On extreme shocks and generalizations for modelling the probability of firms’ default

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Content:

- Shocks and break downs
- Urn model approach to break downs
- Firms default

Joint work with A. Gut and P. Cirillo
1. Shocks and break downs

Simple models:

\( X_i, i \geq 1, \text{ iid. } F \text{ c.d.f., with upper endpoint } x_F \text{ (a cont. point) } \)

A) Extreme shock model

Breakdown of a structure of a material if \( X_i \geq \alpha \)

\( \tau \) stopping time: \( \tau = \min\{i : X_i \geq \alpha\} \)

\[ P\{\tau > m\} = F^m(\alpha) \]

geometric distribution depending on \( F(\alpha) \).

Asympt. result: Let \( \alpha \to x_F \), then

\[ P\{\tau > z/F(\alpha)\} \to \exp(-z) \text{ for any } z \geq 0. \]
B) Cumulative shock models

\[ S_n = \sum_{i=1}^{n} X_i, \text{ breakdown if } S_n \geq \alpha \]

Assume \( \mu_X \) exists
\[ \tau = \min\{n : S_n \geq \alpha\} \text{ stopping time} \]

Obviously: \( \tau \approx \alpha/\mu_x \) for large \( \alpha \)

Some asymptotic known results:

i) \( \tau/\alpha \to 1/\mu_X \) a.s. as \( \alpha \to \infty \).

ii) \( S_\tau/\alpha \to 1 \) a.s.

iii) \( (\tau - \alpha/\mu_X)/\sqrt{\alpha\sigma^2/\mu_X^3} \to Z \sim N(0, 1) \) as \( \alpha \to \infty \).
C) Time of the shocks

time of occurrence of a shock is not $n$, but $T_n$:
let $Y_i$ iid. r.v. with mean $\mu_Y$
interarrival times
define the partial sum $T_n = \sum_{k=1}^{n} Y_i$

Failure time $T_{\tau}$
But $(X_i, Y_i)$ iid., not independent components

Extreme shock model:

Result: $\bar{F}(\alpha) T_{\tau} \xrightarrow{d} \mu_Y \text{Exp}(1)$ as $\alpha \to x_F$

consider $\bar{F}(\alpha) \frac{T_{\tau}}{\tau}$
Cumulative shock model:

\[ S_n = \sum_{i=1}^{n} X_i, \text{ with breakdown if } S_n \geq \alpha \]

\( \tau \) stopping time and \( T_\tau \) failure time

Asympt. result:

If \( \mu_X > 0 \) and \( \mu_Y < \infty \), then as \( \alpha \to \infty \)

\[ T_\tau / \alpha \to \mu_Y / \mu_X \text{ a.s.} \]

\[ (T_\tau - \mu_Y \alpha / \mu_X) / \sigma_\alpha \overset{d}{\to} N(0, 1) \]

where \( \sigma^2_\alpha = \text{Var}(\mu_Y X_1 - \mu_X Y_1) \alpha / \mu_Y^3 \)

(A. Gut and S. Janson)
Extensions, more realistic models

• Delayed sums or recovering from shocks:
  \[ S_{k,n} = \sum_{i=n-k+1}^{n} X_j \]

• Fatal and non-fatal shocks
  
  no effect if \( X_j < \gamma \)
  
  non-fatal, harmful if \( \alpha(L(j)) > X_1 \geq \gamma \)
  
  fatal if \( X_1 \geq \alpha(L(j)) \)

  where \( \alpha(l) \) decreasing sequence with \( \alpha(l) \geq \gamma \).

  stopping time: \( \tau = \min\{n : X_n \geq \alpha(L(n))\} \)
Exact distribution for the model with harmful shocks

Asympt. results:
If $\bar{F}(\alpha(k))/\bar{F}(\gamma) \to c_k$ and $\bar{F}(\gamma) \to 0$
($\gamma$ and $\alpha(k)$ tend to the endpoint $x_F$)
then

$$P\{\bar{F}(\gamma)\tau > z\} \to \sum_{j \geq 0} e^{-z}z^j \frac{j^{-1}}{j!} \prod_{k=0}^{j-1} (1-c_k) = e^{-z} + \sum_{j \geq 1} e^{-z}z^j \frac{j^{-1}}{j!} \prod_{k=0}^{j-1} (1-c_k)$$

Note $c_k \in [0, 1]$, and $\prod_{k=0}^{-1} = 1.$
Further extension

A certain stress improves the material at the beginning

$$X_i$$

$$\alpha_i$$  fatal

$$\alpha$$  harmful

$$\beta$$

$$\gamma$$  improving

$$i$$  $$\tau$$
Exact distribution and asymptotic results:

for $\tau, N_+(\tau), N_-(\tau)$ and $T_\tau$

where

$N_+(\tau)$ number of strengthening strokes

$N_-(\tau)$ number of harmful strokes

depending on conditions of $\alpha_i, \beta$ and $\gamma$.

Extensions:

Mixed models: Mixture of sum and extreme shock models
2. Urn model approach to break downs

Consider an urn containing balls of three different colors: black, blue and red or $x$, $y$, and $w$ each color represents a possible state of risk for the process: $x$-balls – safe state, $y$-balls – risky state and $w$-balls – default state.
Evolution of the process:

1. At time $n$ a ball is random sampled from the urn, with the content depending on the urn composition at time $n - 1$;

2. According to the color of the ball, the process $X_n = x, y$ or $w$;

3. The urn is then changed according to the reinforcement matrix.
The reinforcement matrix

To model the positive dependence between the risky and the default states, we choose a balanced matrix constant over time:

\[
RM = \begin{bmatrix}
    x & y & w \
    x & \theta & 0 & 0 \\
    y & 0 & \delta & \lambda \\
    w & 0 & 0 & \theta
\end{bmatrix}, \text{ where } \lambda = \theta - \delta
\]  

1. If an \( x \)-ball is sampled, \( \theta \) balls of type \( x \) are added;
2. if an \( y \)-ball is sampled, the urn is reinforced with \( \delta \) \( y \)-balls and \( \lambda \) \( w \)-balls (to model dependence);
3. if a \( w \)-ball is picked up, \( \theta \) balls of the same color are added.
Example of a simulated urn process

with $a_k = a_0 + k\theta$, $b_k = b_0 + k\delta$ and $c_k = c_0 + k\lambda$.

Content of the urn after the $n$th-drawing

$w$-balls: $c_0$ $c_0$ .. $c_1$ .. $c_2$ $c_3$ .. $c_4$ .. $c_5$ $c_5 + \theta$

$y$-balls: $b_0$ $b_0$ .. $b_1$ .. $b_2$ $b_3$ .. $b_4$ .. $b_5$ $b_5$

$x$-balls: $a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$ $a_7$ $a_7$ $a_8$ $a_9$ $a_9$ $a_{10}$ $a_{10}$ $a_{10}$
Assumptions:

Condition 1:
Let $\theta \delta \neq 0$, not to have degenerate cases.

Condition 2: Let $\lambda = \theta - \delta \geq 0$,
to model the positive dependence between $y$ and $w$ balls.

Theory for discrete-time balanced urn process
with a $3 \times 3$ reinforcement matrix

Relation (isomorphism) to ordinary differential equation system.
Generating function $H$ of the urn history.
Our model:

\[
\sum = \begin{cases} 
\dot{x} = x^{\theta+1} \\
\dot{y} = y^{\delta+1}w^\lambda \\
\dot{w} = w^{\theta+1}
\end{cases} \quad \text{with i.c.} \quad \begin{cases} 
x(0) = x_0 \\
y(0) = y_0 \\
w(0) = w_0
\end{cases} , \quad (2)
\]

simple integration for \( x \) and \( y \):

\[
x(t) = x_0(1 - \theta x_0^\theta t)^{-\frac{1}{\theta}} \quad (3)
\]

\[
w(t) = w_0(1 - \theta w_0^\theta t)^{-\frac{1}{\theta}} . \quad (4)
\]

Since \( \dot{y}y^{-\delta-1} = w^\lambda \), the solution is:

\[
y(t) = y_0(1 - y_0^\delta \left( w_0^{-\delta} - \left[ w_0(1 - \theta w_0^\theta t)^{-\frac{1}{\theta}} \right]^{-\delta} \right)^{-\frac{1}{\delta}} .
\]

Hence
**Proposition:**

Consider an urn process with a reinforcement matrix $RM$ as in 1, that satisfies **Conditions 1 and 2**, and with an initial composition $(a_0, b_0, c_0)$ of balls.

The 4-variables generating function of urn histories is:

$$H(z; x, y, w) = x^{a_0} y^{b_0} w^{c_0} (1 - \theta x^\theta z)^{-a_0 \theta} (1 - \theta w^\theta z)^{-c_0 \theta}$$

$$\times \left(1 - y^\delta w^{-\delta} \left(1 - (1 - \theta w^\theta z)^{\delta / \theta}\right)\right)^{-b_0 \delta}.$$
Proposition for the moments:

\( X_n, Y_n \) and \( W_n \): number of \( x, y \) and \( w \) balls in the urn at time \( n \).

Their moments: hypergeometric functions, finite linear combinations of product and quotients of Euler Gamma functions.

In particular:

\[
E [X_n] = \frac{a_0}{t_0} (t_0 + n\theta),
\]

\[
E [Y_n] = b_0 \frac{\Gamma \left( \frac{t_0}{\theta} \right)}{\Gamma \left( \frac{t_0 + \delta}{\theta} \right)} n^{\delta} + O(n^{\delta-1}),
\]

\[
E [W_n] = \left[ (t_0 - a_0) \frac{\lambda}{\theta} \right] \frac{\Gamma \left( \frac{t_0}{\theta} \right)}{\Gamma \left( \frac{t_0 + \lambda}{\theta} \right)} n^{\delta} + O(n^{\delta-1}),
\]

where \( t_0 = a_0 + b_0 + c_0 \) and \( \lambda = \theta - \delta \)
**Limit result:**

For any compact set $S$ of $\mathbb{R}^+$ and any $\gamma \in S$ such that $\gamma n^{\delta}$ is an integer, we have that

$$P\left[Y_n = b_0 + \delta \gamma n^{\delta}\right] = n^{-\delta \theta} g(\gamma) + O(n^{-2\delta \theta})$$

(5)

where the error term holds uniformly with respect to $\gamma \in S$.

**Function $g(\cdot)$ (gen. Mittag-Leffler) is defined on $\mathbb{R}^+$ by**

$$g(\gamma) = \frac{\Gamma\left(\frac{t_0}{\theta}\right)}{\Gamma\left(\frac{b_0}{\delta}\right)} \gamma^{\frac{b_0}{\delta} - 1} \sum_{k\geq 0} (-1)^k \frac{\gamma^k}{\Gamma(k + 1) \Gamma\left(\frac{c_0 - k\delta}{\theta}\right)}.$$

An analogous reasoning is valid for $W_n$ with a different $g(\gamma)$ or for $X_n$, as a standard Poly urn.
Remarks

1. For $c_0 = 0$, no balls in the initial composition, no immediate failing, the function $g(\gamma)$ represents a Paretian stable law of index $\frac{\delta}{\theta}$.
   So $Y_n$ has a power law, asympt., in accordance with the Zipf’s law in econometrics.

2. If $c_0 < \theta$, on the contrary, $g(\gamma)$ becomes a Gamma distribution (even an exponential for $b_0 = \delta$).
Joint limit distribution:

Consider the whole process $U_n = (X_n, Y_n, W_n)$ with $U_0 = (a_0, b_0, c_0)$

Then $U_n/(\theta n)$ converges to a random vector, depending on $(V_1, V_2, V_3)$ has a Dirichlet distribution, whose density on the simplex \((u_x \geq 0, u_y \geq 0, u_z \geq 0, u_x + u_y + u_w = 1)\) given by

\[
\frac{\Gamma\left(\frac{t_0}{\theta}\right) u_x^{a_0+c_0} u_y^{b_0-c_0} u_w^{1/\theta [\lambda c_0 - \delta a_0]}}{\Gamma(a_0 + c_0) \Gamma(b_0 - c_0) \Gamma\left(\frac{1}{\theta} [\lambda c_0 - \delta a_0]\right)},
\]

\((u_x: \text{proportion of } x \text{ balls in the urn})\)

and on the RM.
3. Firms default

Example of a simulated urn process

with \( a_k = a_0 + k\theta \), \( b_k = b_0 + k\delta \) and \( c_k = c_0 + k\lambda \).

Content of the urn after the \( n \)th-drawing

- **w-balls**: \( c_0 \quad c_0 \quad \ldots \quad c_1 \quad \ldots \quad c_2 \quad c_3 \quad \ldots \quad c_4 \quad c_5 \quad c_5 + \theta \)
- **y-balls**: \( b_0 \quad b_0 \quad \ldots \quad b_1 \quad \ldots \quad b_2 \quad b_3 \quad \ldots \quad b_4 \quad b_5 \quad b_5 \)
- **x-balls**: \( a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_7 \quad a_8 \quad a_9 \quad a_9 \quad a_{10} \quad a_{10} \quad a_{10} \quad \ldots \)
Data

data from the CEBI database: CEBI comprehensive database first developed by the Bank of Italy and now maintained by Centrale dei Bilanci Srl. biggest Italian industrial dataset, with firm-level observations and balance sheets of thousands of firms.

Subset of 380 manufacturing firms with the conditions:

1. All firms’ data: active in the period 1982-2000;

2. Every firm: more than 100 employees with reliable information about capital and financial ratios;

3. Under bank control for possible insolvency at least once.
Selected firms are comparable with those originally used by Altman (1968) (famous paper): the benchmark for our work.

For every firm: standard balance ratios:

$r_1$ : working capital / total assets
$r_2$ : retained earnings / total assets
$r_3$ : EBIT / total assets
$r_4$ : market value of equity / book value of total liabilities
$r_5$ : sales / total assets
$r_6$ : equity ratio
$r_7$ : debt ratio
Initialization of the process

We need initialized values and RM?

RM:

We simply set $\theta = 3$ and $\delta = 2$ for every firm

’Best fit’ by a simple grid search.

A first attempt

RM has good properties, essentially a Poly urn.
Initial composition of firms’ urns:

heuristic method, based on well-known stylized facts of industrial economics.

First $x$-balls: equity ratio $r_6$ as a proxy of the proportion of $x$-balls. Firm with $r_6 \geq 0.5$ can be considered as financially robust. Hence, for every firm set $a_0 = [r_6 \times 100]$.

Second: $y$ and $w$-balls: risky and the default states, combine debt ratio $r_7$ and complement with equity ratio $r_6$.

a higher debt ratio: signal of danger for firms’ reliability set $b_0 = [r_7 \times (1 - r_6) \times 100]$ and $c_0 = [(1 - r_6)(1 - r_7) \times 100]$.

So, initial number of balls in the urn: 100.
Examples:

**Initial urn composition for some firms of the dataset**

<table>
<thead>
<tr>
<th>firm code</th>
<th>year</th>
<th>equity ratio</th>
<th>debt ratio</th>
<th>(a_0)</th>
<th>(b_0)</th>
<th>(c_0)</th>
<th>default in (t + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IM223A</td>
<td>1982</td>
<td>0.42</td>
<td>0.71</td>
<td>42</td>
<td>41</td>
<td>17</td>
<td>0</td>
</tr>
<tr>
<td>IM298A</td>
<td>1982</td>
<td>0.62</td>
<td>0.52</td>
<td>62</td>
<td>20</td>
<td>18</td>
<td>0</td>
</tr>
<tr>
<td>IM567B</td>
<td>1982</td>
<td>0.68</td>
<td>0.37</td>
<td>68</td>
<td>12</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>IM1031B</td>
<td>1982</td>
<td>0.39</td>
<td>0.66</td>
<td>39</td>
<td>40</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>IM1988A</td>
<td>1982</td>
<td>0.72</td>
<td>0.57</td>
<td>72</td>
<td>16</td>
<td>12</td>
<td>0</td>
</tr>
</tbody>
</table>

**All firms together:**

average numbers of initialized \(x\), \(y\) and \(w\)−balls: 48, 39, 13
distributions for $a_0$, $b_0$ and $c_0$ of all the firms:
For every firm we can compute all the probabilities at an time $n$.

In this experiment assume that a firm fails at time $n + 1$ if the probability of extracting a $w$-ball is $\geq 0.20$ at time $n$, very common threshold

For every firm, in every period, we can prediction failure, compare it
– with actual data and
– with simple Altman’s ones.
Altman’s $Z$-score (1968) popular measure, based on discriminant analysis, to classify firms’ riskiness.

In particular, using Altman’s original formulation we have

$$Z = 0.012r_1 + 0.014r_2 + 0.033r_3 + 0.006r_4 + 0.999r_5.$$ 

According to this score, a firm is likely to default if $Z < 1.8$,
safe if $Z > 3$,
’gray’ otherwise.

Estimated the $Z$-score on CEBI data set to understand its general formulation on Italian data.
using standard regression techniques

\[ Z^* = 0.014r_1 + 0.013r_2 + 0.052r_3 + 0.007r_4 + 0.955r_5, \]

\[(Z = 0.012r_1 + 0.014r_2 + 0.033r_3 + 0.006r_4 + 0.999r_5).\]

Similar values

\(r_3\) EBIT larger, but not significant.

Use both \(Z\)-Scores, only small differences.

So Altman’s \(Z\) Scores used, traditional.
Results:

For 245 firms from 380, both methods correctly predict firms’ default.

For 72 firms from 380, our model (UGESM) seems to behave better.

Remaining 63 firms: both models do not predict default in the right way.
Comparison of the number of correctly predicted defaults for UGESM and Altman’s Z-score

<table>
<thead>
<tr>
<th></th>
<th>UGESM</th>
<th>Z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td>correct</td>
<td>83%</td>
<td>66%</td>
</tr>
<tr>
<td>no correct</td>
<td>17%</td>
<td>34%</td>
</tr>
</tbody>
</table>

Remaining cases:

Altman’s method usually underestimates the possibility of a failure, UGSEM seems to be more pessimistic: for 40 firms from 63 generally predict failure 2-3 periods before actual default.

Improve
A good result with such a simple model, more prudent behavior is required for banks and similar companies (e.g. Basel II).

**Remaining 63 cases:**
number of underestimated and overestimated defaults before and after actual failure with averages of number of wrong periods

<table>
<thead>
<tr>
<th></th>
<th>UGESM</th>
<th>Z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>underestimated</strong></td>
<td>63% (2.7)</td>
<td>28% (1.4)</td>
</tr>
<tr>
<td><strong>overestimated</strong></td>
<td>37% (3.2)</td>
<td>72% (2.3)</td>
</tr>
</tbody>
</table>
Distributions of time of default?

Kernel estimates using the Epanechnikov kernel

Kernel estimates of the df of the number of defaults over time