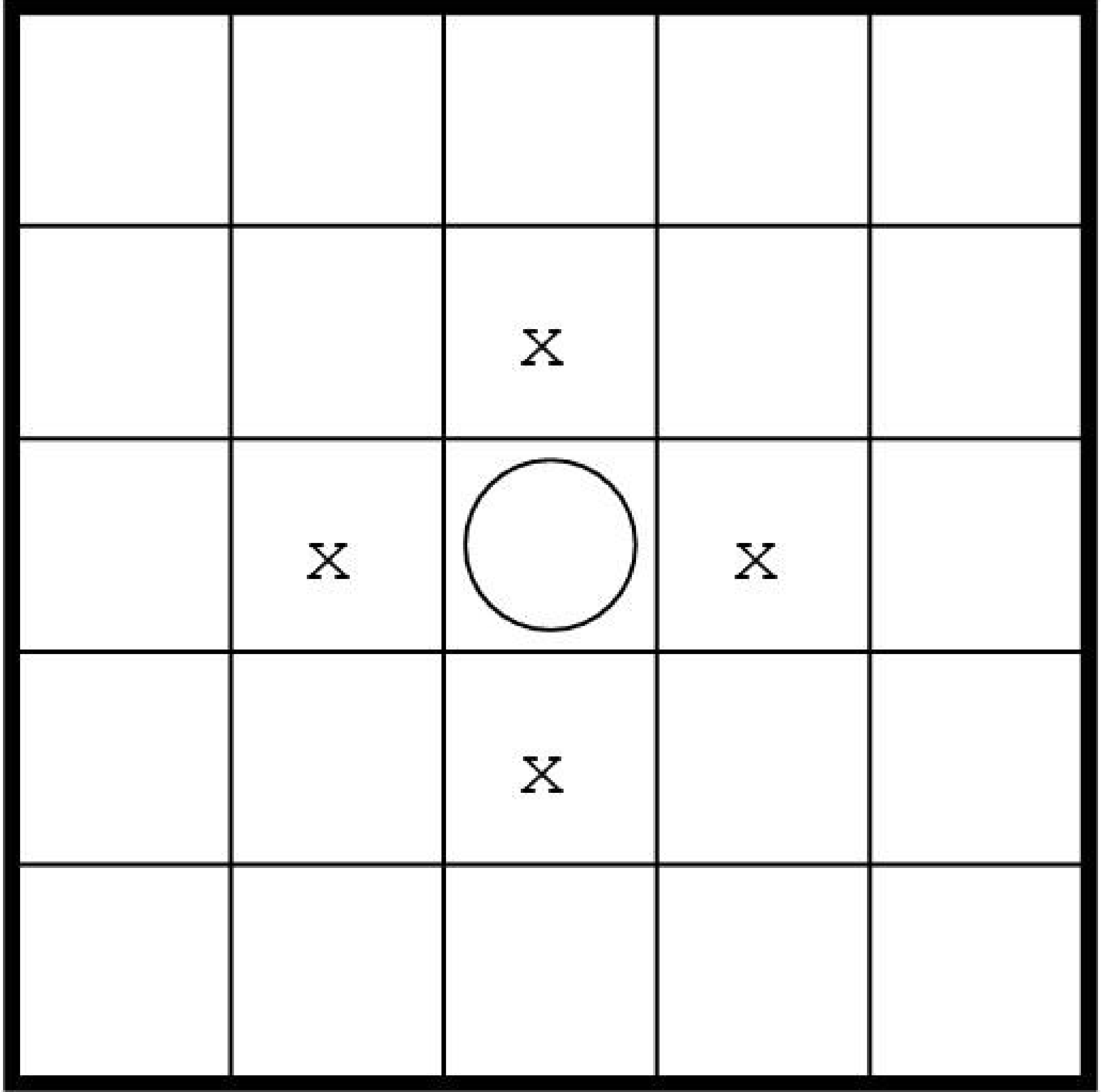

289 169 81 25
153 91 45 15
81 49 25 9
72 42 20 6
64 36 16 4
8 6 4 2
3 5 7 9
9 25 49 81
12 30 56 90
16 36 64 100
28 66 120 190
49 121 225 361

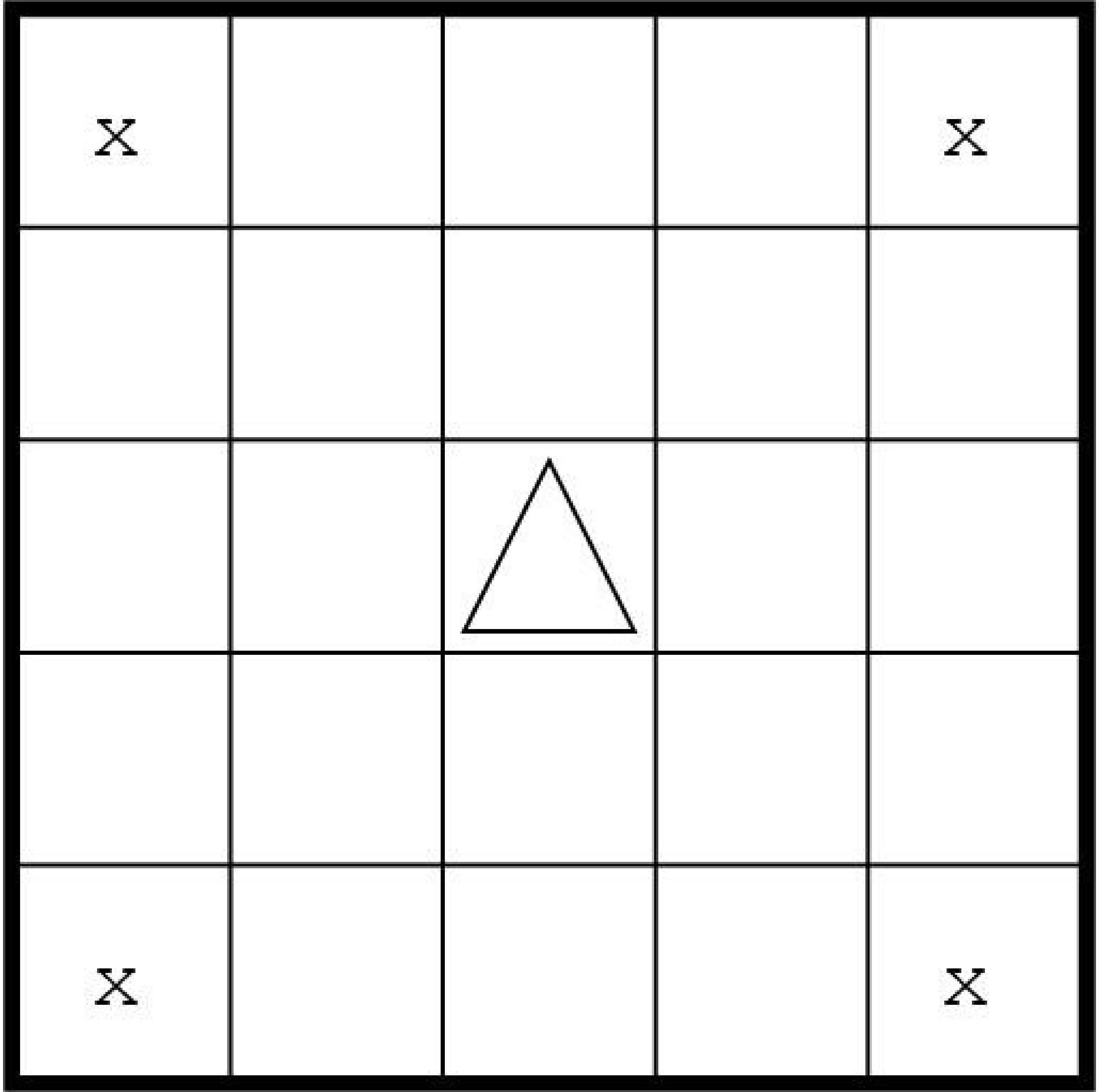


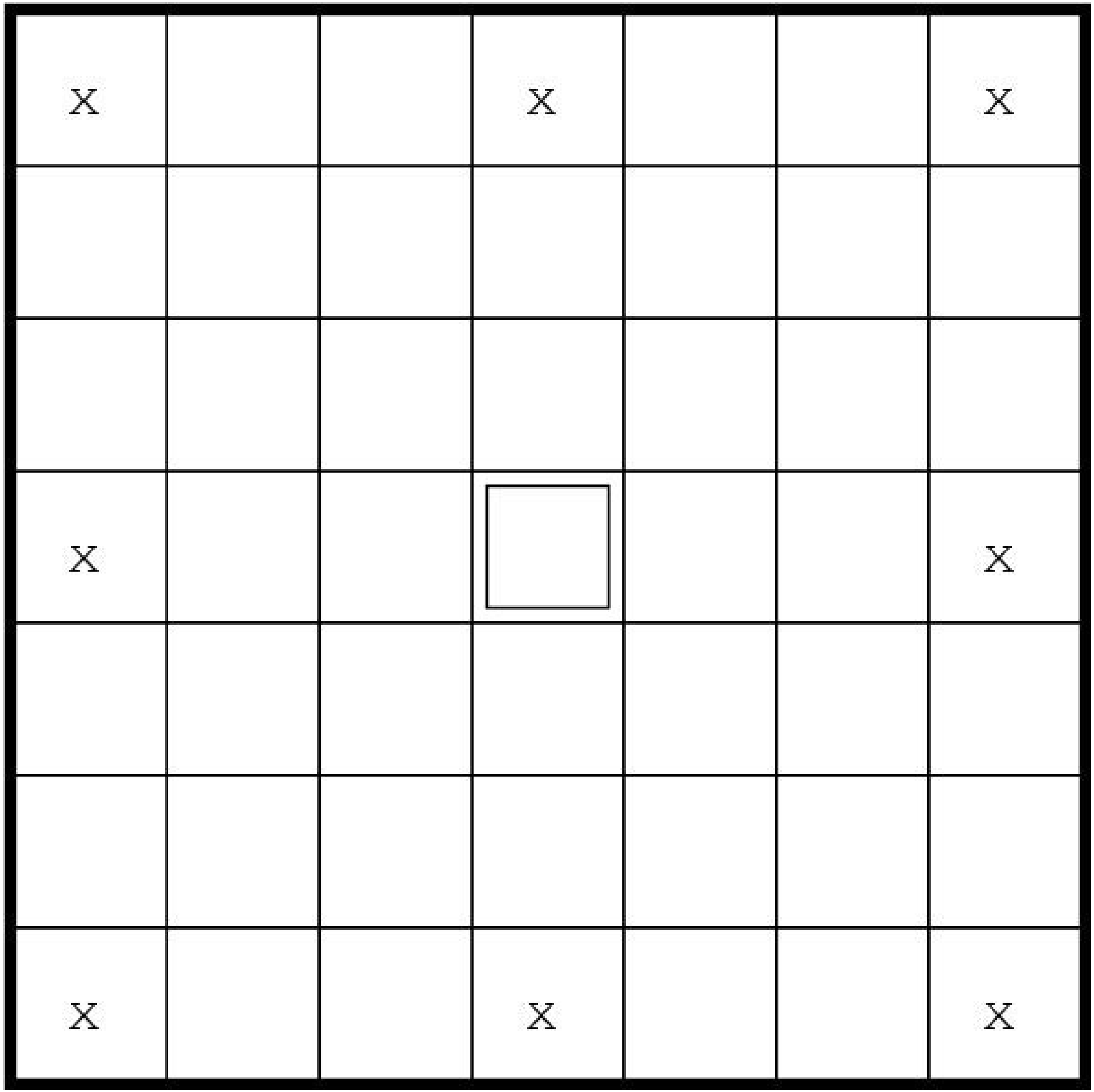
49 121 225 361
28 66 120 190
16 36 64 100
12 30 56 90
9 25 49 81
3 5 7 9
8 6 4 2
64 36 16 4
81 72 6 9
49 42 20 25
153 91 45 15
289 169 81 25

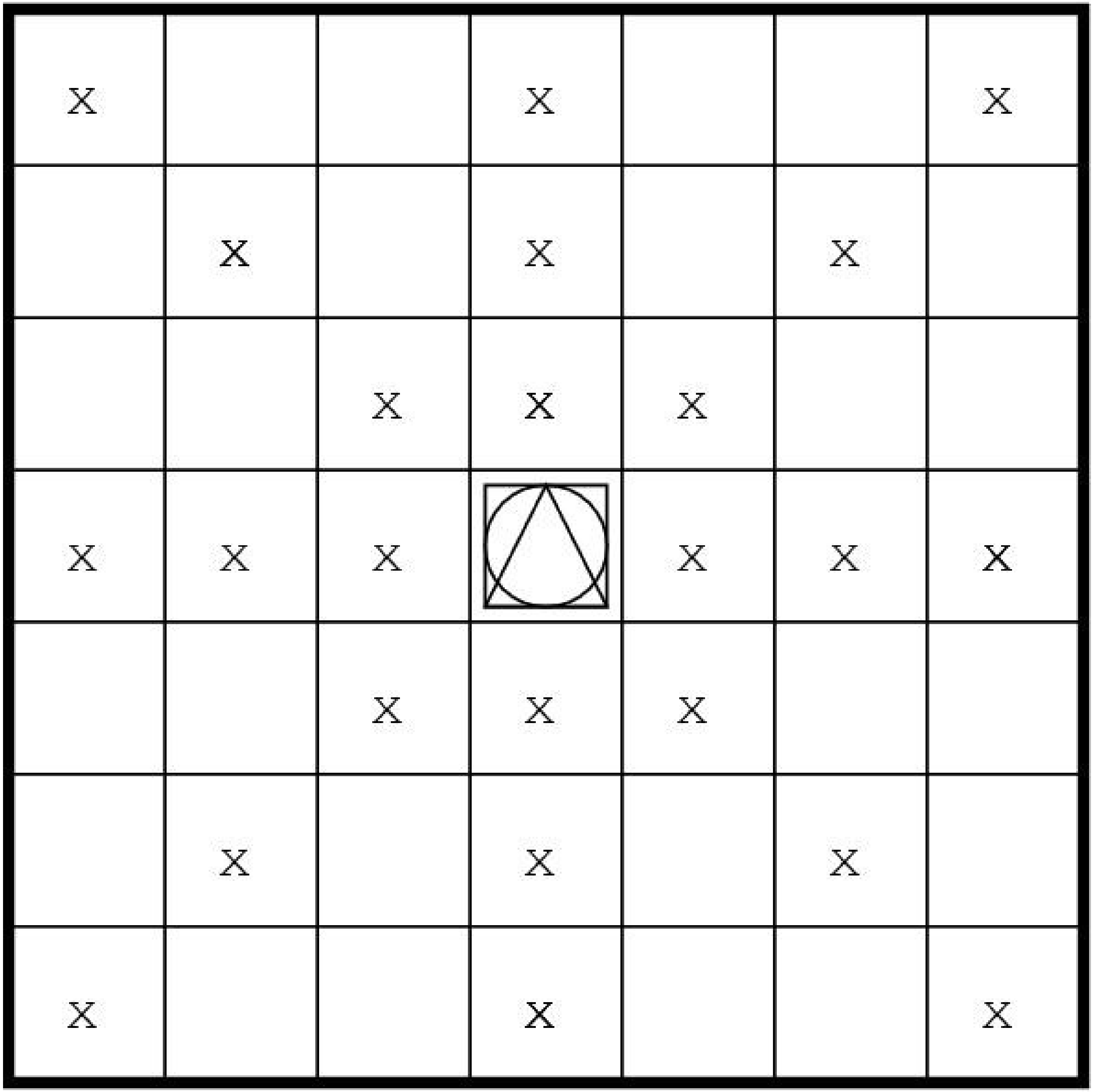
How pieces move

- Rounds move one step orthogonally (similar to Pawns in Chess).
- Triangles move two steps diagonally (similar to Chess Bishops).
- Squares move orthogonally and diagonally three steps (similar to Chess Queen).
- Pyramids move exactly like Chess Queens.

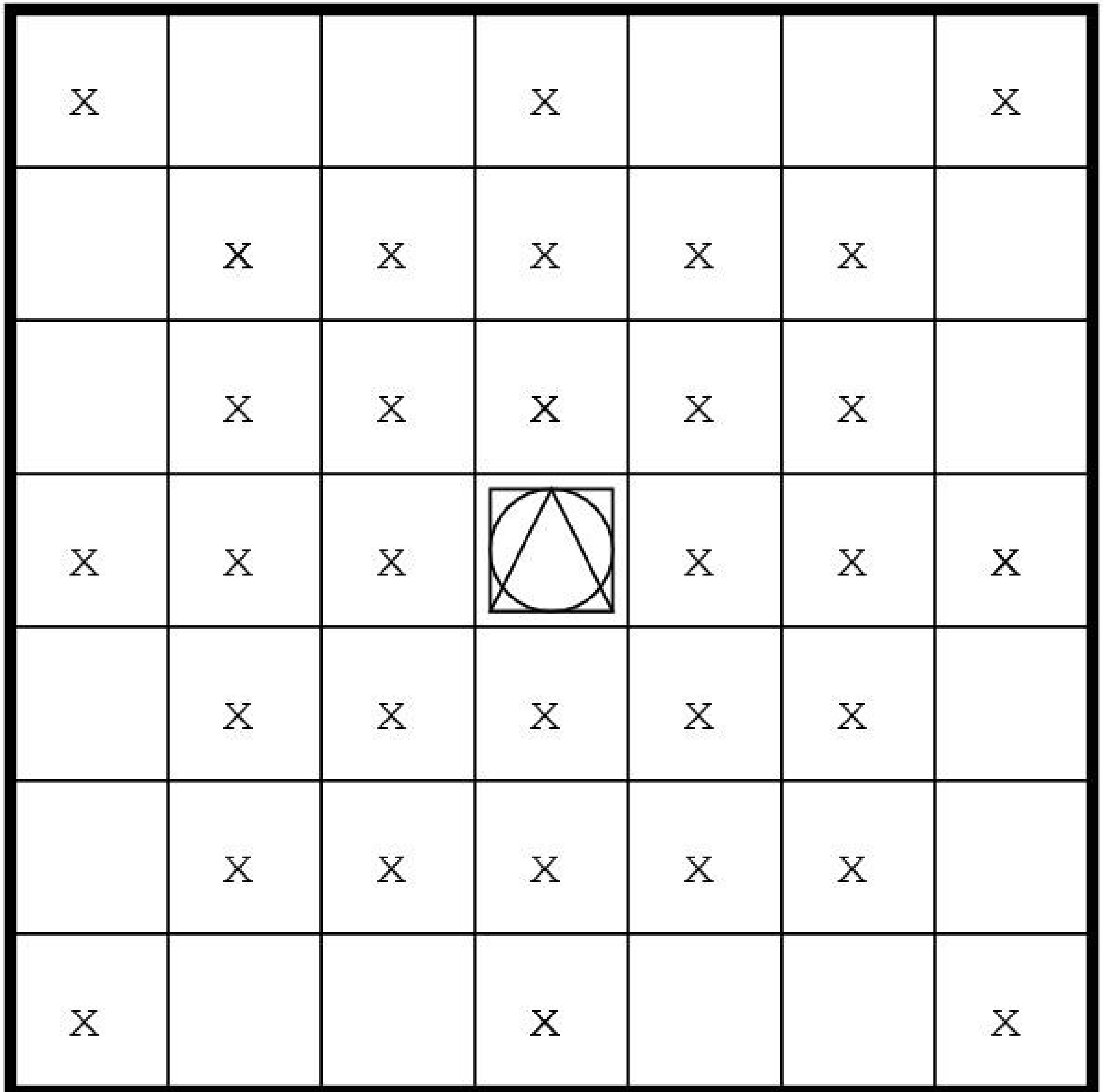

X
X
X
X


x
x
x
x


X
X
X
X
X
X
X
X

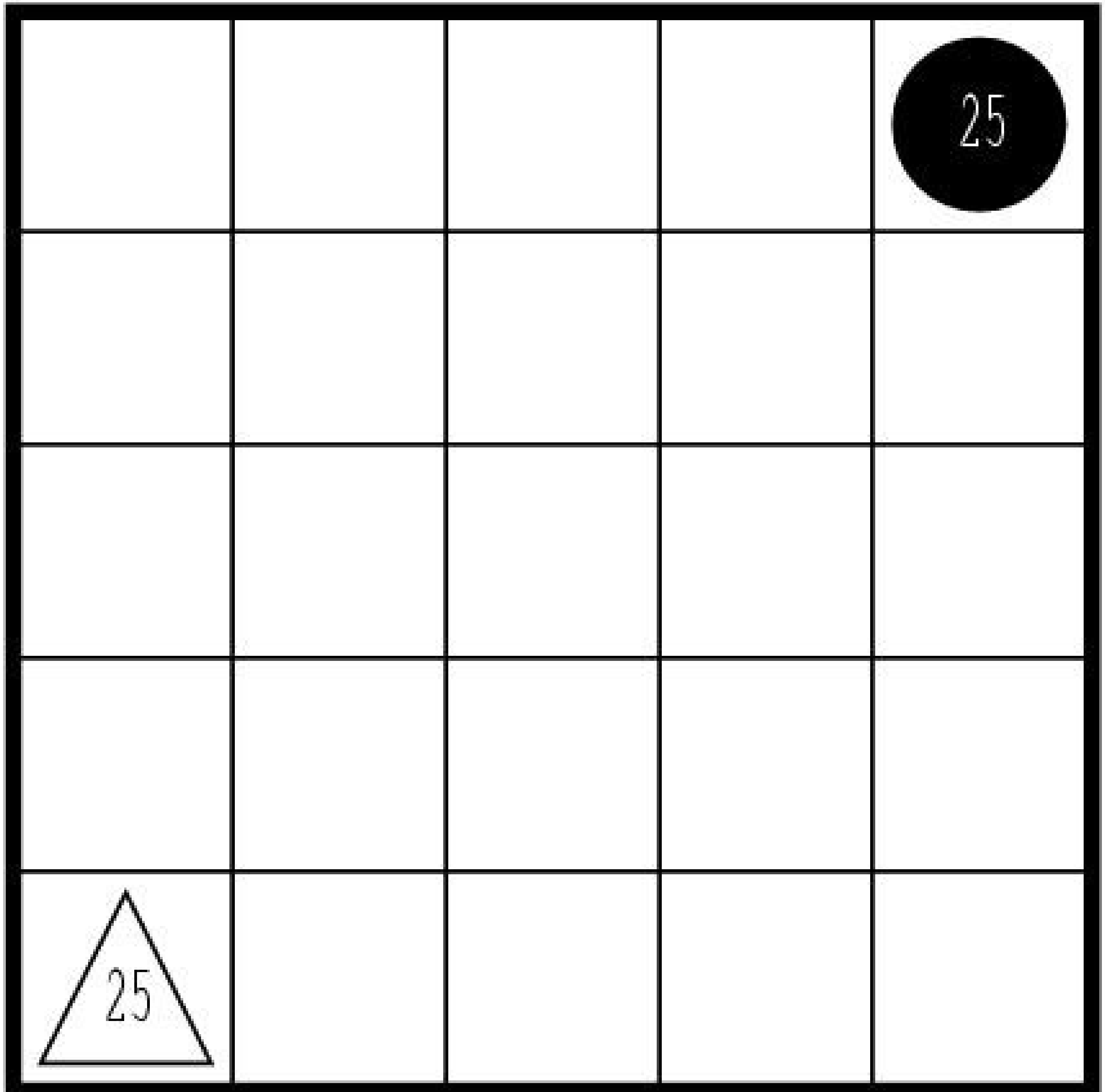


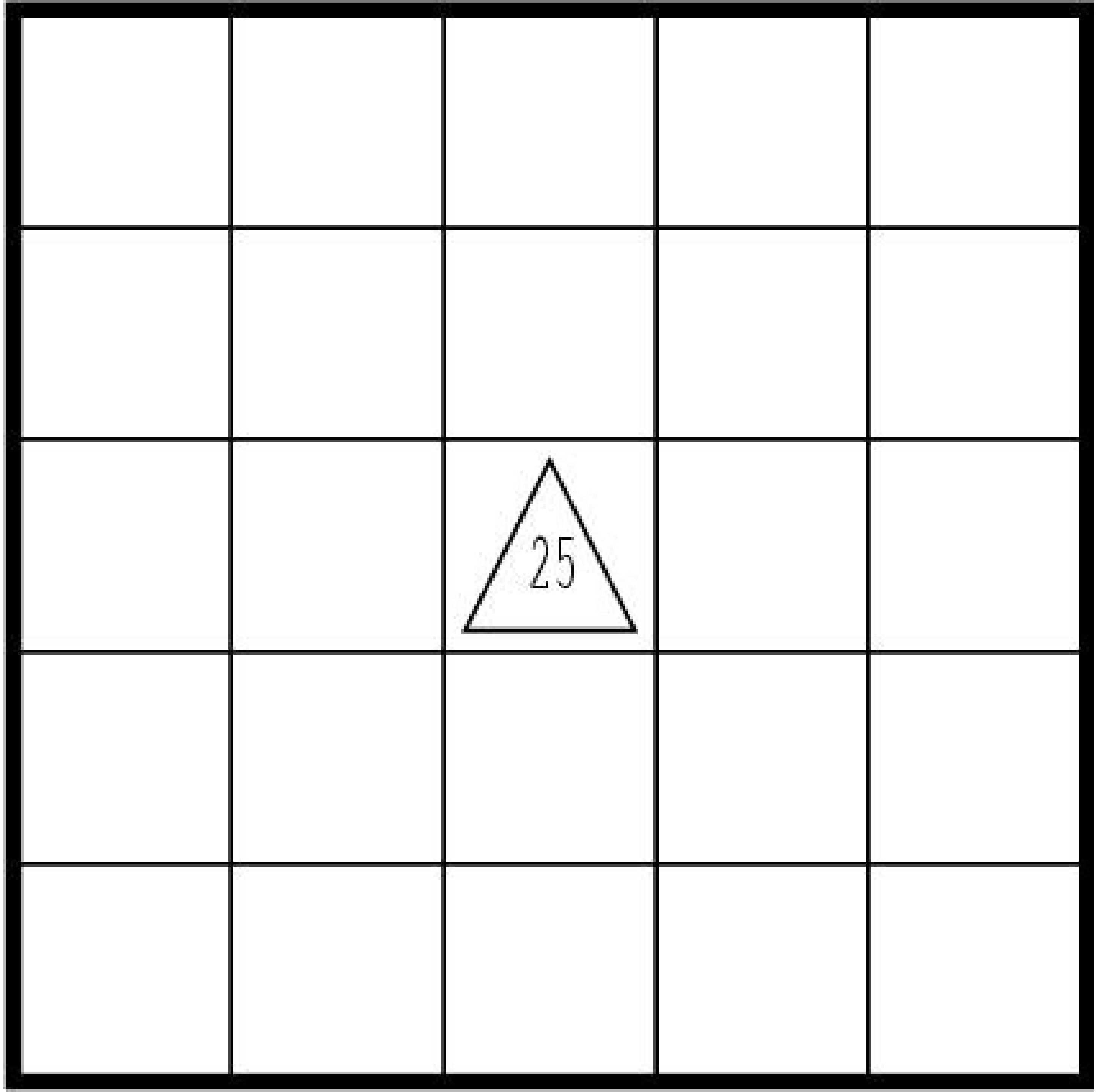
- When attacked, the Pyramid can play also as a Chess Knight.
- It can also capture one of the attacking pieces with a friendly one.
- If captured all its levels are turned, it becomes a pyramid for the opponent.



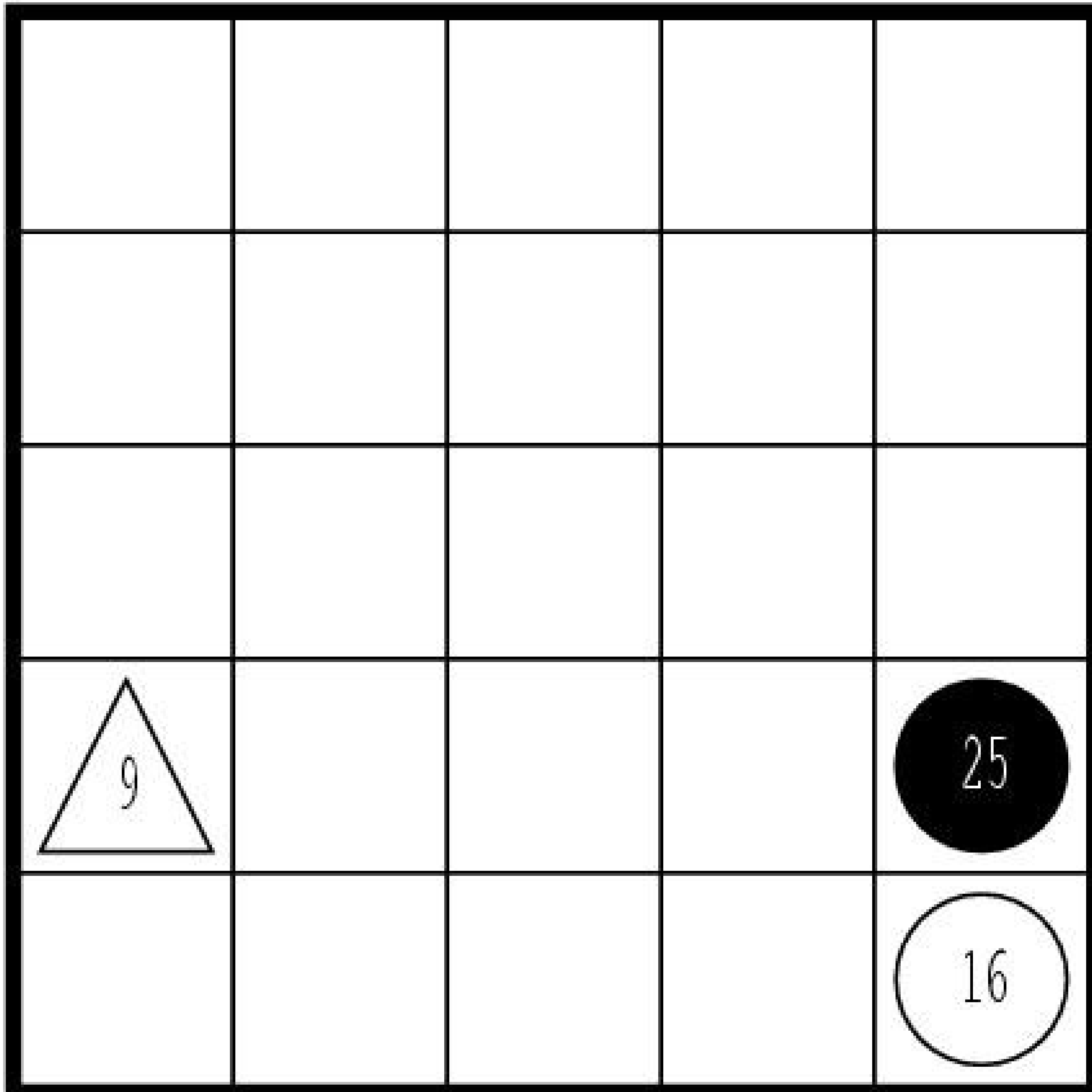
Captures

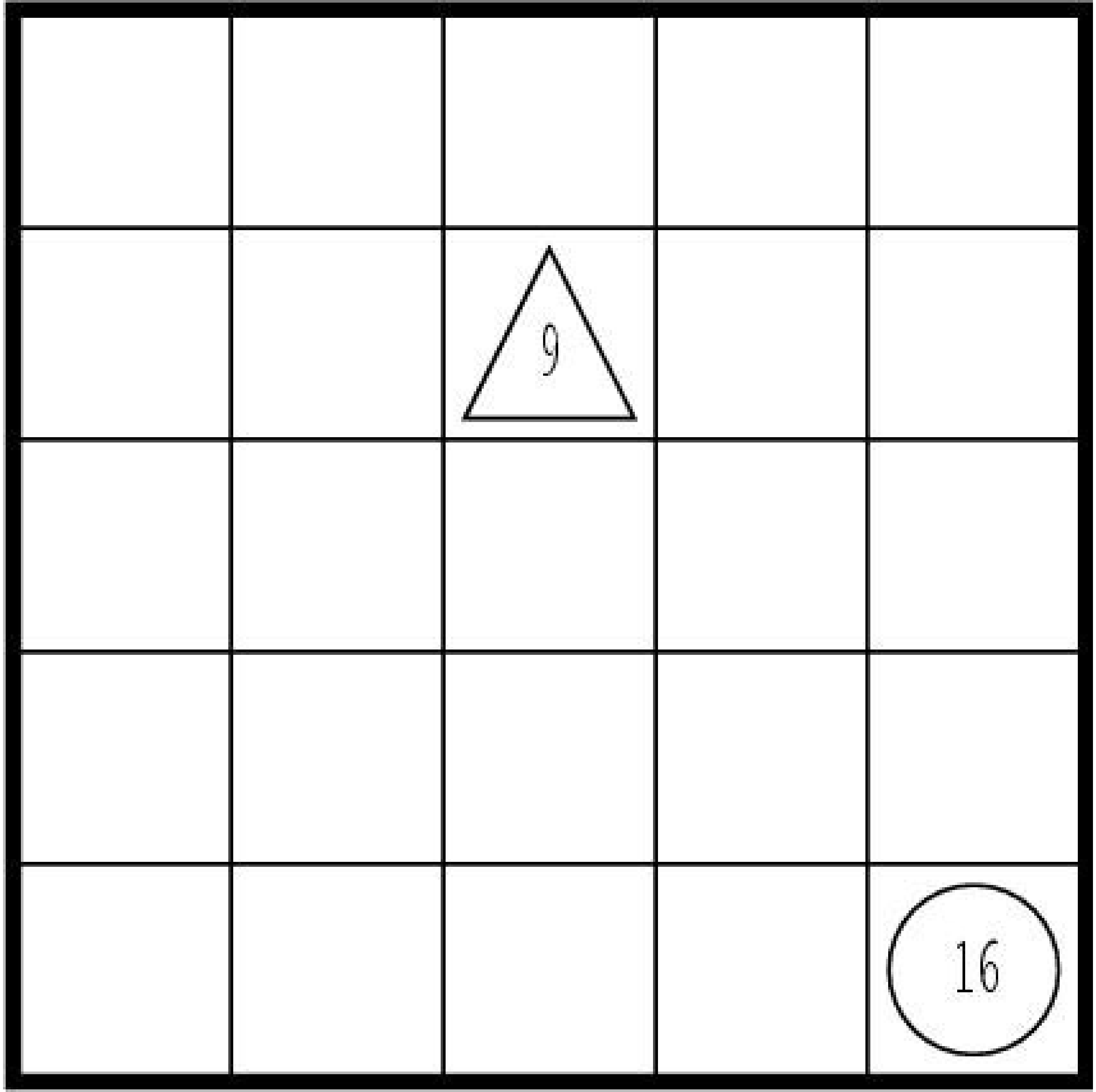
- **Numerar**: equal value capture without replacement.
- From...


25
25

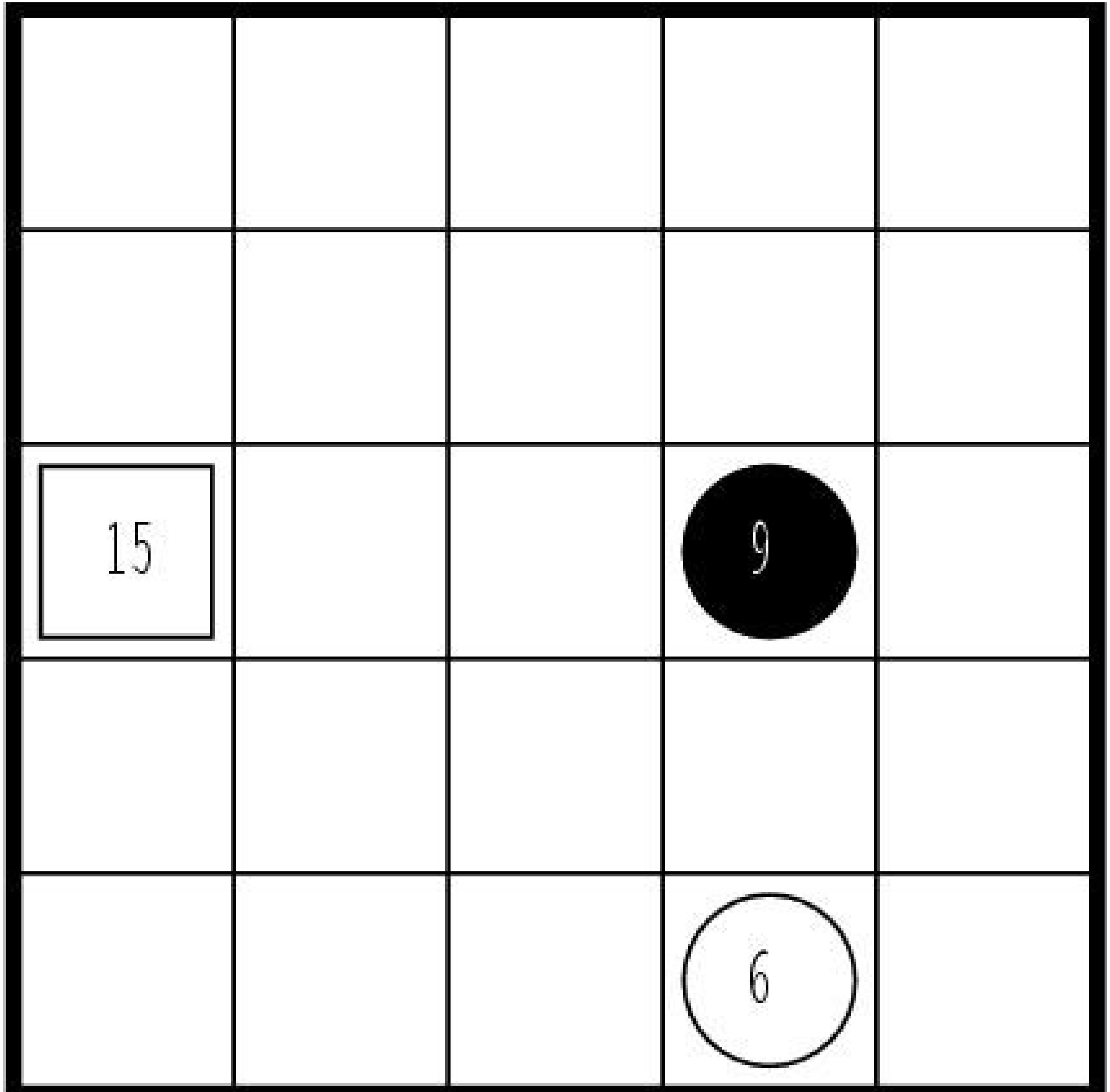

25

- **Summar:** If two pieces can reach a cell occupied by an adversary’s piecce, and the sum of the values of the first two equals the number of the third, this one is taken from the board.
- No replacement.


9
25
16


9
16

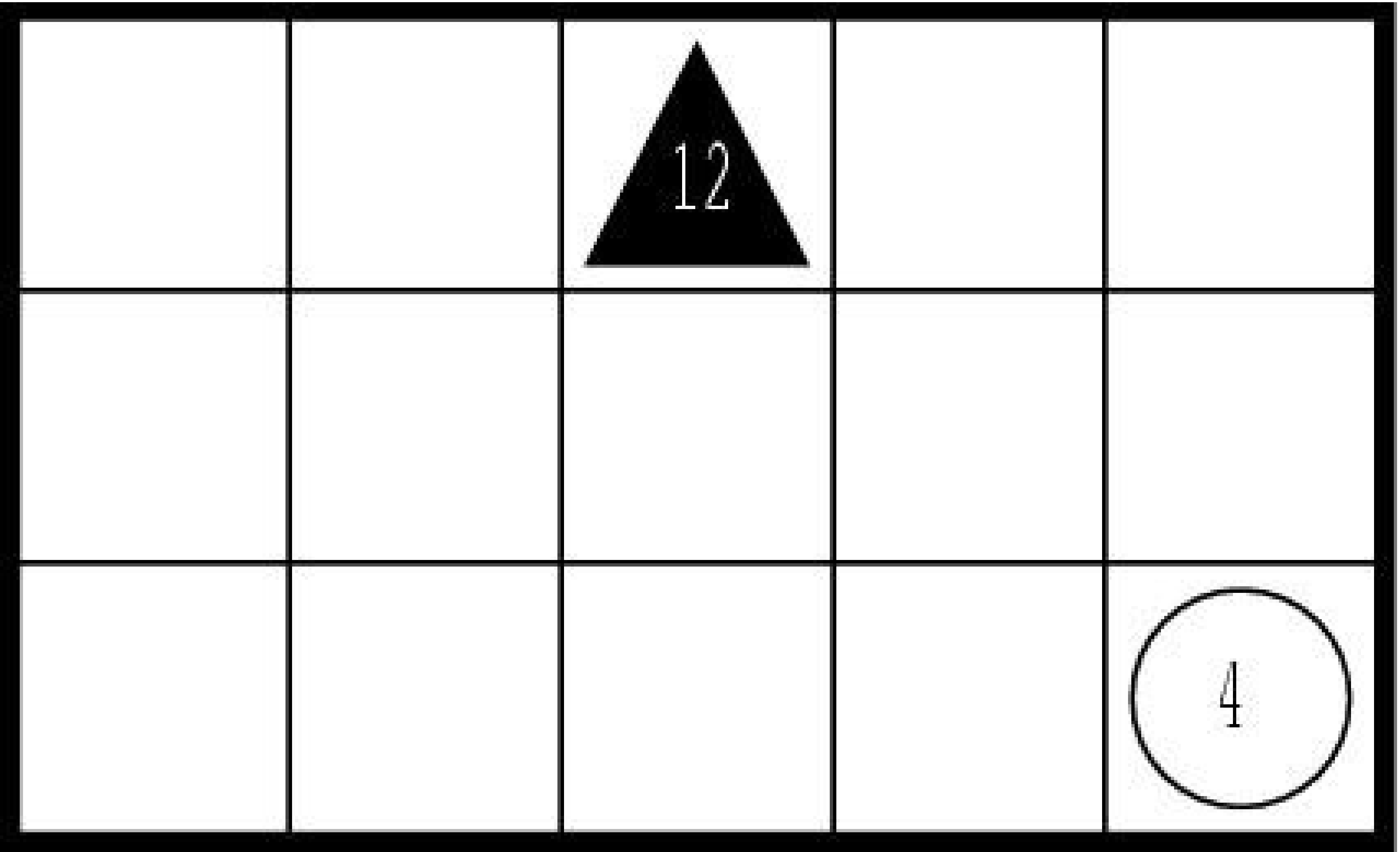
- **Sotrar:** Similar to summar, but with subtraction.


15
9
6

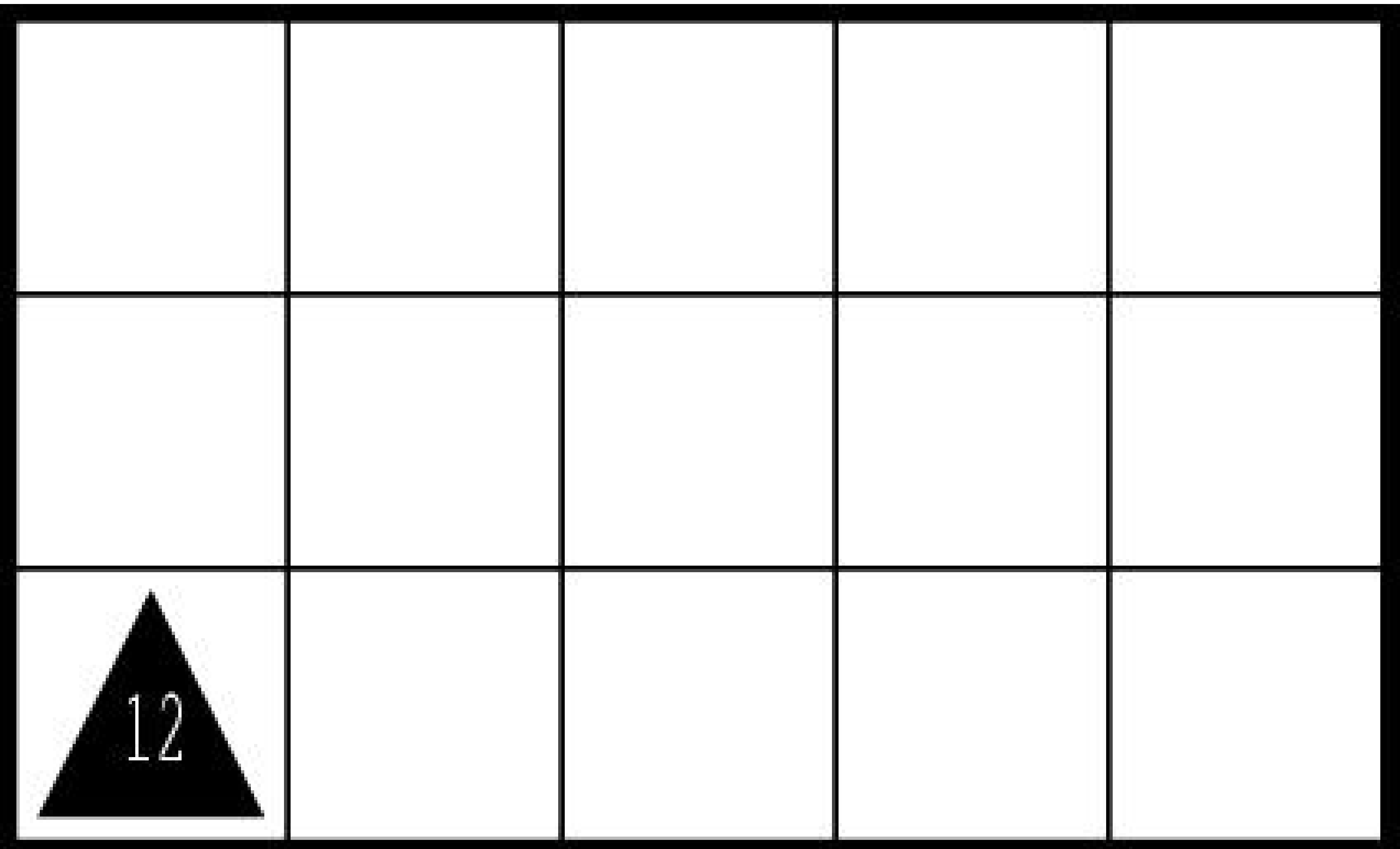
15				
			6	

- **Moltiplicar/Partir:** A piece with a number that equals an adversary’s piece number multiplied by the number of cells between them is taken or takes, according to whoever just played.

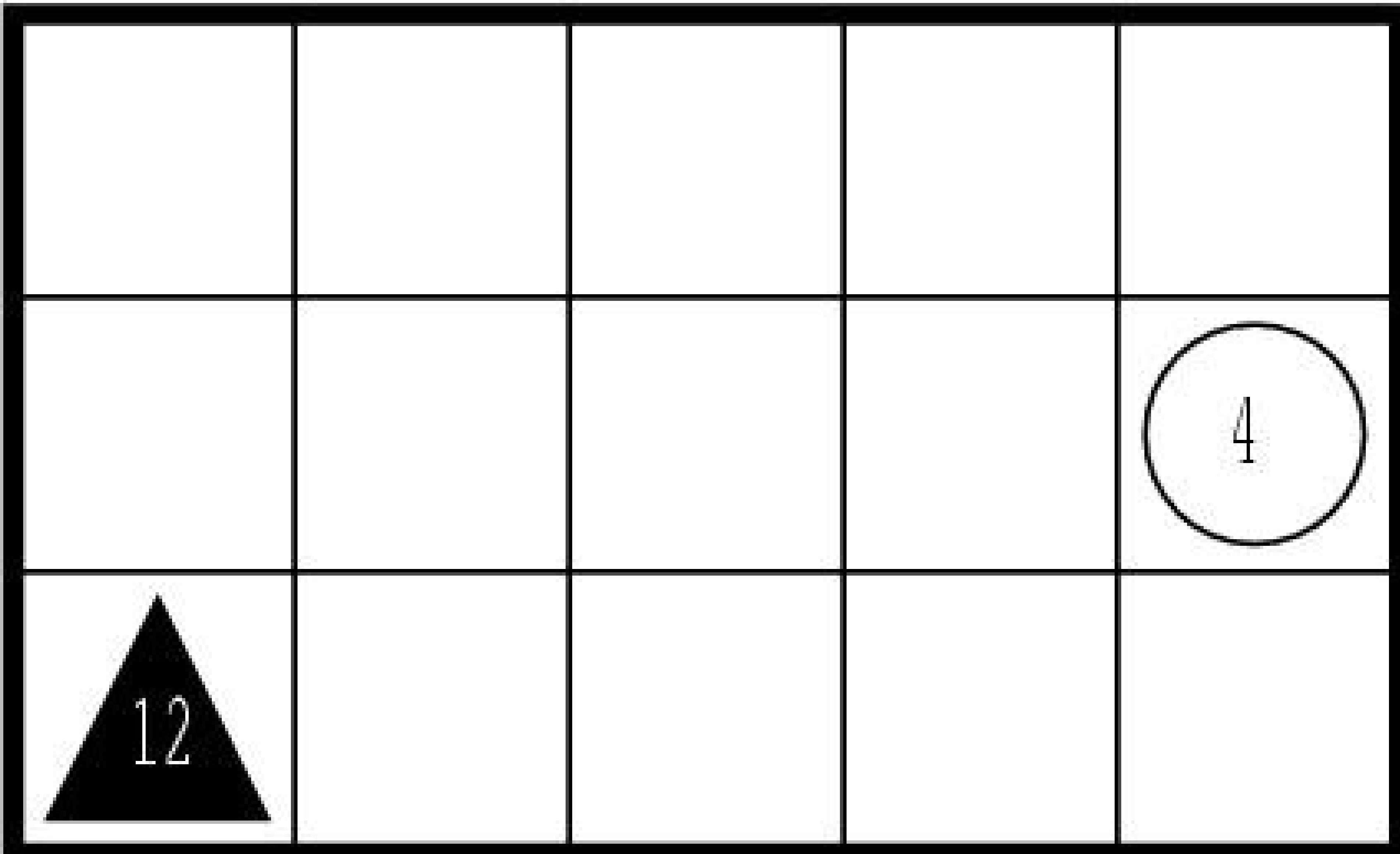
Black plays from



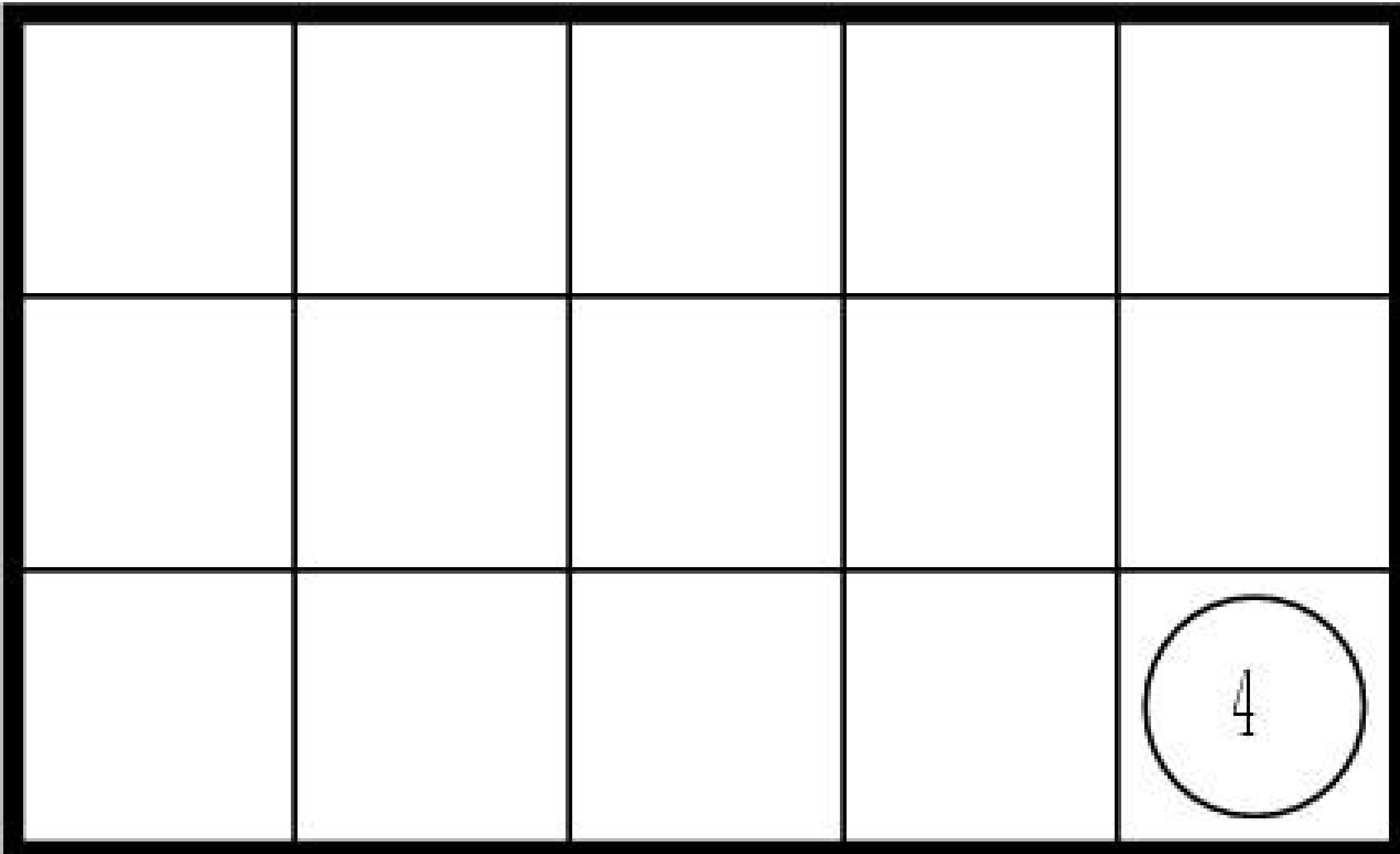
and captures



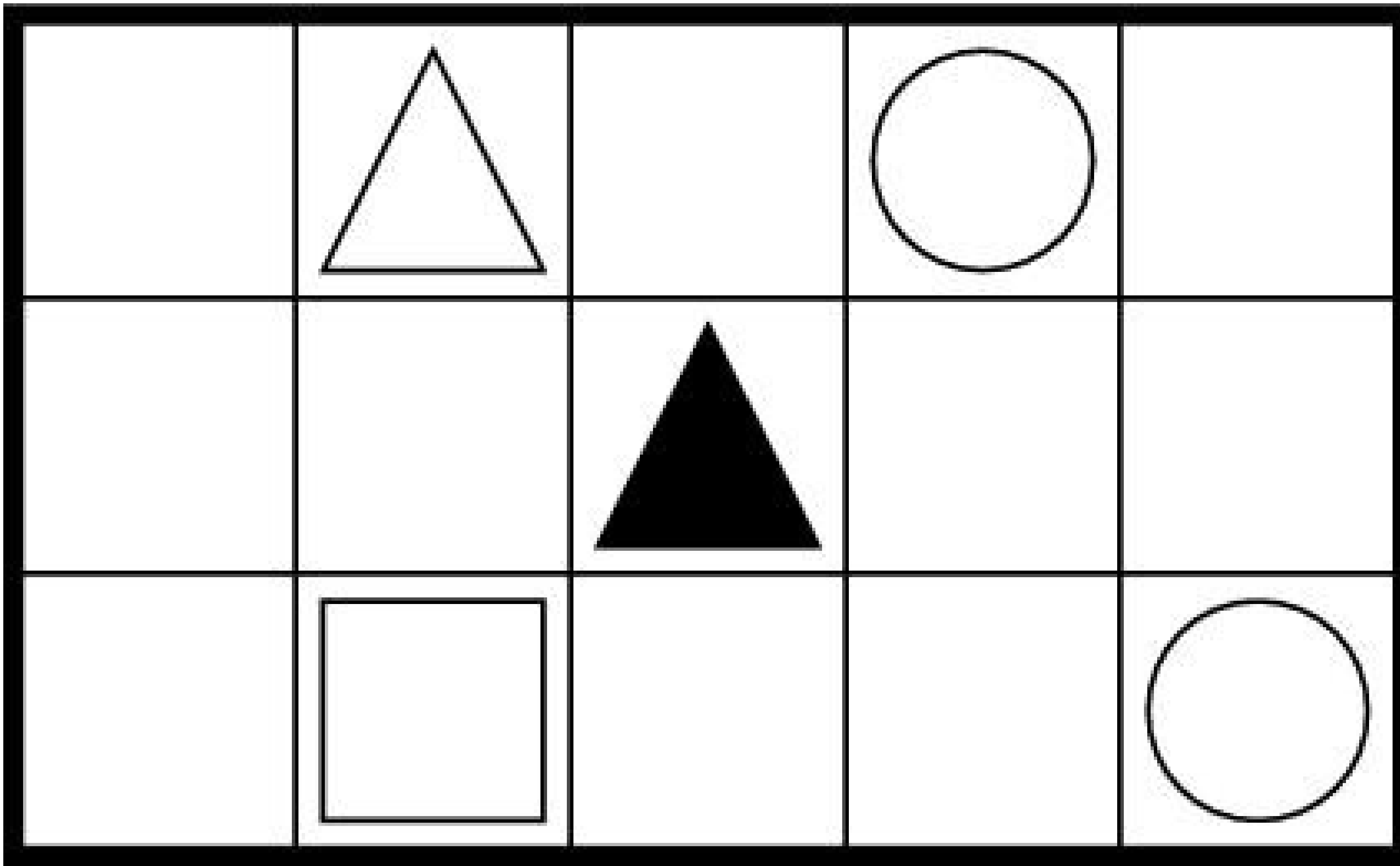
White plays from



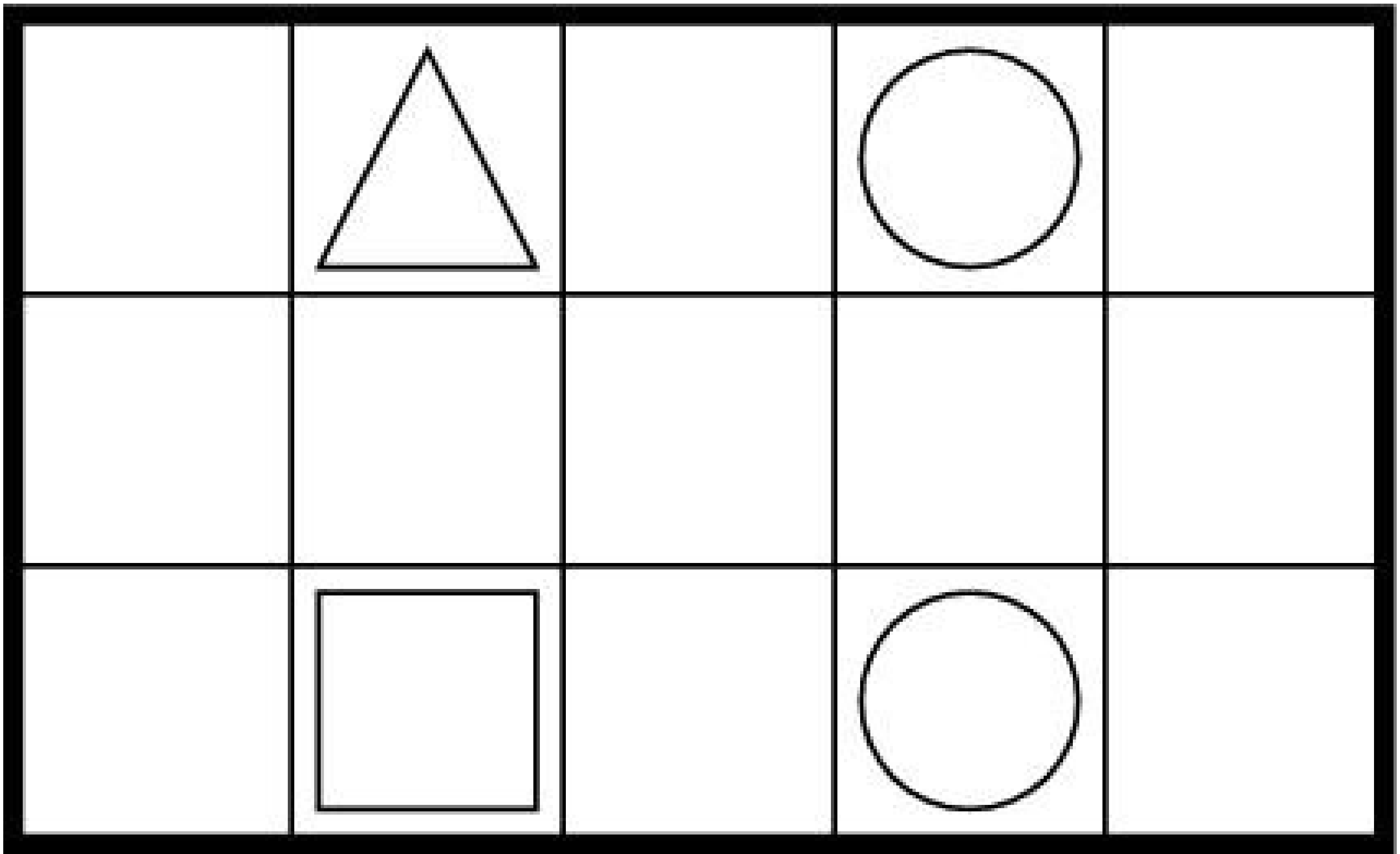
and captures



- **Assedio:** capture by stalemating a piece.
- White plays from



and captures



- Pyramids can be captured by levels.
- A part of a Pyramid just captured can be ramsoned with a piece with the same number.
- Captured pieces are turned and entered in the game, at the players choice (even several at the same time).
- An attacked Pyramid can capture one of the attackers with a friendly piece.

Smaller Victories

- ***First***: Capture more pieces.
- ***Second***: Add the values of the captured pieces.
- ***Third:*** Add the number of digits of the captured pieces.
- ***Fourth:*** First and second combined.
- ***Fifth:*** First and third combined.
- **Sixth:** Second and third combined.
- **Seventh:** 1st, 2nd and 3rd combined.

Victories

- Were based on progressions. Three numbers $x < y < z$, form a progression said to be
  - Arithmetic if $(z-y)/(y-x)=x/x$
  - Geometric if $(z-y)/(y-x)=y/x$
  - Harmonic if $(z-y)/(y-x)=z/x$

Victories

- **Grande**: Place three pieces in the adversary's half board forming one progression. Examples:

2	15	28
9	15	25
9	15	45

Victories

- **Maggior**: Place four pieces in the adversary's half board forming two progressions of three numbers. Examples:

2	3	4	8	(2-3-4, 2-4-8)
3	4	5	6	(3-4-5, 3-4-6)
2	3	6	12	(3-6-12, 2-3-6)

Victories

- **Massima:** Place four pieces in the adversary's half board forming three progressions of three numbers. Example:

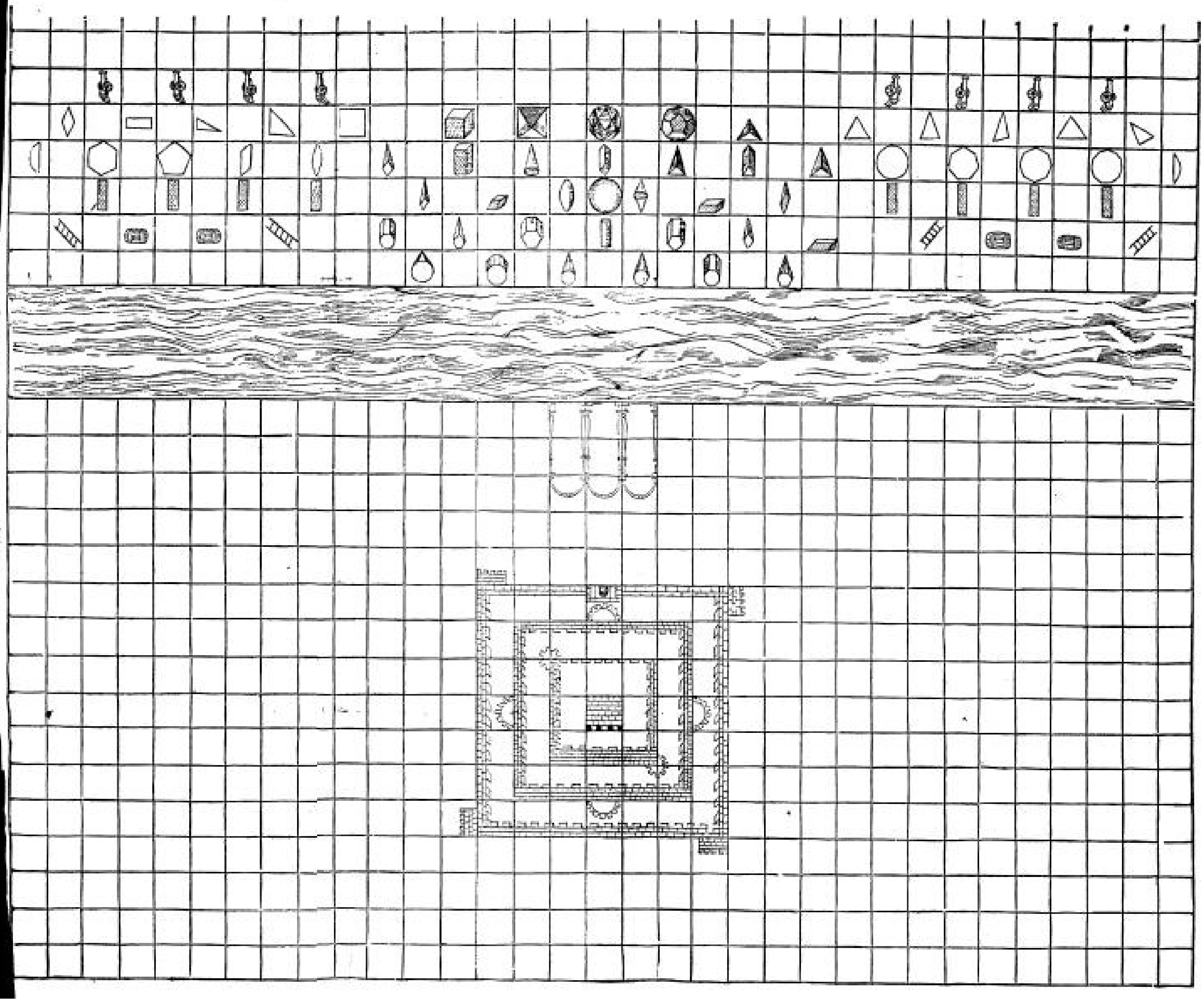
4 6 9 12

(6-9-12, 4-6-9, 4-6-12)

The end of Rythmomachia

- It was a pedagogical game.
- Invented in the 11th century.
- It was popular everywhere where Boethius Arithmetic was taught.
- It vanished, naturally, in the 17th century, as mathematics developed in a different way.
- Chess took over.

Metromachia (XVI) was created as a pedagogical game, to teach Geometry.



NIM

Players alternate taking matches from one group.
Whoever takes the last match wins.

3 5 7

Bouton's Theorem

A position is safe if the nim-sum of the numbers of matches is 0. If you can, move to a safe position.

NIM-SUM

With the numbers expressed in binary, operate by:

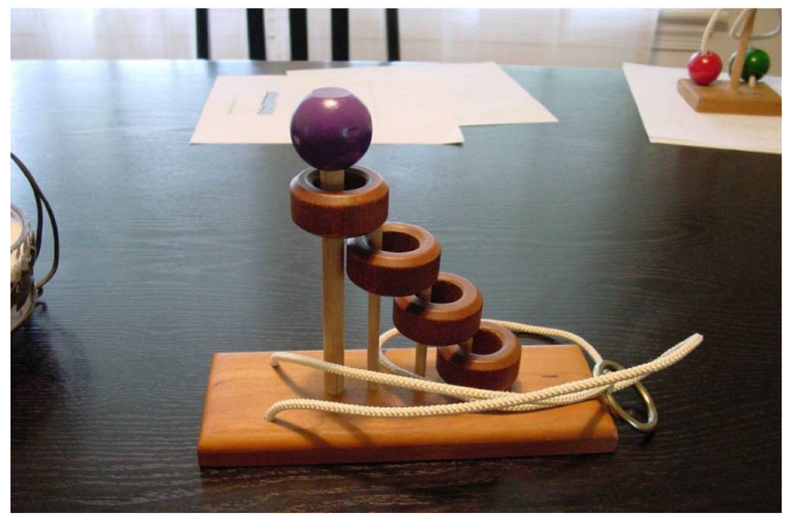
$0 + 0 = 0$ $1 + 0 = 1$ $0 + 1 = 1$ $1 + 1 = 0$

In our example:

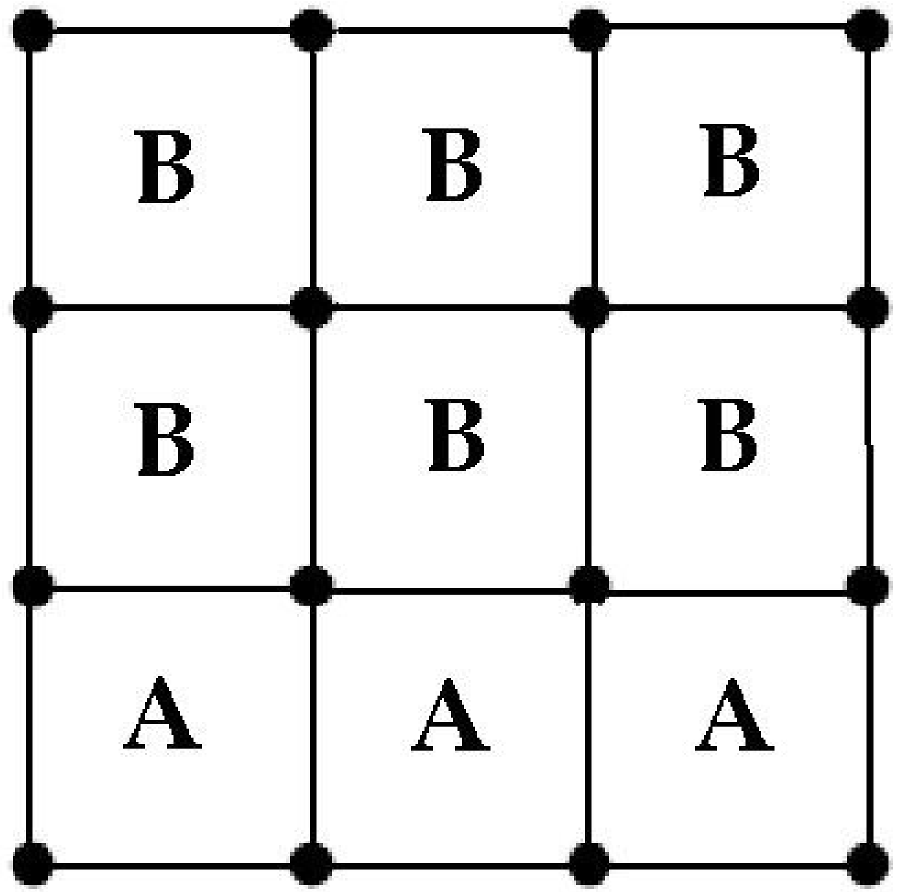
$$3 \oplus 5 \oplus 7 = (011)_2 + (101)_2 + (111)_2 = (001)_2$$

To win, next player takes one match from any group. If we take a match from the 7-group we get 3, 5, 6, and $3 \oplus 5 \oplus 6 = (011)_2 + (101)_2 + (110)_2 = (000)_2 = 0$

Some classic puzzles are related to binary notation: Towers of Hanoi, Chinese rings, ...

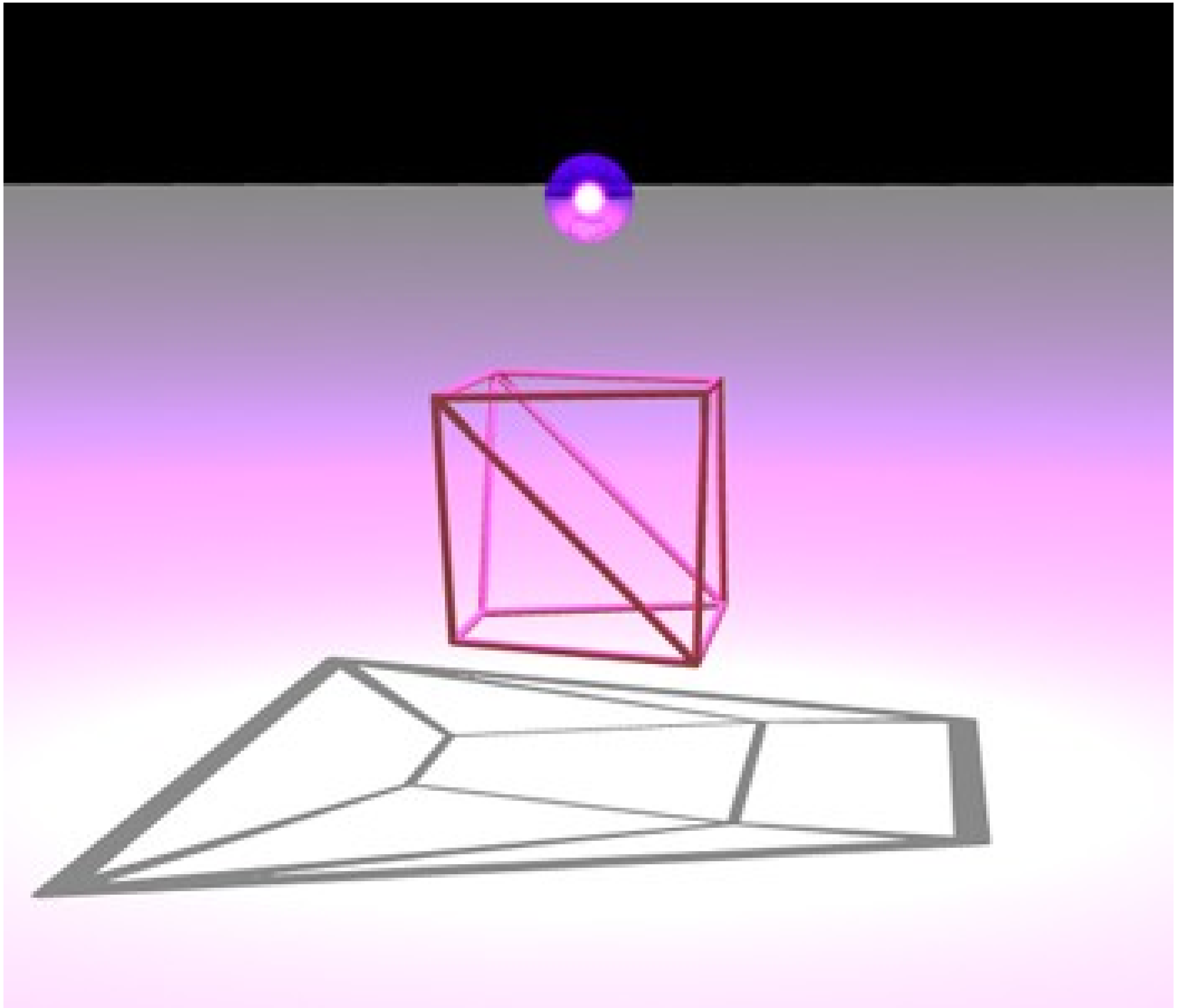


Dots & Boxes



Players alternate connecting orthogonally adjacent points. Whoever finishes a square inscribes in it his initial and plays again. The player with more squares at the end is the winner.

Euler: **E = F + V – 1** (w/ infinite face)



In D&B, Euler:

$$\mathbf{L = Q + P - 1}$$

Another way of counting:

$$\mathbf{L = Q - D + J - 1}$$

where: L = lines, Q = Squares, P = Points,
D = Double moves, J = moves.

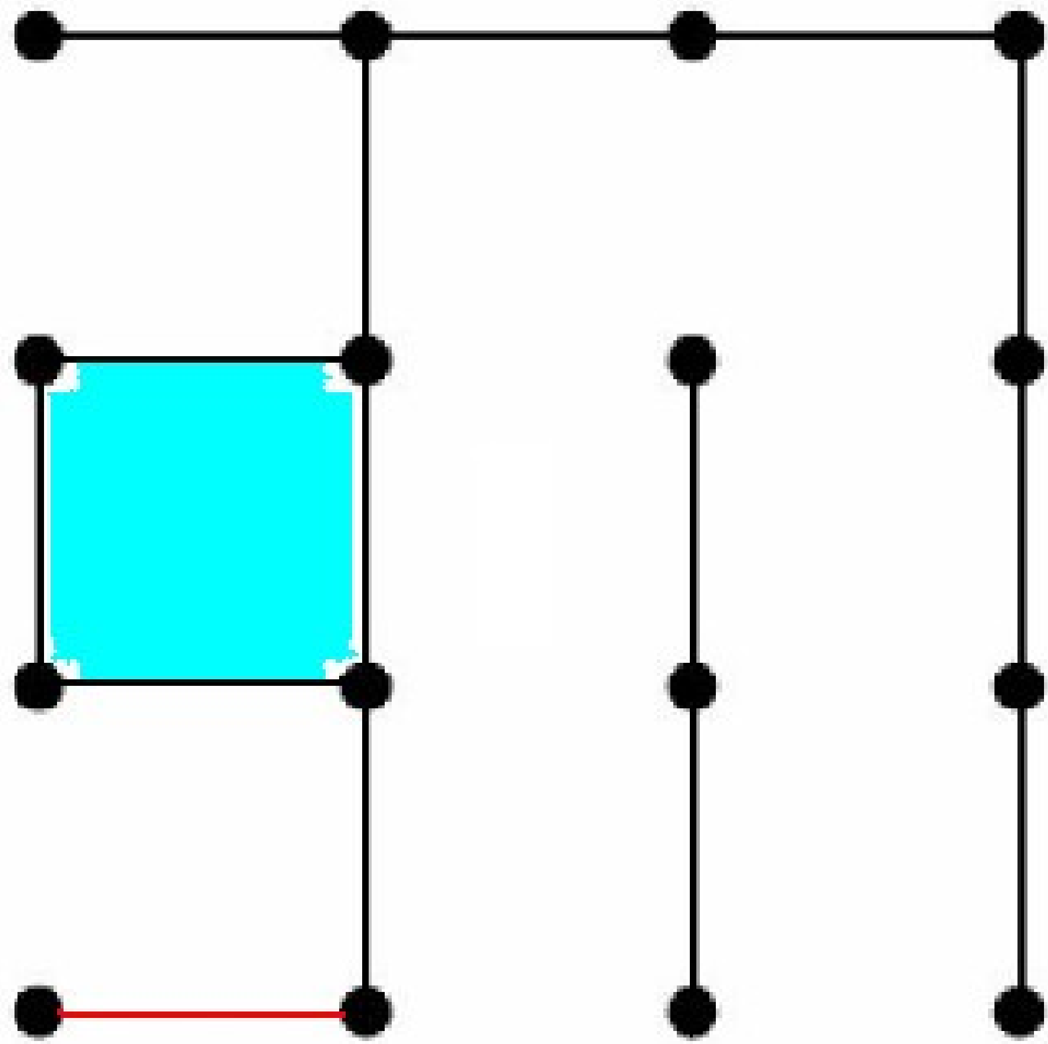
Therefore J = P + D. Thus, A wants P+D odd, B wants it even. (Both want to be the last).

Usually, the number D is one less than the number of long chains of squares, therefore A wants P+C even, B wants P+C odd, where C stands for the number of long chains.

(Long means at least 3).

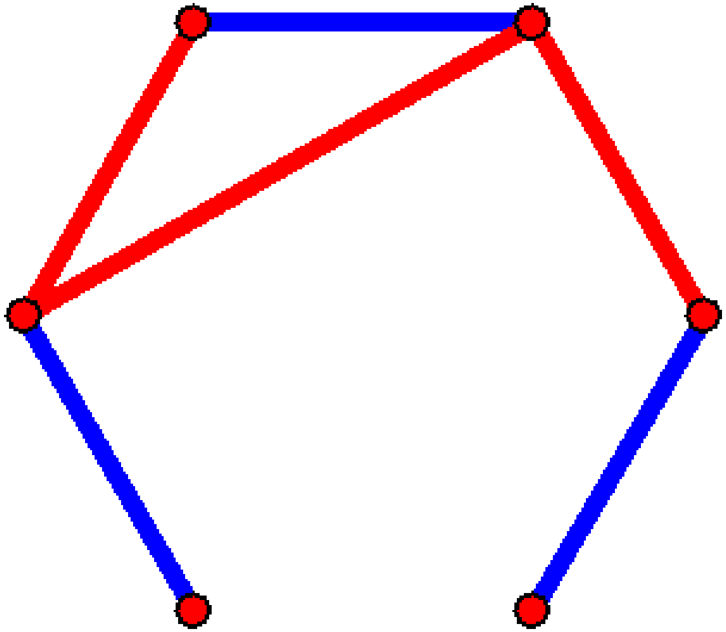
This gives a strategy to play the position above.

Bad Move: Just 1 long chain



SIM

Given six points, two players alternate connecting a pair of them using different colors. Whoever finishes a monochromatic triangle looses.



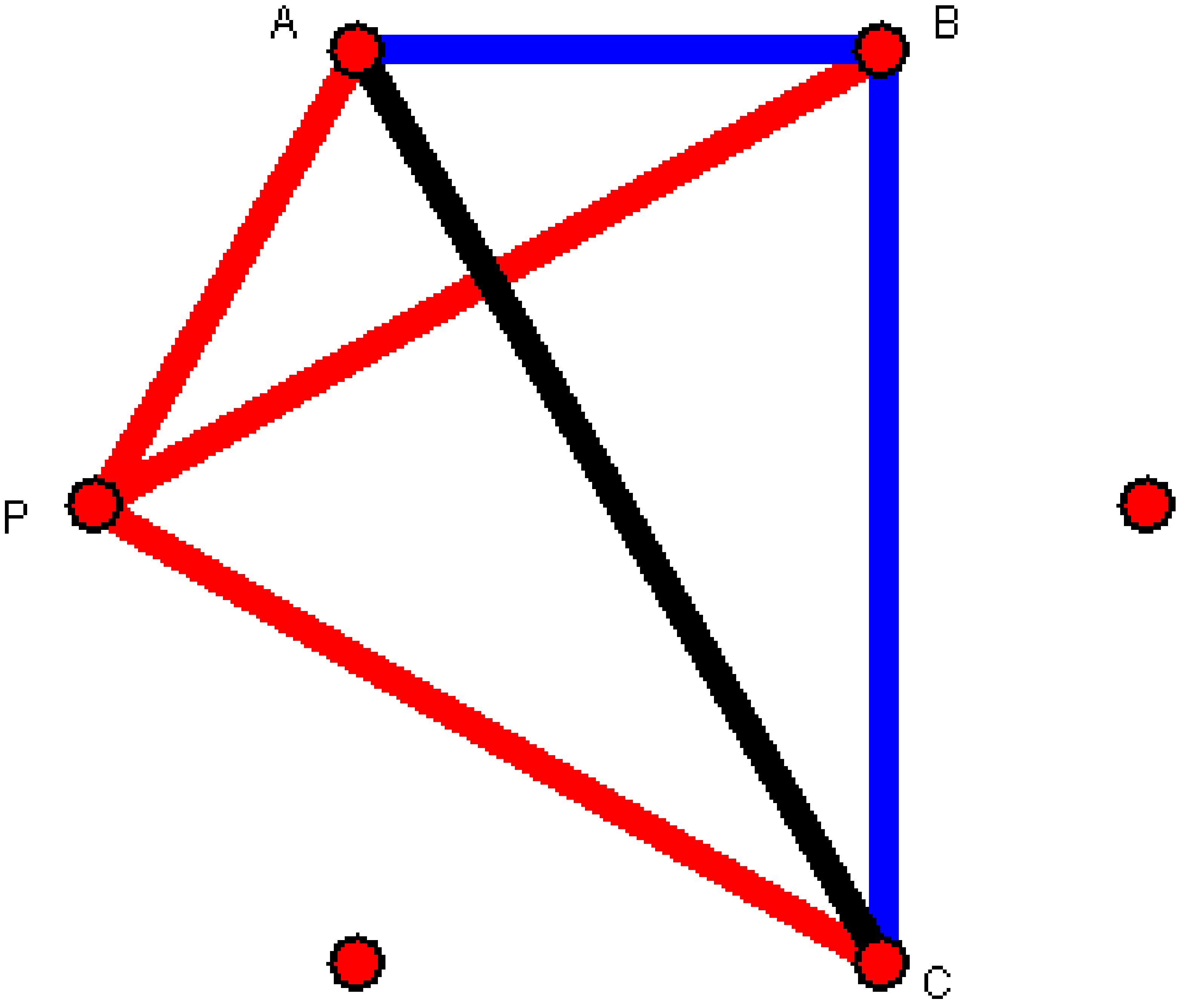
Pigeon-Hole Principle

If we have N+1 objects to put in N boxes, then at least one box will contain at least two objects.

Dirichlet used this principle to prove an important result in number theory (approximation of real numbers by rationals) in the middle XIX.

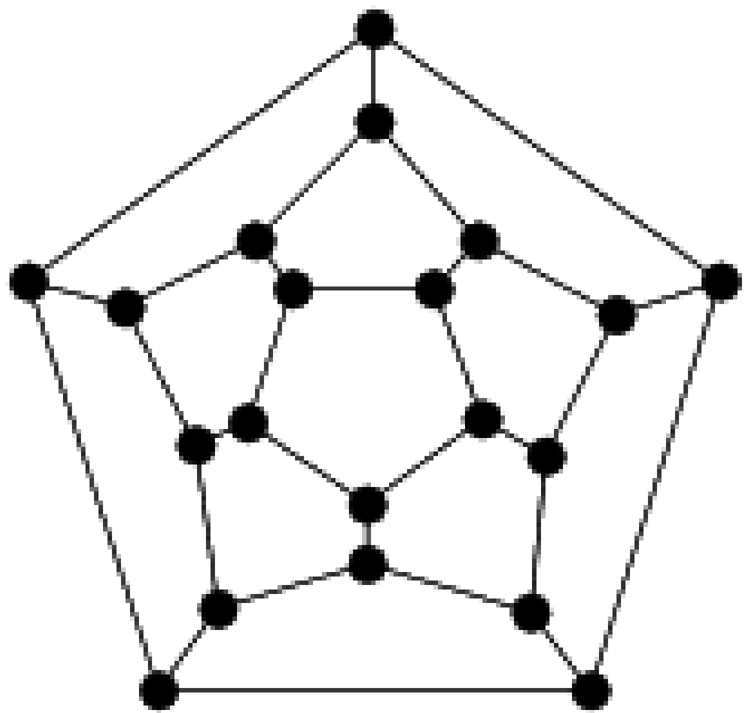
SIM has no draws

- Suppose all lines have been used, then each point is the end of five lines. Choose one point, P. By the PH Principle, at least three of the lines from P have the same color, red say. Let A, B, C be the other ends of such lines. Then AB, BC are blue.
- What color does AC have?


A
B
P
C

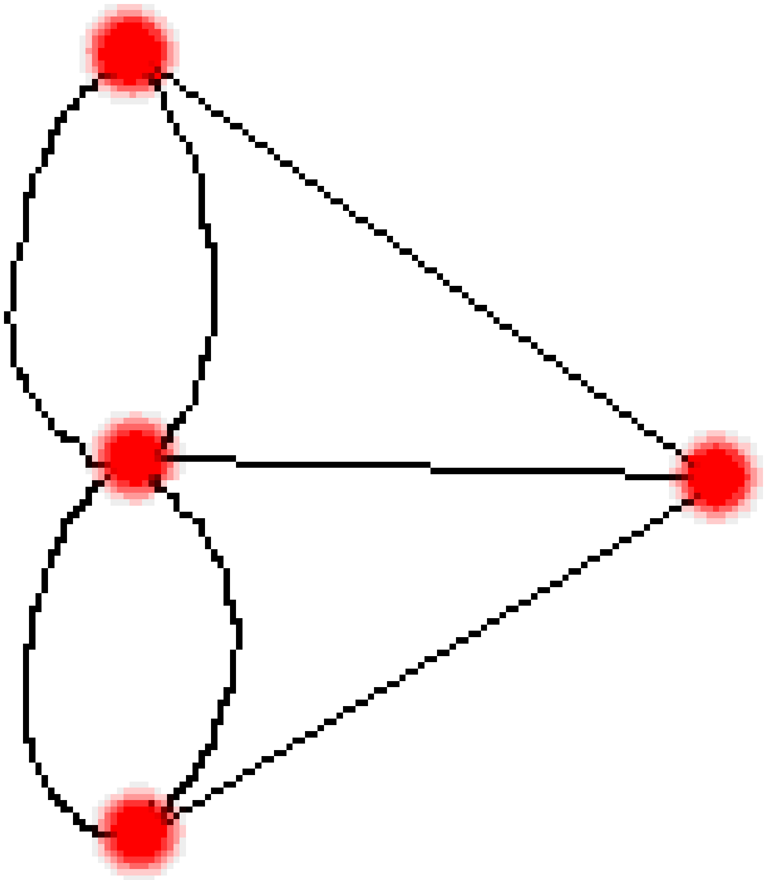
Hamilton

- Icosian game: visit each vertice once.



Euler

- Is it possible to go through each bridge once?



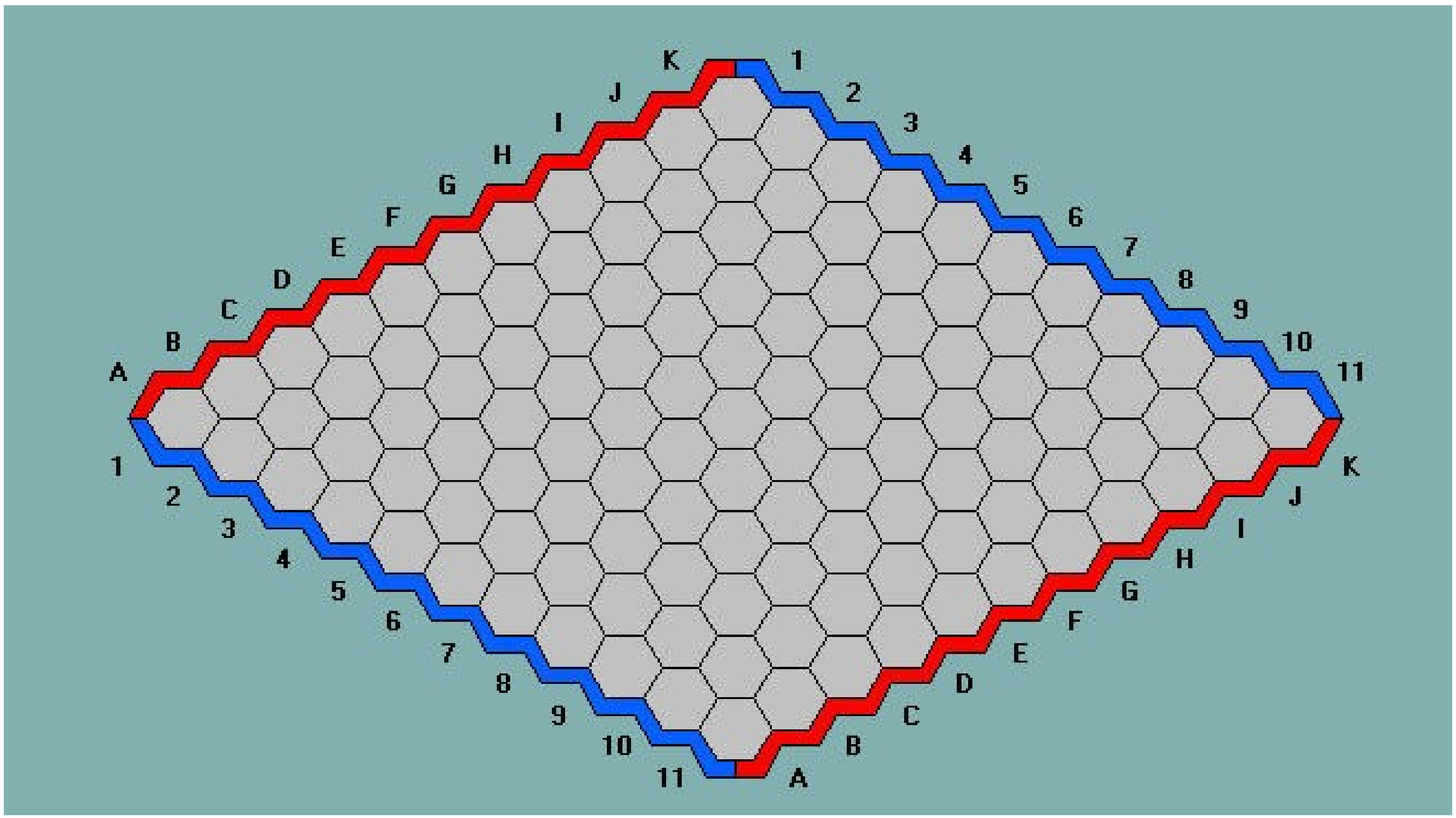
- A graph has an Euler path only if it has at most two vertices with odd degree.

- Unlike Eulerian graphs, Hamiltonian graphs are very difficult to characterize.

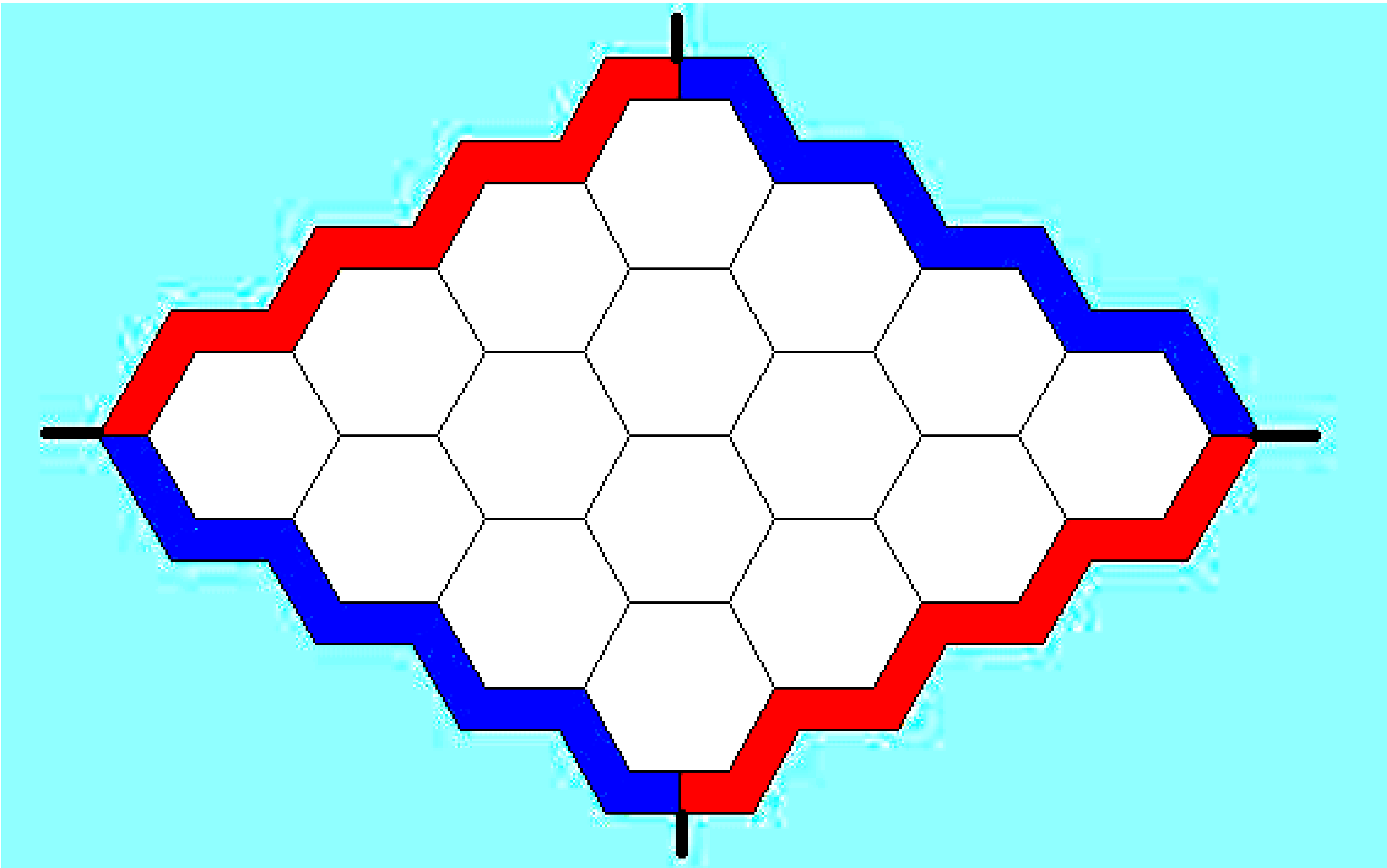
(Testing whether a graph is Hamiltonian is an NP-complete problem).

HEX

Players alternate placing counters of different colors, trying to connect parallel margins of the board



Hex's Theorem: There are no ties!

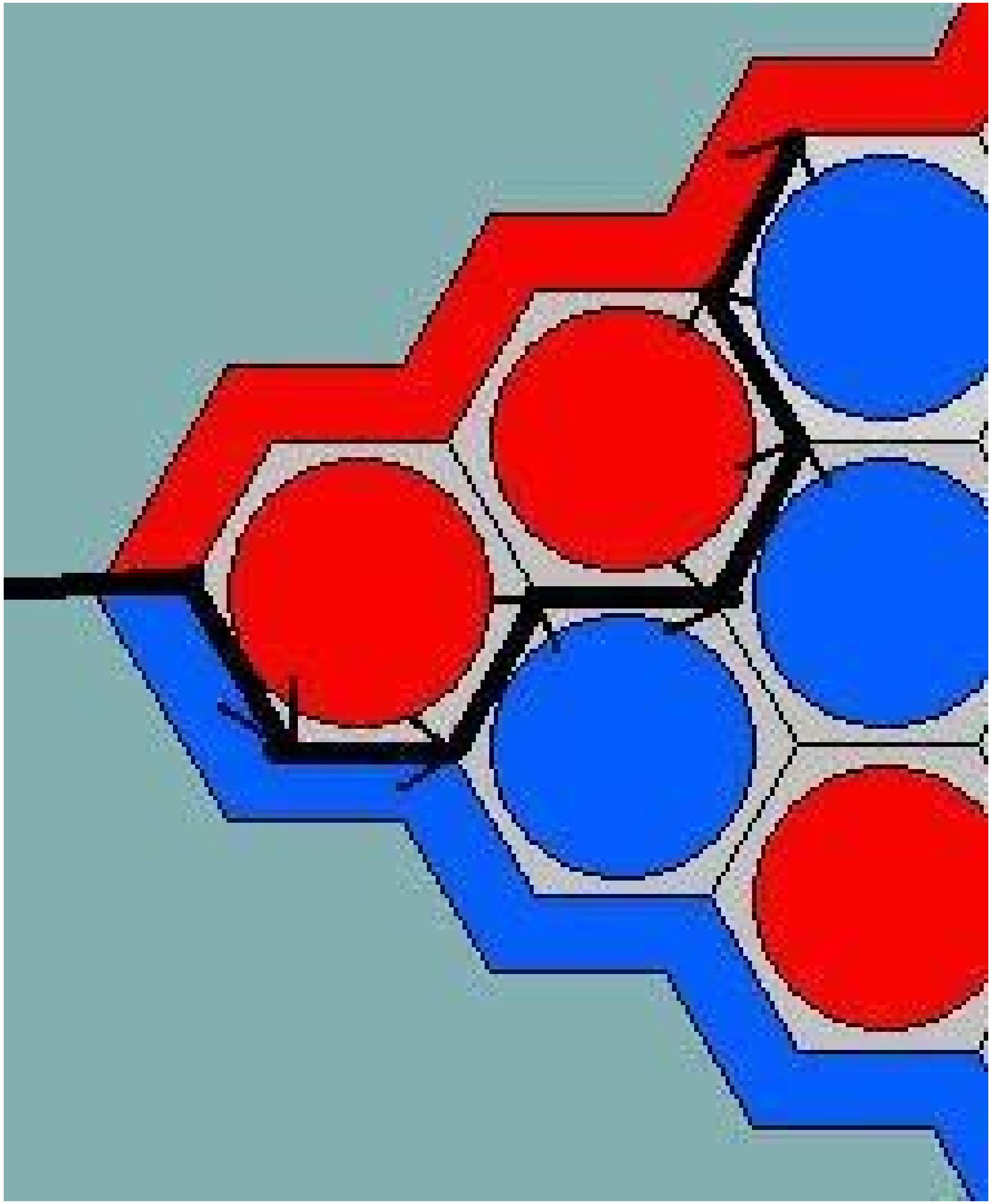


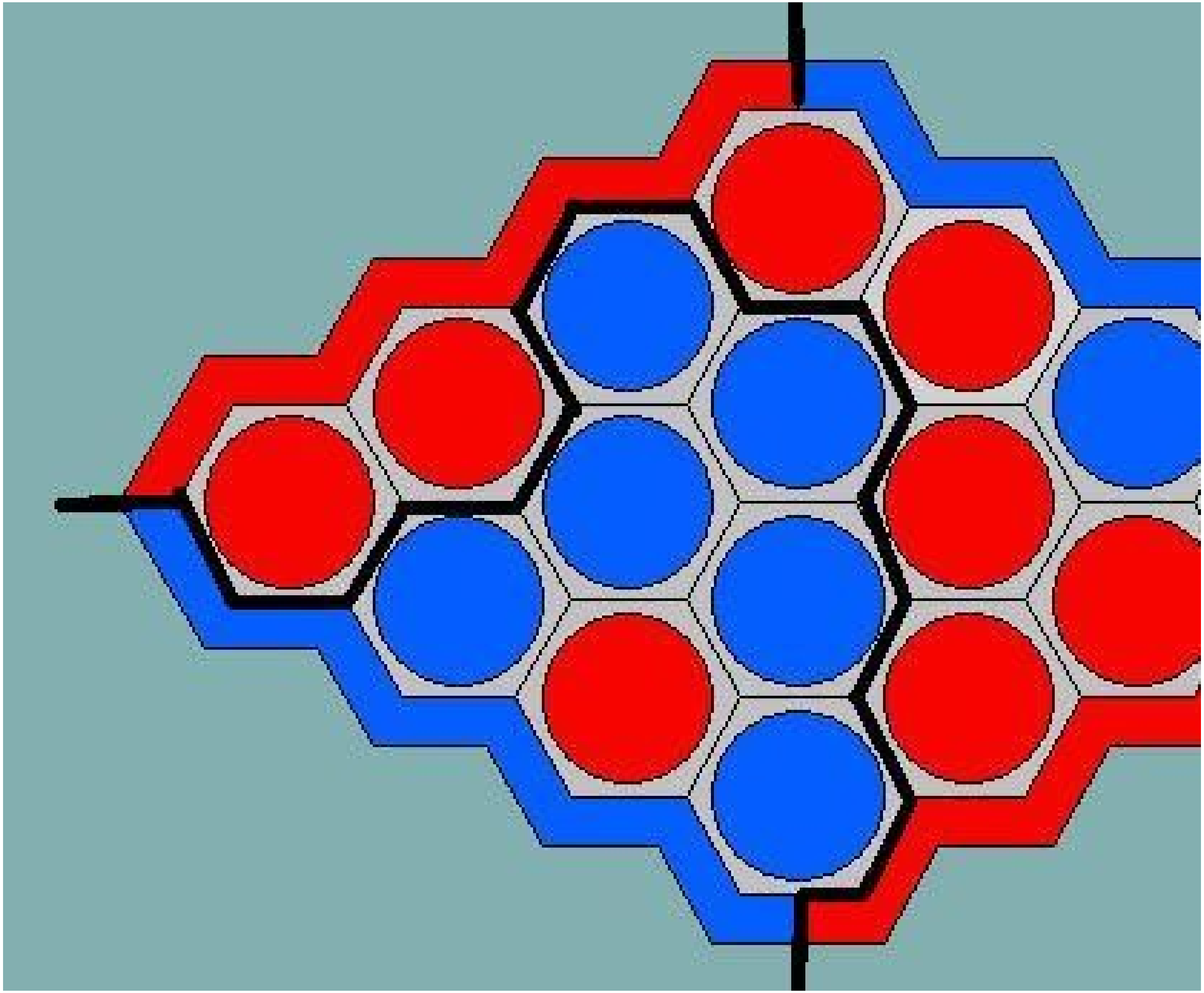
Start at a corner, move always between hexagons of different color.

Facts:

4. This path cannot visit a vertice more than once.
5. This path must end in another corner

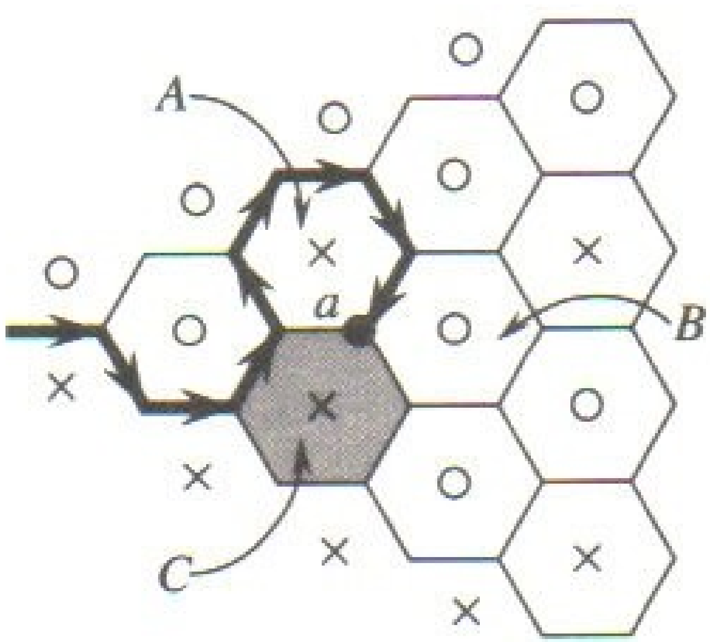
Conclusion: someone won!

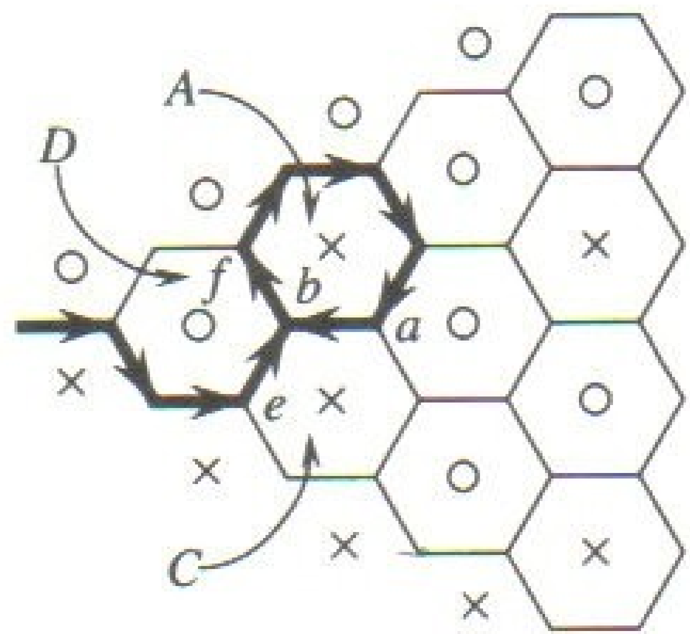




The path cannot finish inside the board.

No vertice can be visited twice.



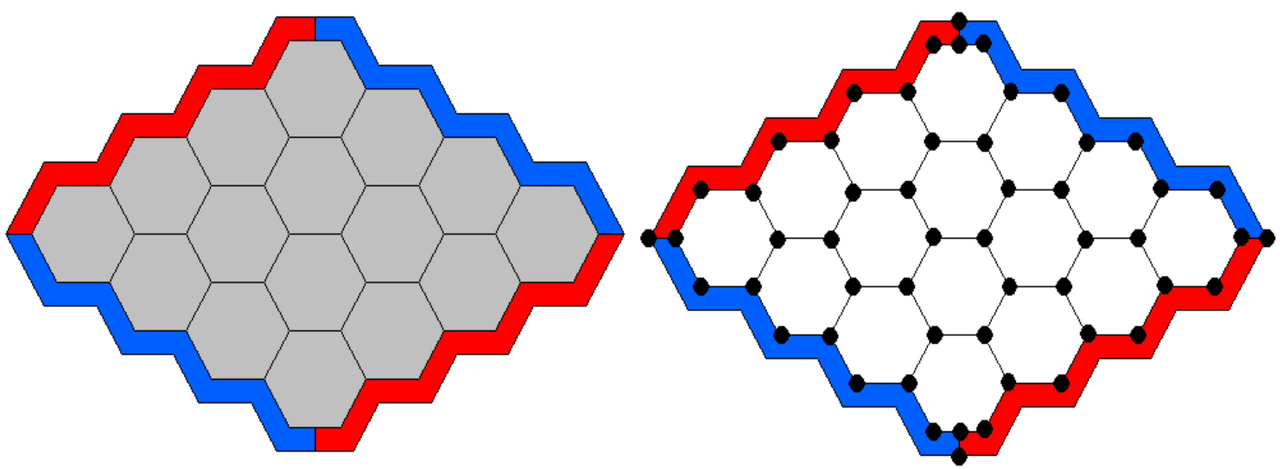


There are no ties!

If there were, it would be possible to fill the board without any connecting path.

Let's see that this is impossible.

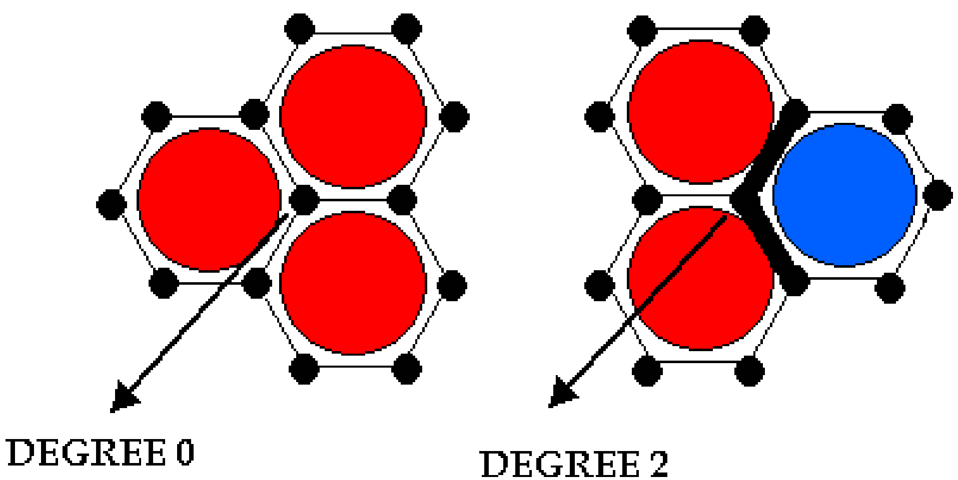
Let G be the planar graph associated with the Hex Board.



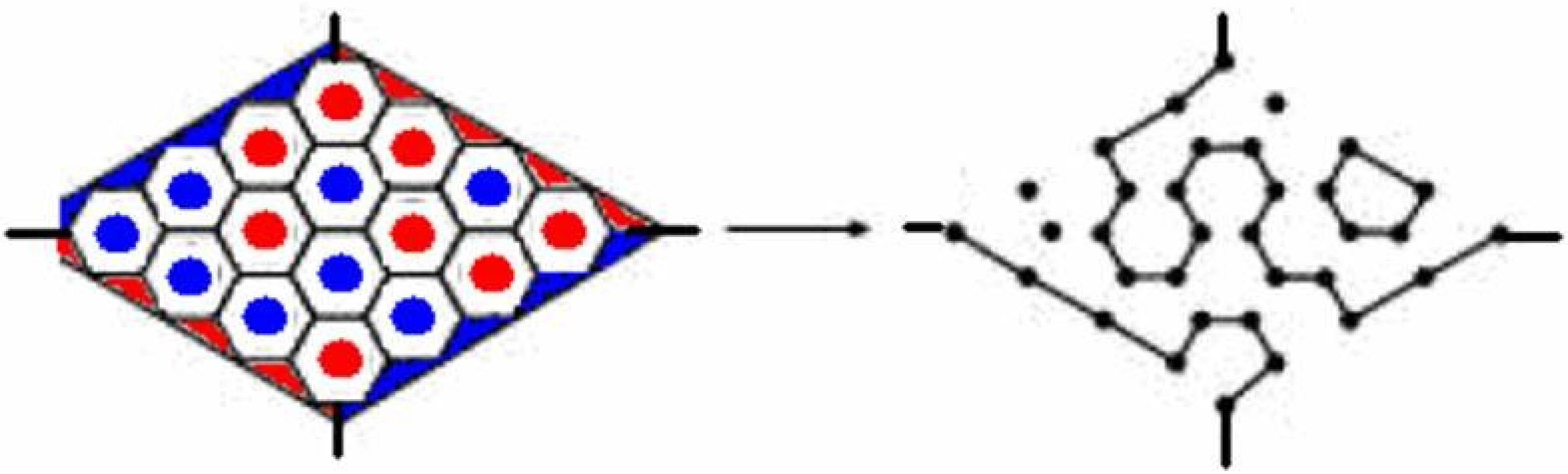
Let’s consider a full hex board.

We can now create a subgraph G´ of G by considering all edges lying between two hexagons of different colors.

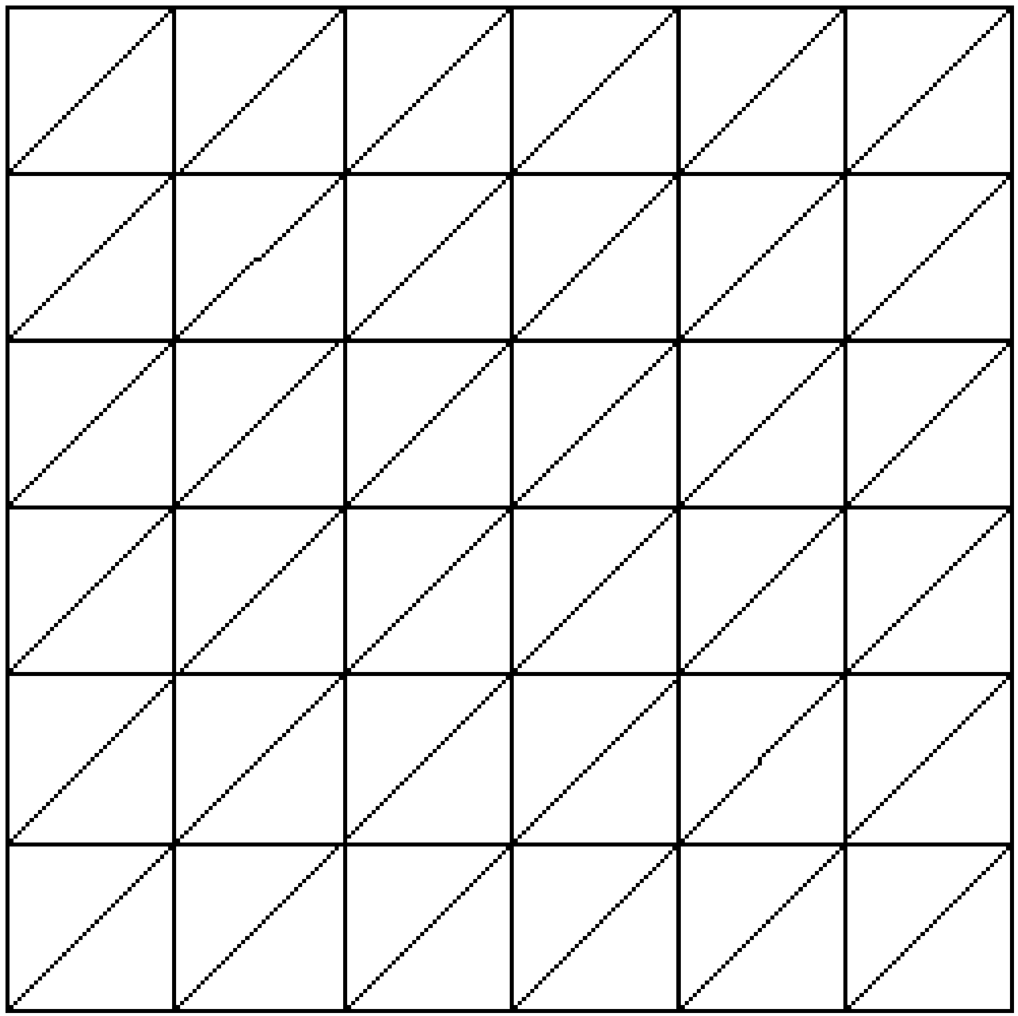
All inner vertices in G will be of degree 0 or 2 in G´



G′ will consist purely of disjoint simple cycles, simple paths and isolated points. Since we have exactly four vertices of degree 1, there must be exactly two simple paths and as they cannot intersect, these must both connect East/West or North/South.



Nash's Hex
(X plays E-W, O plays N-S)

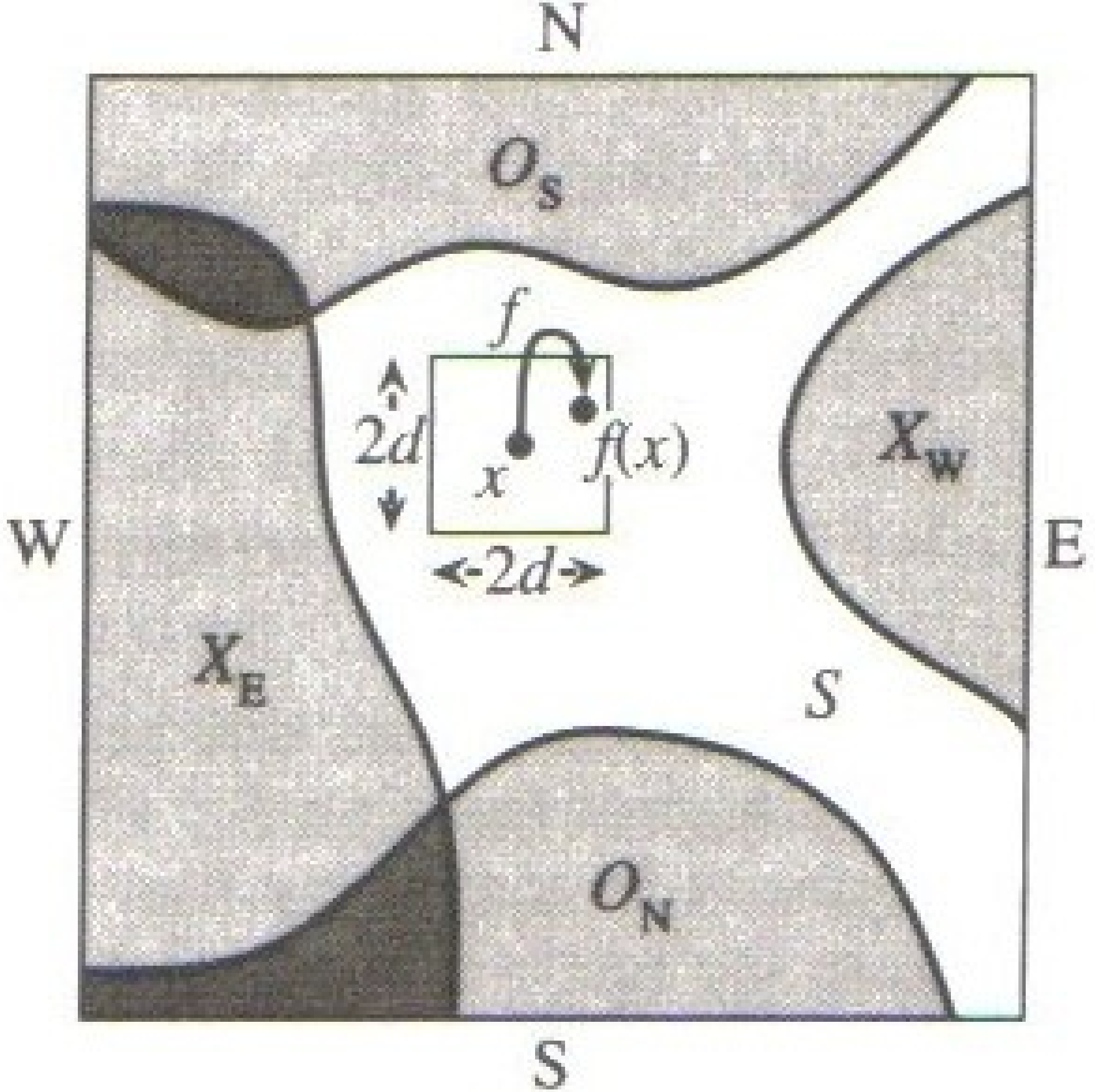


Brouwer's Fixed Point Theorem

- Let $f:Z \rightarrow Z$ be continuous, where Z is a compact convex of R^n. Then $f(x)=x$ for some x in Z.

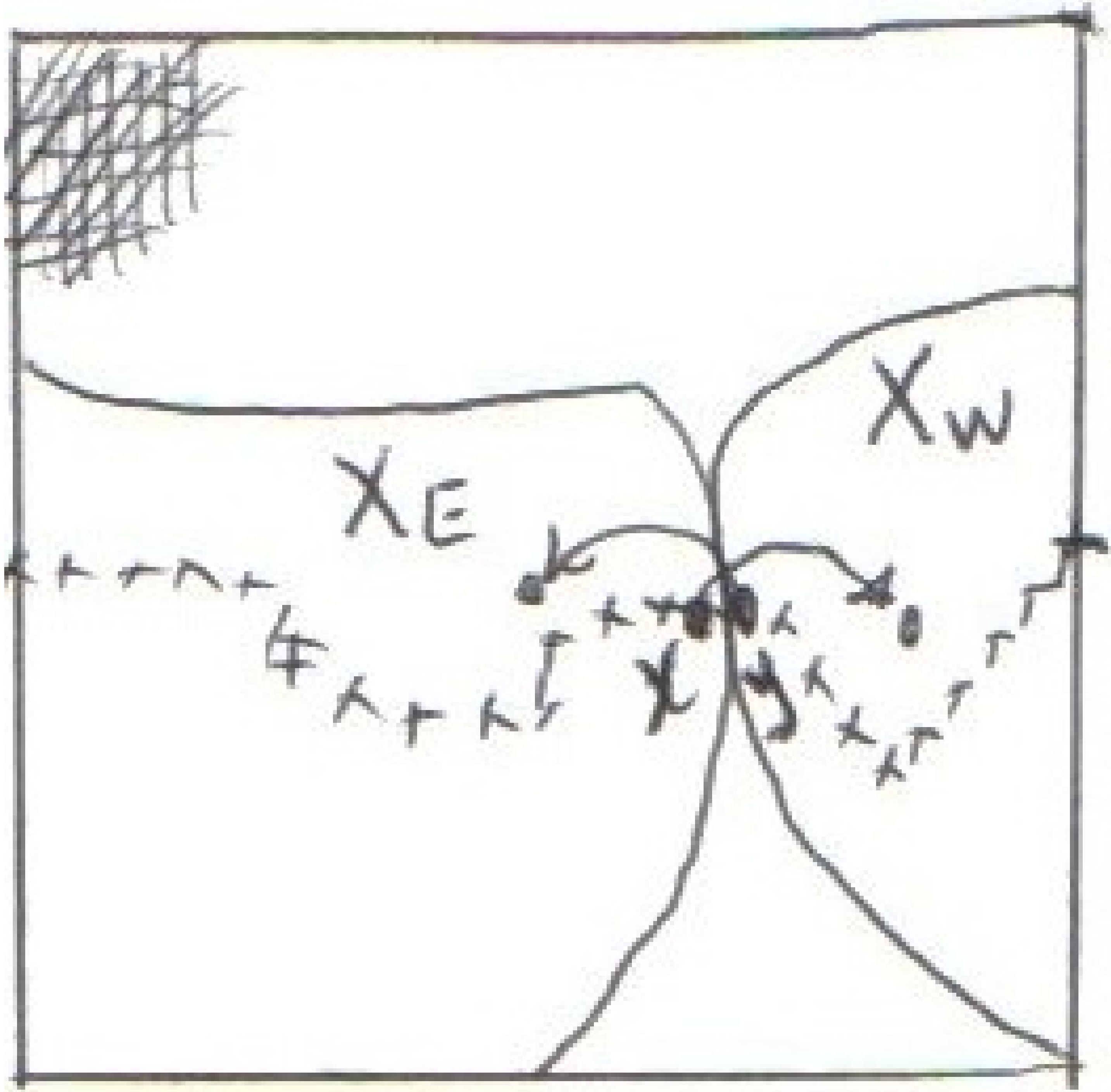
Proof (Using Hex's Theorem!): wlog Z is the unit square of the plane (n=2).

For d>0 let X_E the set of points f moves east at least the distance d. (X_W, O_N, O_S defined similarly)


N
O_S
f
2d
x
f(x)
X_W
W
E
2d
X_E
S
O_N
S

- Suppose for some d>o these four sets contain all of the square.
- Put an Hex board (Nash's style) with small mesh on the square.
- Suppose X (East-West) won that game.

Then for adjacent points x, y we have x in X_E, y in X_W. As we can have the mesh as small as we choose to, this contradicts the continuity of f.



XE
Xw
x
y

For each value of a vanishing sequence, d_n (values for d), we can take an x_n such that

$$|f(x_n)-x_n|<2d_n.$$

The compacity of Z lets us assume that x_n converge to some x.

The inequality above gives, taking limits,

$$f(x)=x.$$

QED

We can also show that

Brouwer's theorem implies that there are no tie

.

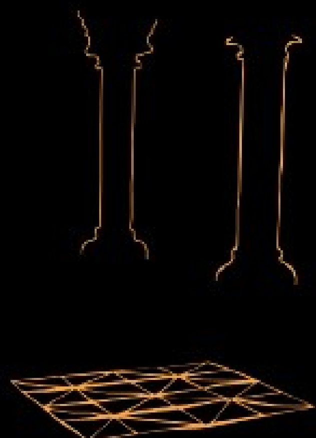

JOGOS MATEMÁTICOS
1.º CAMPEONATO NACIONAL
FINAL, 26 NOVEMBRO 2004
Pavilhão do Conhecimento - Ciência Viva
Parque das Nações, Lisboa
ludicum.org
Mais informações:
http://ludicum.org
Organização:
spm
Ciência Viva
AGÊNCIA NACIONAL PARA A CULTURA CIENTÍFICA E TECNOLÓGICA
POCTI


jogosmatemáticos
2.º Campeonato Nacional - Final
fábrica centro ciência viva de aveiro
10 de março 2006
http://ludicum.org
organização
apoios
FÁBRICA
CENTRO CIÊNCIA VIVA
aveiro
ludicum.org
APM
spm

4º Campeonato Nacional

29 de fevereiro de 2008

jogos matemáticos



alquerque dos 12

localização | Claustro da Igreja da Colegiada de Sta. Maria da Oliveira, em Guimarães, onde actualmente se encontra instalado o Museu de Alberto Sampaio

cronologia | o tabuleiro do jogo deve ter sido realizado por clérigos que aqui viviam, durante o século XVII ou XVIII

Pavilhão Desportivo <<

Campus de Gualtar | Braga

>> Universidade do Minho

apoio |

ASSOCIAÇÃO LUDUS

http://ludicum.org

ludus@ludicum.org