On extreme shocks and generalizations for modelling the probability of firms' default

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Content:

- **Shocks and break downs**
- **Urn model approach to break downs**
- **Firms default**

Joint work with A. Gut and P. Cirillo

1. Shocks and break downs

Simple models:

X_i, $i \geq 1$, **iid.** F **c.d.f., with upper endpoint** x_F **(a cont. point)**

A) Extreme shock model

Breakdown of a structure of a material if $X_i \geq \alpha$

τ **stopping time:** $\tau = \min\{i : X_i \geq \alpha\}$

$P\{\tau > m\} = F^m(\alpha)$

geometric distribution depending on $F(\alpha)$**.**

Asympt. result: Let $\alpha \to x_F$**, then**

$P\{\tau > z/\bar{F}(\alpha)\} \to \exp(-z)$ **for any** $z \geq 0$**.**

B) Cumulative shock models

$S_n = \sum_{i=1}^n X_i$, **breakdown if** $S_n \geq \alpha$

Assume μ_X **exists**

$\tau = \min\{n : S_n \geq \alpha\}$ **stopping time**

Obviously: $\tau \approx \alpha/\mu_x$ **for large** α

Some asymptotic known results:

i) $\tau/\alpha \to 1/\mu_X$ **a.s. as** $\alpha \to \infty$**.**

ii) $S_\tau/\alpha \to 1$ **a.s.**

iii) $(\tau - \alpha/\mu_X)/\sqrt{\alpha\sigma^2/\mu_X^3} \to Z \sim N(0,1)$ **as** $\alpha \to \infty$**.**

C) Time of the shocks

time of occurence of a shock is not n, but T_n:

let Y_i iid. r.v. with mean μ_Y

interarrival times

define the partial sum $T_n = \sum_{k=1}^{n} Y_i$

Failure time T_τ

But (X_i, Y_i) iid., not independent components

Extreme shock model:

Result: $\bar{F}(\alpha) T_\tau \xrightarrow{d} \mu_Y \mathrm{Exp}(1)$ **as** $\alpha \to x_F$

consider $\bar{F}(\alpha)\, \tau \; \times \dfrac{T_\tau}{\tau}$

Cumulative shock model:

$S_n = \sum_{i=1}^{n} X_i$, **with breakdown if** $S_n \geq \alpha$

τ **stopping time and** T_τ **failure time**

Asympt. result:

If $\mu_X > 0$ **and** $\mu_Y < \infty$**, then as** $\alpha \to \infty$

$T_\tau/\alpha \to \mu_Y/\mu_X$ **a.s.**

$(T_\tau - \mu_Y\alpha/\mu_x)/\sigma_\alpha \xrightarrow{d} N(0,1)$

where $\sigma_\alpha^2 = \mathrm{Var}(\mu_\mathrm{Y}\mathrm{X}_1 - \mu_\mathrm{X}\mathrm{Y}_1)\alpha/\mu_\mathrm{Y}^3$

(A. Gut and S. Janson)

Extensions, more realistic models

- **Delayed sums or recovering from shocks:**

  $S_{k,n} = \sum_{i=n-k+1}^{n} X_j$

- **Fatal and non-fatal shocks**

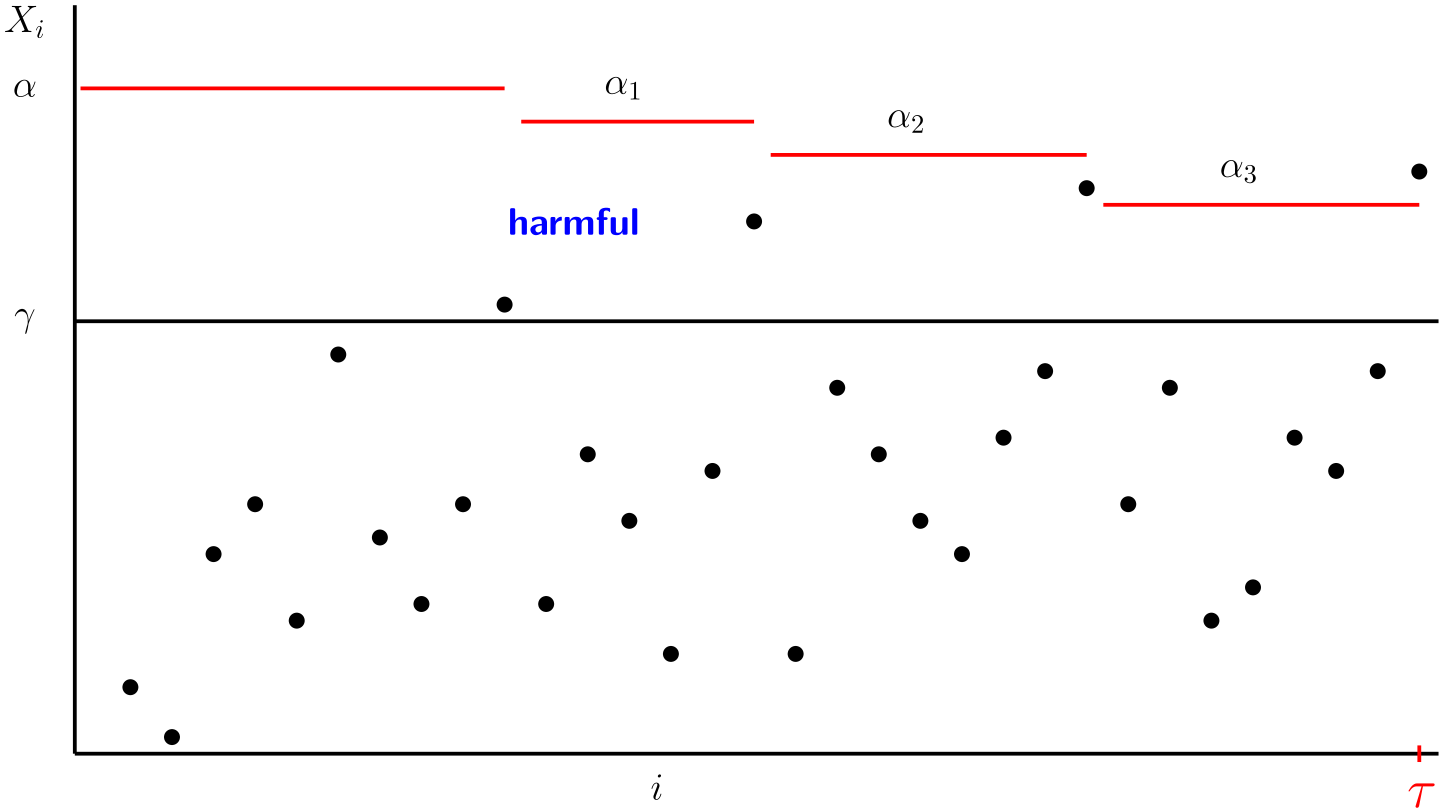
  no effect **if** $X_j < \gamma$

  non-fatal, harmful **if** $\alpha(L(j)) > X_1 \geq \gamma$

  fatal **if** $X_1 \geq \alpha(L(j))$

  where $\alpha(l)$ **decreasing sequence with** $\alpha(l) \geq \gamma$**.**

  stopping time: $\tau = \min\{n : X_n \geq \alpha(L(n))\}$



X_i
α
α_1
α_2
α_3
harmful
γ
i
τ

Exact distribution for the model with harmful shocks

Asympt. results:

If $\bar{F}(\alpha(k))/\bar{F}(\gamma) \to c_k$ **and** $\bar{F}(\gamma) \to 0$

(γ **and** $\alpha(k)$ **tend to the endpoint** x_F**)**

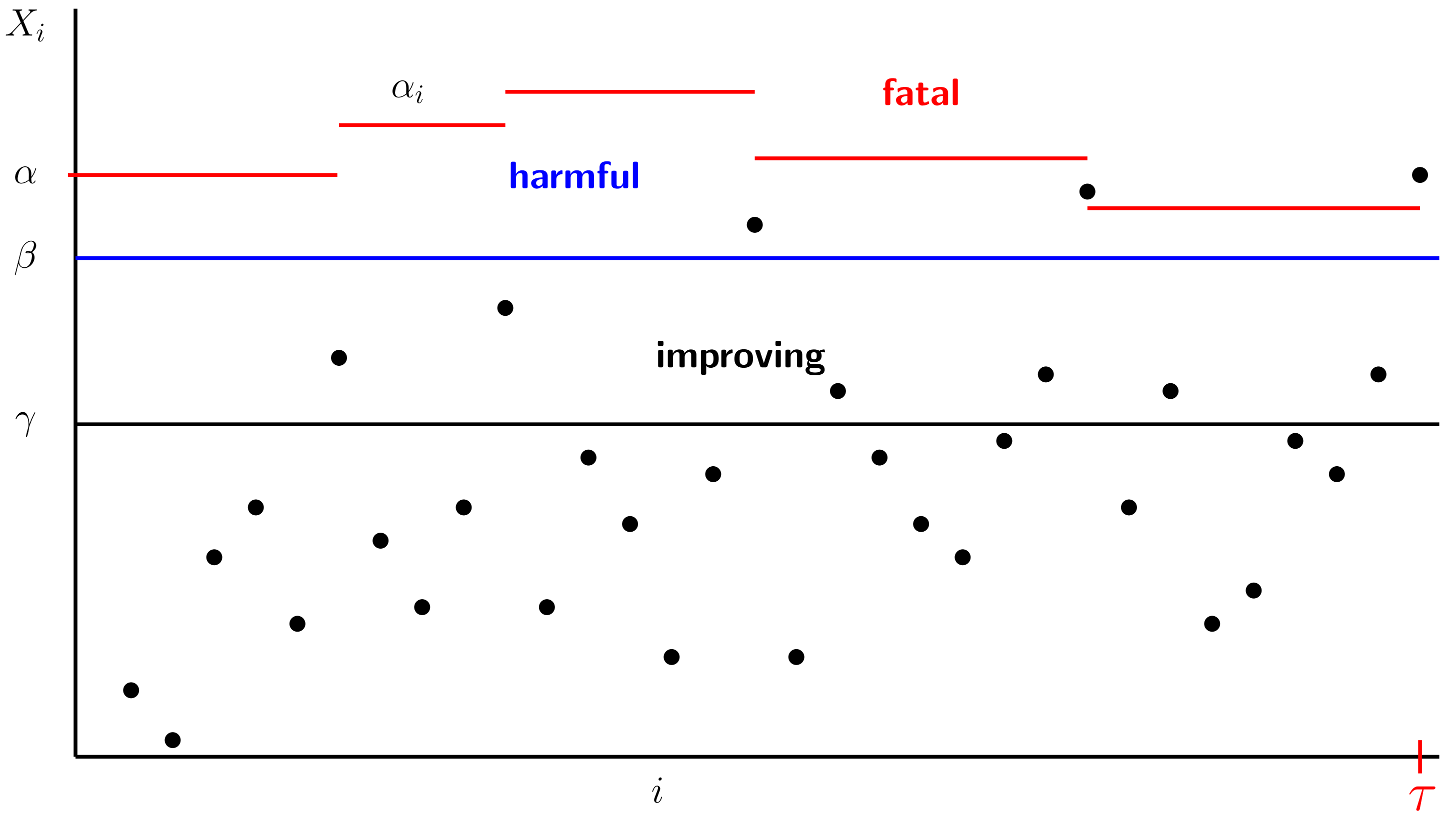
then

$$P\{\bar{F}(\gamma)\tau > z\} \to \sum_{j\geq 0} e^{-z}\frac{z^j}{j!}\prod_{k=0}^{j-1}(1-c_k) = e^{-z}+\sum_{j\geq 1} e^{-z}\frac{z^j}{j!}\prod_{k=0}^{j-1}(1-c_k)$$

Note $c_k \in [0,1]$**, and** $\prod_{k=0}^{-1} = 1$**.**

Further extension

A certain stress improves the material at the beginning



Exact distribution and asymptotic results:

for $\tau, N_+(\tau), N_-(\tau)$ **and** T_τ

where

$N_+(\tau)$ **number of strengthening strokes**

$N_-(\tau)$ **number of harmful strokes**

depending on conditions of α_i**,** β **and** γ**.**

Extensions:

Mixed models: Mixture of sum and extreme shock models

2. Urn model approach to break downs

Consider an urn containing balls of three different colors:

black, blue and red or

x**,** y**, and** w

each color represents a possible state of risk for the process:

x**-balls – safe state,**

y**-balls – risky state and**

w**-balls – default state.**

Evolution of the process:

1. **At time n a ball is random sampled from the urn, with the content depending on the urn composition at time $n-1$;**
2. **According to the color of the ball, the process $X_n = x, y$ or w;**
3. **The urn is then changed according to the reinforcement matrix.**

The reinforcement matrix

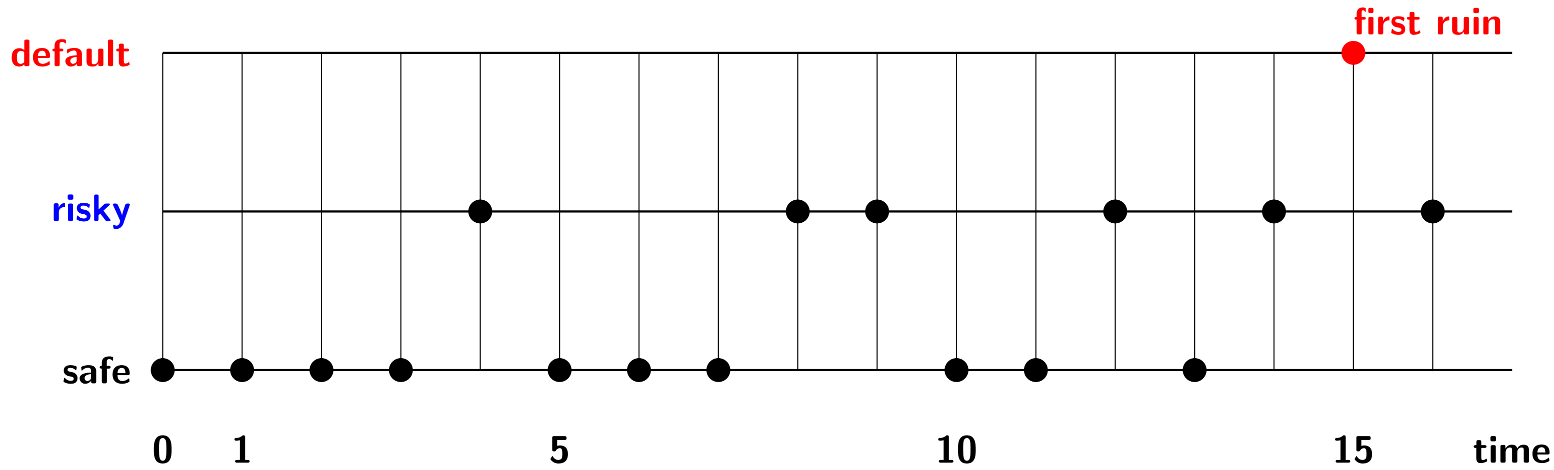
To model the positive dependence between the risky and the default states, we choose a balanced matrix constant over time:

$$RM = \begin{array}{c} \\ x \\ y \\ w \end{array} \begin{array}{c} \begin{array}{ccc} x & y & w \end{array} \\ \begin{bmatrix} \theta & 0 & 0 \\ 0 & \delta & \lambda \\ 0 & 0 & \theta \end{bmatrix} \end{array}, \quad \textbf{where } \lambda = \theta - \delta \qquad (1)$$

1. **If an x-ball is sampled, θ balls of type x are added;**
2. **if an y-ball is sampled, the urn is reinforced with δ y-balls and λ w-balls (to model dependence);**
3. **if a w-ball is picked up, θ balls of the same color are added.**

Example of a simulated urn process

with $a_k = a_0 + k\theta$, $b_k = b_0 + k\delta$ and $c_k = c_0 + k\lambda$.



Content of the urn after the nth-drawing

w-**balls:**	c_0	c_0	.	.	c_1	.	.	.	c_2	c_3	.	.	c_4	.	c_5	$c_5 + \theta$
y-**balls:**	b_0	b_0	.	.	b_1	.	.	.	b_2	b_3	.	.	b_4	.	b_5	b_5
x-**balls:**	a_1	a_2	a_3	a_4	a_4	a_5	a_6	a_7	a_7	a_7	a_8	a_9	a_9	a_{10}	a_{10}	a_{10}

Assumptions:

Condition 1:

Let $\theta\delta \neq 0$, not to have degenerate cases.

Condition 2: Let $\lambda = \theta - \delta \geq 0$,

to model the positive dependence between y and w balls.

Theory for discrete-time balanced urn process

with a 3×3 reinforcement matrix

Relation (isomorphism) to ordinary differential equation system.

Generating function H of the urn history.

Our model:

$$\sum = \begin{cases} \dot{x} = x^{\theta+1} \\ \dot{y} = y^{\delta+1} w^{\lambda} \\ \dot{w} = w^{\theta+1} \end{cases} \quad \textbf{with i.c.} \quad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \\ w(0) = w_0 \end{cases}, \tag{2}$$

simple integration for x and y:

$$x(t) = x_0(1 - \theta x_0^{\theta} t)^{-\frac{1}{\theta}} \tag{3}$$

$$w(t) = w_0(1 - \theta w_0^{\theta} t)^{-\frac{1}{\theta}}. \tag{4}$$

Since $\dot{y}y^{-\delta-1} = w^{\lambda}$, the solution is:

$$y(t) = y_0(1 - y_0^{\delta} \left(w_0^{-\delta} - \left[w_0(1 - \theta w_0^{\theta} t)^{-\frac{1}{\theta}} \right]^{-\delta} \right)^{-\frac{1}{\delta}}.$$

Hence

Proposition:

Consider an urn process with a reinforcement matrix RM as in 1, that satisfies Conditions 1 and 2, and with an initial composition (a_0, b_0, c_0) of balls.

The 4-variables generating function of urn histories is:

$$H(z; x, y, w) = x^{a_0} y^{b_0} w^{c_0} (1 - \theta x^\theta z)^{-\frac{a_0}{\theta}} (1 - \theta w^\theta z)^{-\frac{c_0}{\theta}} \times \left(1 - y^\delta w^{-\delta} \left(1 - (1 - \theta w^\theta z)^{\frac{\delta}{\theta}}\right)\right)^{-\frac{b_0}{\delta}}.$$

Propostion for the moments:

X_n, Y_n **and** W_n: **number of** x, y **and** w **balls in the urn at time** n.

Their moments: hypergeometric functions, finite linear combinations of product and quotients of Euler Gamma functions.

In particular:

$$E\left[X_n\right] = \frac{a_0}{t_0}(t_0 + n\theta),$$

$$E\left[Y_n\right] = b_0 \frac{\Gamma\left(\frac{t_0}{\theta}\right)}{\Gamma\left(\frac{t_0+\delta}{\theta}\right)} n^{\frac{\delta}{\theta}} + O(n^{\frac{\delta}{\theta}-1}),$$

$$E\left[W_n\right] = \left[(t_0 - a_0)\frac{\lambda}{\theta}\right] \frac{\Gamma\left(\frac{t_0}{\theta}\right)}{\Gamma\left(\frac{t_0+\lambda}{\theta}\right)} n^{\frac{\delta}{\theta}} + O(n^{\frac{\delta}{\theta}-1}),$$

where $t_0 = a_0 + b_0 + c_0$ **and** $\lambda = \theta - \delta$

Limit result:

For any compact set S of $\mathbb{R}^+$ and any $\gamma \in S$ such that $\gamma n^{\frac{\delta}{\theta}}$ is an integer, we have that

$$P\left[Y_n = b_0 + \delta\gamma n^{\frac{\delta}{\theta}}\right] = n^{-\frac{\delta}{\theta}} g(\gamma) + O(n^{-2\frac{\delta}{\theta}}) \qquad (5)$$

where the error term holds uniformly with respect to $\gamma \in S$.

Function $g(\cdot)$ (gen. Mittag-Leffler) is defined on $\mathbb{R}^+$ by

$$g(\gamma) = \frac{\Gamma\left(\frac{t_0}{\theta}\right)}{\Gamma\left(\frac{b_0}{\delta}\right)} \gamma^{\frac{b_0}{\delta}-1} \sum_{k \geq 0} (-1)^k \frac{\gamma^k}{\Gamma(k+1)\Gamma\left(\frac{c_0 - k\delta}{\theta}\right)}.$$

An analogous reasoning is valid for W_n with a different $g(\gamma)$ or for X_n, as a standard Poly urn.

Remarks

1. **For $c_0 = 0$, no w balls in the initial composition, no immediate failing, the function $g(\gamma)$ represents a Paretian stable law of index $\frac{\delta}{\theta}$.**
   So Y_n has a power law, asympt., in accordance with the Zipf's law in econometrics.

2. **If $c_0 < \theta$, on the contrary, $g(\gamma)$ becomes a Gamma distribution (even an exponential for $b_0 = \delta$).**

Joint limit distribution:

Consider the whole process $U_n = (X_n, Y_n, W_n)$ **with** $U_0 = (a_0, b_0, c_0)$

Then $U_n/(\theta n)$ **converges to a random vector, depending on**

(V_1, V_2, V_3) **has a Dirichlet distribution, whose density on the simplex** $(u_x \geq 0, u_y \geq 0, u_z \geq 0, u_x + u_y + u_w = 1)$ **given by**

$$\Gamma(\frac{t_0}{\theta})\frac{u_x^{a_0+c_0}}{\Gamma\left(a_0+c_0\right)}\frac{u_y^{b_0-c_0}}{\Gamma\left(b_0-c_0\right)}\frac{u_w^{\frac{1}{a}[\lambda c_0-\delta a_0]}}{\Gamma\left(\frac{1}{\theta}\left[\lambda c_0-\delta a_0\right]\right)},$$

(u_x**: proportion of** x **balls in the urn)**

and on the RM.

3. Firms default

Example of a simulated urn process

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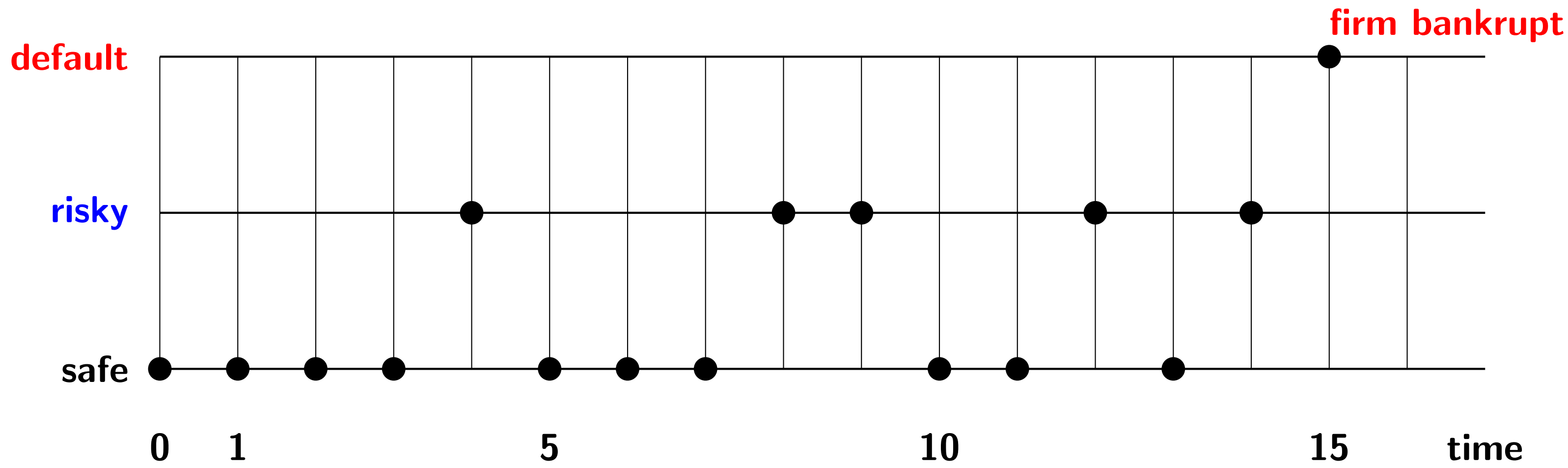

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x-**balls:**	a_1	a_2	a_3	a_4	a_4	a_5	a_6	a_7	a_7	a_7	a_8	a_9	a_9	a_{10}	a_{10}	a_{10}

Data

data from the CEBI database: CEBI
comprehensive database first developed by the Bank of Italy and now maintained by Centrale dei Bilanci Srl.
biggest Italian industrial dataset, with firm-level observations and balance sheets of thousands of firms.

Subset of 380 manufacturing firms with the conditions:

1. All firms' data: active in the period 1982-2000;
2. Every firm: more than 100 employees with reliable information about capital and financial ratios;
3. Under bank control for possible insolvency at least once.

Selected firms are comparable with those originally used by Altman (1968) (famous paper):
the benchmark for our work.

For every firm: standard balance ratios:

r_1 **: working capital / total assets**

r_2 **: retained earnings / total assets**

r_3 **: EBIT / total assets**

r_4 **: market value of equity / book value of total liabilities**

r_5 **: sales / total assets**

r_6 **: equity ratio**

r_7 **: debt ratio**

Initialization of the process

We need initialized values and RM?

RM:

We simply set $\theta = 3$ and $\delta = 2$ for every firm

'Best fit' by a simple grid search.

A first attempt

RM has good properties, essentially a Poly urn.

Initial composition of firms' urns:

heuristic method,

based on well-known stylized facts of industrial economics.

First x-balls: equity ratio r_6 as a proxy of the proportion of $x-$balls.

Firm with $r_6 \geq 0.5$ can be considered as financially robust

Hence, for every firm set $a_0 = [r_6 * 100]$.

Second: y and w-balls: risky and the default states,

combine debt ratio r_7 and complement with equity ratio r_6

a higher debt ratio: signal of danger for firms' reliability

set $b_0 = [r_7 * (1 - r_6) * 100]$ and $c_0 = [(1 - r_6)(1 - r_7) * 100]$.

So, initial number of balls in the urn: 100.

Examples:

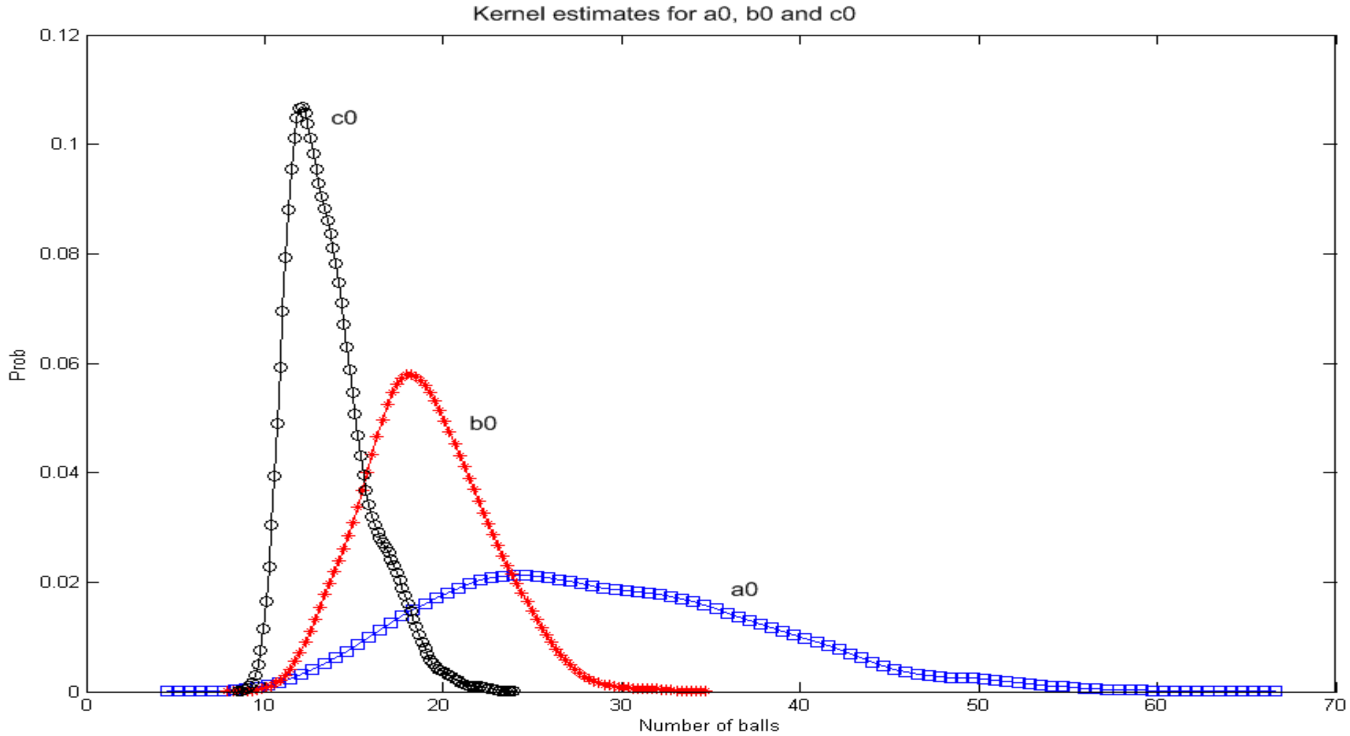
Initial urn composition for some firms of the dataset

firm code	year	equity ratio r_6	debt ratio r_7	a_0	b_0	c_0	default in $t+1$
$IM223A$	1982	0.42	0.71	42	41	17	0
$IM298A$	1982	0.62	0.52	62	20	18	0
$IM567B$	1982	0.68	0.37	68	12	20	1
$IM1031B$	1982	0.39	0.66	39	40	21	1
$IM1988A$	1982	0.72	0.57	72	16	12	0

All firms together:

average numbers of initialized x, y and $w-$balls: 48, 39, 13

distributions for a_0, b_0 and c_0 of all the firms:



For every firm we can compute all the probabilities at an time n.

In this experiment assume that a firm fails at time $n+1$ if the probability of extracting a w-ball is ≥ 0.20 at time n, very common threshold

For every firm, in every period, we can prediction failure, compare it

– with actual data and

– with simple Altman's ones.

Altman's Z-score (1968) popular measure,
based on discriminant analysis,
to classify firms' riskiness.

In particular, using Altman's original formulation we have

$$Z = 0.012r_1 + 0.014r_2 + 0.033r_3 + 0.006r_4 + 0.999r_5.$$

According to this score, a firm is likely
to default if $Z < 1.8$,
safe if $Z > 3$,
'gray' otherwise.

Estimated the Z-score on CEBI data set to understand its general formulation on Italian data.

using standard regression techniques

$$Z^* = \underset{(0.0062)}{0.014}\, r_1 + \underset{(0.0057)}{0.013}\, r_2 + \underset{(0.039)}{0.052} r_3 + \underset{(0.0028)}{0.007}\, r_4 + \underset{(0.3413)}{0.955}\, r_5,$$

$$(Z = 0.012r_1 + 0.014r_2 + 0.033r_3 + 0.006r_4 + 0.999r_5).$$

Similar values

r_3 **EBIT larger, but not significant.**

Use both Z**-Scores, only small differences.**

So Altman's Z **Scores used, traditional.**

Results:

For 245 firms from 380, both methods correctly predict firms' default.

For 72 firms from 380, our model (UGESM) seems to behave better.

Remaining 63 firms: both models do not predict default in the right way.

Comparison of the number of correctly predicted defaults for UGESM and Altman's Z-score

	UGESM	**Z-score**
correct	83%	66%
no correct	17%	34%

Remaining cases:

Altman's method usually underestimates the possibility of a failure,

UGSEM seems to be more pessimistic: for 40 firms from 63 generally predict failure 2-3 periods before actual default.

Improve

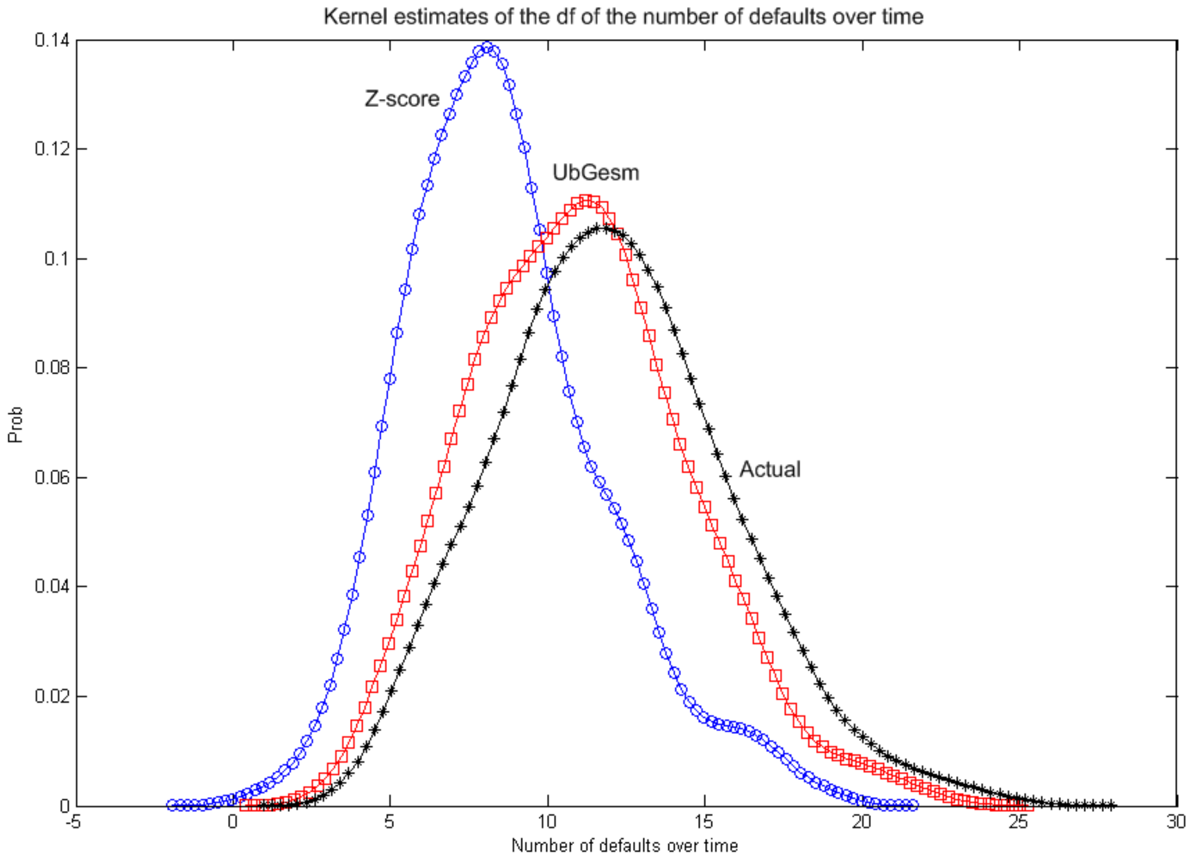
A good result with such a simple model, more prudent behavior is required for banks and similar companies (e.g. Basel II).

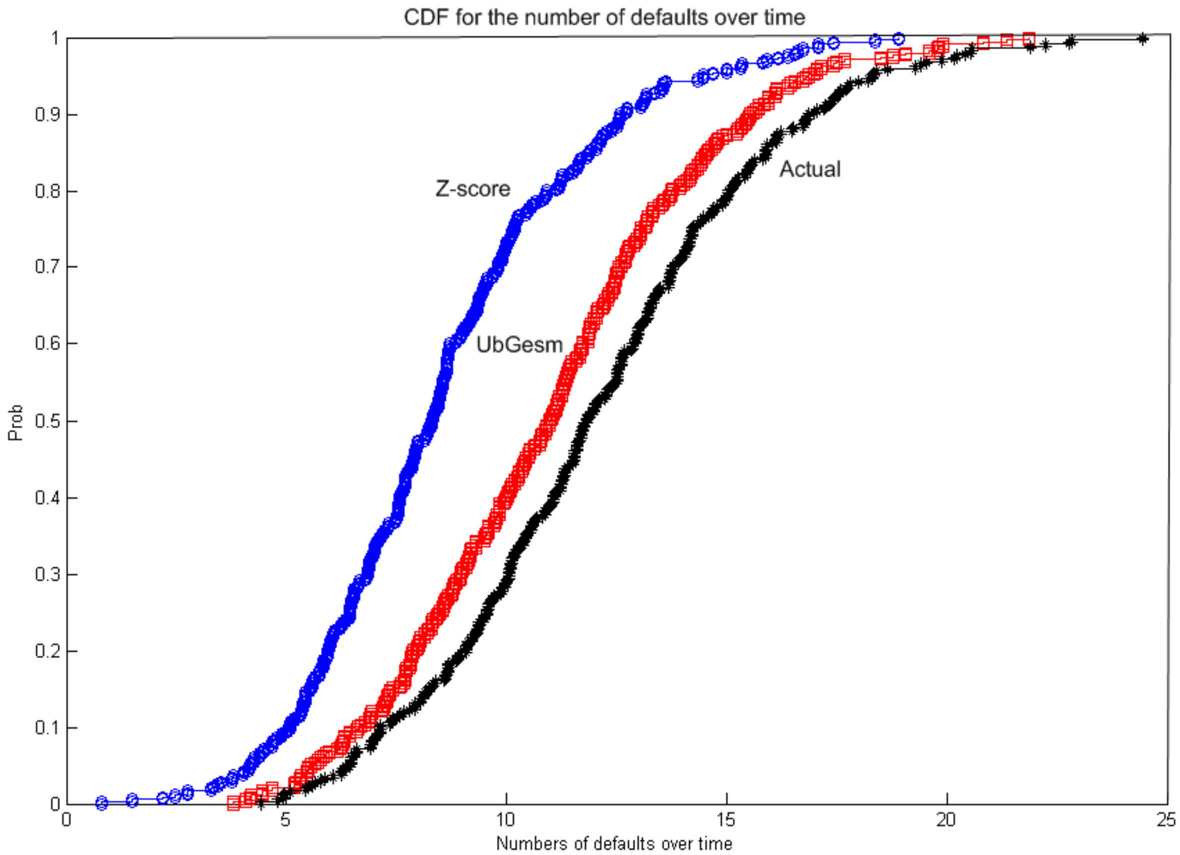
Remaining 63 cases:
number of underestimated and overestimated defaults before and after actual failure with averages of number of wrong periods

	UGESM	Z**-score**
underestimated	63% (2.7)	28% (1.4)
overestimated	37% (3.2)	72% (2.3)

Distributions of time of default ?

Kernel estimates using the Epanechnikov kernel





CDF for the number of defaults over time
Prob
Numbers of defaults over time
Z-score
UbGesm
Actual
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25