Geometry of Singular Symplectic Quotients

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Introduction: Symplectic Reduction

Let $(\mathcal{P}, \omega, G, \mathbf{J})$ be a Hamiltonian space, where

- (M, ω) is a symplectic manifold,
- $G \times \mathcal{P} \to \mathcal{P}$ is a smooth and proper Hamiltonian action, and
- $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ equivariant momentum map

$$\omega(\xi_{\mathcal{P}}, \cdot) = \mathbf{d}\langle \mathbf{J}, \xi \rangle, \quad \xi \in \mathfrak{g}.$$

Symplectic Reduction: If $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$ and G_μ acts freely on $\mathbf{J}^{-1}(\mu)$ then the quotient space

$$\mathcal{P}_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$$

is a smooth symplectic manifold. (Marsden and Weinstein, 1974)

The reduced symplectic form ω_μ on $\mathcal{P}_\mu$ is defined by

$$i_\mu^* \omega = \pi_\mu^* \omega_\mu$$

where

- $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow \mathcal{P}$ is the inclusion, and

- $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathcal{P}_\mu : \mathbf{J}^{-1}(\mu)/G_\mu$ is the projection.

If μ is not regular or G_μ does not act freely on $\mathbf{J}^{-1}(\mu)$ then $\mathcal{P}_\mu$ is a **symplectic stratified space** (it is partitioned in smooth symplectic manifolds with reduced symplectic forms like in the regular case).

Goal: Explain this symplectic stratification of $\mathcal{P}_\mu$ when μ is not regular (singular μ).

Reduction in Mechanics and Geometry

- Symmetric Hamiltonian dynamics: The components of $\mathbf{J}$ are conserved quantities (Theorem of Nöether), $\mathcal{P}_\mu$ is the space of symmetric equivalence classes of dynamical states with fixed momentum μ.
  The original dynamics on $\mathcal{P}$ can be dropped to $\mathcal{P}_\mu$ reducing the dimensionality of the problem. (for example N-body problem, $\mathcal{P} = T^*(\mathbb{R}^{3N})$, $G = \mathrm{SO}(3)$, $\mathbf{J}$=angular momentum).

- Coadjoint orbits: $\mathcal{P} = T^*G = G \times \mathfrak{g}^*$ with action $g \cdot (g', \nu) = (gg', \nu)$ and momentum $\mathbf{J}(g, \nu) = \mathrm{Ad}^*_{g^{-1}}\nu$. Then $\mathcal{P}_\mu = \mathcal{O}_\mu$ (coadjoint orbit through μ and ω_μ is the $(-)$ Konstant-Kirillov-Souriau form, i.e.

$$\omega_\mu(\lambda)(\mathrm{ad}_\xi\lambda, \mathrm{ad}_\eta\lambda) = -\langle \lambda, [\xi, \eta] \rangle,$$

  for $\lambda \in \mathcal{O}_\mu$, $\mathrm{ad}_\xi\lambda$, $\mathrm{ad}_\eta\lambda \in T_\lambda\mathcal{O}_\mu$, with $\xi, \eta \in \mathfrak{g}$.

- Moduli space of flat connections: K compact and $\zeta : K \to M \to \Sigma$ a principal bundle over a closed oriented surface Σ. The space $\mathcal{A}$ of connections of ζ has a symplectic form

$$\omega(A)(\alpha, \beta) = \int_{\Sigma} \kappa(\alpha \wedge \beta),$$

G^{ζ} acts on $\mathcal{A}$ by $g \cdot A = g^{-1}Ag + g^{-1}dg$. with momentum map $J(A) = F_A$. Then

$$\mathcal{P}_0 = \{A \in \mathcal{A} : F_A = 0\}/G^{\zeta}$$

has a reduced symplectic structure (Chern-Simons theory, low-dimensional topology).

- Toric manifolds: $\mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n$ as

$$(\theta_1, \ldots, \theta_n) \cdot (z_1, \ldots, z_n) = (e^{2\pi i \theta_n} z_1, \ldots, e^{2\pi i \theta_1} z_n).$$

$\mathbb{T}^k \hookrightarrow \mathbb{T}^n$ subtorus acting on $\mathbb{C}^n$ by restriction with momentum map $\mathbf{J} : \mathbb{C}^n \to \mathbb{R}^k$ corresponding to $\omega = \frac{i}{2} \Sigma_k dz_k \wedge d\overline{z_k}$.

$M = \mathbf{J}^{-1}(0)/G$ is a toric manifold for $\mathbb{T}^{n-k}$ (Delzant construction).

Bifurcation Lemma

Singular reduction starts with the **Bifurcation Lemma** (Arms, Marsden, Gotay 1981):

$$\text{range}\,(T_z\mathbf{J}) = (\mathfrak{g}_z)^\circ.$$

In other words: μ is a singular value of $\mathbf{J}$ iff $\mathbf{J}^{-1}(\mu)$ contains a point with continuous stabilizer.

The study of singularities of the momentum map is equivalent to the study of singularities of the Hamiltonian group action on $\mathcal{P}$.

Slice Theorem

Associated Bundle: Let $H \subset G$ compact act on a vector space A. H acts on $G \times A$ by

$$h \cdot (g, a) = (gh^{-1}, h \cdot a)$$

We denote the quotient space as

$$G \times_H A := (G \times A)/H.$$

- $G \times_H A$ is an associated bundle to $G \to G/H$ over G/H with fiber A.

- G acts on $G \times_H A$ by $g' \cdot [g, a] = [g'g, a]$.

- **Slice Theorem:** $G \times M \to M$ proper action. $x \in M$, $\mathbf{S} = T_x M / \mathfrak{g} \cdot x$. Then

  $$\phi : G \times_{G_x} \mathbf{S} \to M$$

  is an equivariant tubular neighborhood of $G \cdot x$ (Palais 1961).

Symplectic Slice Theorem

$(\mathcal{P}, \omega, G, \mathbf{J})$ Hamiltonian G-space, $\mathbf{J}(z) = \mu$.

- $N = \ker T_z\mathbf{J}/\mathfrak{g}_\mu \cdot z$ (**symplectic normal space**). $(N, \omega|_N, H, \mathbf{J}_N)$ Hamiltonian linear H-space,
$$\langle \mathbf{J}_N(v), \xi \rangle = \frac{1}{2}\omega_N(\xi \cdot v, v).$$

- $\phi : Y := G \times_{G_z} ((\mathfrak{g}_\mu/\mathfrak{g}_z)^* \oplus N) \to \mathcal{P}$. ϕ is a G-equivariant symplectomorphism with respect to a natural symplectic form ω_Y.

- (Marle 1985, Guillemin and Sternberg 1984) $(Y, \omega_Y, G, \mathbf{J}_Y)$ is a Hamiltonian G-space with
$$\mathbf{J}_Y([g, \nu, v]) = \mathrm{Ad}^*_{g^{-1}}(\mu + \nu + \mathbf{J}_N(v)).$$

- **Lerman-Bates Lemma** (1997): There exists a neighborhood $Y_0 \subset Y$ such that
$$\mathbf{J}_Y^{-1}(\mu) \cap Y_0 = \left(G_\mu \times_{G_z} (0 \times \mathbf{J}_N^{-1}(0))\right) \cap Y_0.$$

Stratified Spaces

X topological space. A locally finite disjoint parition $X = \coprod_i X_i$ is a **stratification** of X if

- **smoothness:** X_i are smooth manifolds,
- **frontier condition:** $X_i \cap \overline{X_j} \neq \varnothing \Rightarrow X_i \subseteq \partial X_j$ ($\partial X_j = \overline{X_j} \backslash X_j$).

Application: $G \times M \to M$ proper action. Then

$$M/G = \coprod_{(H)} M_{(H)}/G, \quad \text{where}$$

- (H) is the conjugacy class of H in G, and
- $M_{(H)} = \{x \in M \ : \ G_x \in (H)\}$ (**orbit type**).

Why?$\longrightarrow$ use slices: near $G \cdot x$ with $G_x = H$, $M \simeq G \times_H \mathbf{S} \simeq G/H \times \mathbf{S}$. Then

$$M_{(H)} \simeq (G \times_H \mathbf{S})_{(H)} = G \times_H \mathbf{S}^H \simeq G/H \times \mathbf{S}^H$$

$\Rightarrow M_{(H)}$ is a G-submanifold of M.

1. **smoothness:** Near $[x] \in M_{(H)}/G$,

$$M_{(H)}/G \simeq (G \times_H \mathbf{S}^H)/G = \mathbf{S}^H/H = \mathbf{S}^H \simeq \mathbb{R}^k.$$

$\Rightarrow M_{(H)}/G$ is a smooth manifold.

2. **frontier conditions:** Analogously,

$$M_{(H)}/G \subseteq \partial(M_{(K)}/G) \Leftrightarrow (K) < (H).$$

(**isotropy stratification of** M/G)

Strategy to study the symplectic stratification: repeat this for a Hamiltonian G-space using the Symplectic Slice Theorem instead.

Symplectic Stratification of $\mathcal{P}_0$

$(\mathcal{P}, \omega, G, \mathbf{J})$ Hamiltonian G-space. Suppose 0 is a singular value of $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$. Then $\mathbf{J}^{-1}(0)$ and $\mathcal{P}_0 = \mathbf{J}^{-1}(0)/G$ are singular spaces.

Theorem: (Sjamaar, Lerman 1991).

(i) The sets $\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)}$ and $(\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)})/G$ are smooth manifolds, and

$$\mathcal{P}_0 = \coprod_{(H)} \left(\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)}\right)/G$$

is a stratification of $\mathcal{P}_0$.

(ii) Each stratum $\mathcal{P}_0^{(H)} := (\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)})/G$ is symplectic with a reduced symplectic form $\omega_0^{(H)}$ defined by

$$i_0^{(H)}\omega = \pi_0^{(H)*}\omega_0^{(H)}, \quad \text{where}$$

- $i_0^{(H)} : \mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)} \hookrightarrow \mathcal{P}$ and
- $\pi_0^{(H)} : \mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)} \to \mathcal{P}_0^{(H)}$.

- Sketch of proof of (i): $z \in \mathcal{P}$ with $G_z = H$. Using Lerman-Bates Lemma, near $G \cdot z$

$$\mathbf{J}^{-1}(0) \simeq \mathbf{J}_{Y_0}^{-1}(0) = G \times_H (0 \times \mathbf{J}_N^{-1}(0)).$$

- $N^H \subseteq \mathbf{J}_N^{-1}(0)$ $(\langle \mathbf{J}_N(v), \xi \rangle = \frac{1}{2}\omega_N(\xi \cdot v, v))$

- Then $\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H)}$ is a manifold:

$$\begin{aligned} \mathbf{J}_{Y_0}^{-1}(0) \cap (Y_0)_{(H)} &= G \times_H (0 \times (\mathbf{J}_N^{-1}(0))^H) \\ &= G \times_H (0 \times N^H) \\ &\simeq G/H \times (0 \times N^H) \\ &\subseteq G/H \times ((\mathfrak{g}/\mathfrak{g}_z)^* \oplus N) \simeq Y_0 \end{aligned}$$

(i) **smoothness:** $\mathcal{P}_0^{(H)}$ is a manifold.

$$(\mathbf{J}_{Y_0}^{-1}(0) \cap (Y_0)_{(H)})/G = N^H/H = N^H \simeq \mathbb{R}^k$$

(ii) **frontier conditions:** follow from frontier conditions for $\mathcal{P}/G$ since $\mathcal{P}_0^{(H)} \subseteq \mathcal{P}_{(H)}/G$.

$$\mathcal{P}_0^{(H)} \subseteq \partial \mathcal{P}_0^{(K)} \Leftrightarrow (K) < (H).$$

- Sketch of proof of (ii): **Sjamaar Principle:** $\mathcal{P}_H$ is a symplectic submanifold of $\mathcal{P}$. $N(H)/H$ acts FREELY and Hamiltonially on $(\mathcal{P}_H, \omega|_{\mathcal{P}_H})$ with momentum map $\mathbf{J}_{\mathcal{P}_H}$. Then there is a diffeomorphism

$$f : \mathcal{P}_0^{(H)} \to \mathbf{J}_{\mathcal{P}_H}^{-1}(0)/(N(H)/H).$$

(Sjamaar, Lerman 1991).

Then $\mathbf{J}_{\mathcal{P}_H}^{-1}(0)/(N(H)/H)$ is a Marsden-Weinstein reduced manifold with reduced symplectic form Ω. Then pull-back

$$\omega_0^{(H)} := f^*\Omega$$

satisfies the requirements of the Sjamaar-Lerman Theorem:

$$i_0^{(H)}\omega = \pi_0^{(H)^*}\omega_0^{(H)}.$$

Cotangent Lifted Actions

- Q smooth manifold, $(\tau : T^*Q \to Q, \omega_Q)$ is canonically a symplectic manifold:
  for $p_x \in T^*_x Q$, $V \in T_{p_x}(T^*Q)$,

  $$\Theta_Q(p_x)(V) = \langle p_x, T_{p_x}\tau(V)\rangle, \quad \omega_Q = -\mathbf{d}\Theta_Q.$$

- $G \times Q \to Q$ base action $\Rightarrow$ $G \times T^*Q \to T^*Q$ lifted action. **A lifted action is always Hamiltonian**.

- If $G \times Q \to Q$ is free, proper, then
  $G \times T^*Q \to T^*Q$ is also free, proper.

- Momentum map $\langle \mathbf{J}(p_x), \xi\rangle = \langle p_x, \xi_Q(x)\rangle$.

Regular Cotangent Bundle Reduction

$G \times Q \to Q$ free and proper action. Then every momentum value is regular. How are the Marsden-Weinstein reduced spaces?$\rightarrow$ **They are bundles:**

- ($\mu = 0$): There is a symplectomorphism $(\mathbf{J}^{-1}(0)/G, \omega_0) \to (T^*(Q/G), \omega_{Q/G})$ (Satzer 1977).

- ($\mu \neq 0$): There is a symplectic embedding

  $(\mathbf{J}^{-1}(\mu)/G, \omega_\mu) \to (T^*(Q/G_\mu), \omega_{Q/G_\mu} - \tau^* B_\mu)$

  onto a subbundle of $T^*(Q/G_\mu)$.
  B_μ is a closed differential 2-form on Q/G_μ obtained from a principal connection on

  $$G_\mu \to Q \to Q/G_\mu.$$

  (Abraham, Marsden 1978).

Singular Cotangent Bundle Reduction

Motivation: $G \times Q \to Q$ not free $\Rightarrow 0 \in \mathfrak{g}^*$ singular momentum value: The smooth cotangent bundle projection $\tau : T^*Q \to Q$ induces a continuous projection $\tau_0 : \mathcal{P}_0 \to Q/G$.

- In the regular case, $\mathcal{P}_0 = T^*(Q/G)$ and τ_0 is a smooth fibration (the cotangent bundle projection $\tau_0 : T^*(Q/G) \to Q/G$).

- Everything is constructible from $G \times Q \to Q$.

- In the singular case we expect τ_0 to be a **stratified fibration** (maps strata to strata and restricts to smooth fibrations). This FAILS! since $\tau_0(\mathcal{P}_0^{(H)}) = \overline{Q_{(H)}/G} \neq Q_{(H)}/G$.

Solution: Substitute the symplectic stratification of $\mathcal{P}_0$ with the finer **coisotropic stratification**.

Seams

Consider one orbit type submanifold $Q_{(H)} \subset Q$. $(T^*Q_{(H)}, \omega_{Q_{(H)}}, G, \mathbf{J}_{(H)})$ is a Hamiltonian G-space obtained by restriction from $(T^*Q, \omega_Q, G, \mathbf{J})$.

$N^*Q_{(H)} \subset T^*_{Q_{(H)}}Q$ conormal bundle to $Q_{(H)}$, inherits a G-action. **Facts:**

- $(N^*Q_{(H)})_{(K)} \neq \varnothing \Leftrightarrow$
$$Q_{(K)} \neq \varnothing \quad \text{and} \quad (K) \leq (H).$$

- $S_{H\to K} := \frac{\mathbf{J}^{-1}_{(H)}(0)\times(N^*Q_{(H)})_{(K)}}{G} \to Q_{(H)}/G$ is a smooth bundle.

- $S_{H\to H} = T^*(Q_{(H)}/G)$ (Emmrich-Romer 1991).

We call $S_{H\to K}$ with $(K) < (H)$ a **seam**.

Decomposition of the Symplectic Strata

In the cotangent bundle case we can write the following decomposition of every symplectic stratum:

$$\mathcal{P}_0^{(K)} = T^*(Q_{(K)}/G) \coprod_{(K)<(H)} S_{H\to K}$$

Furthermore:

- $\mathcal{P}_0^{(K)} \neq \varnothing \Leftrightarrow Q_{(K)} \neq \varnothing$.
- $T^*(Q_{(K)}/G)$ is open and dense in $\mathcal{P}_0^{(K)}$.
- The reduced symplectic form $\omega_0^{(K)}$ is the unique extension of $\omega_{Q_{(K)}/G}$ from $T^*(Q_{(K)}/G)$ to $\mathcal{P}_0^{(K)}$.
- Seams $S_{H\to K}$ are coisotropic in $(\mathcal{P}_0^{(K)}, \omega_0^{(K)})$.

The Coisotropic Stratification of $\mathcal{P}_0$

Let $I_Q = \{(H) : Q_{(H)} \neq \varnothing\}$. Take every cotangent bundle and seam of the form

- $T^*(Q_{(L)}/G), \quad (L) \in I_Q$,
- $S_{K' \to K}, \quad (K), (K') \in I_Q, (K) < (K')$.

then

$$\mathcal{P}_0 = \coprod_{(L)} T^*(Q_{(L)}/G) \coprod_{(K)<(K')} S_{K \to K'}$$

with $(L), (K), (K') \in I_Q$ is a stratification of $\mathcal{P}_0$ (Perlmutter, Sousa-Dias, R-O 2003).

Notice: The strata are bundles over strata of Q/G, indeed

$$\begin{array}{rcl} T^*(Q_{(L)}/G) & \longrightarrow & Q_{(L)}/G \\ S_{K \to K'} & \longrightarrow & Q_{(K)}/G. \end{array}$$

Properties of the Coisotropic Stratification

- The continuous projection $\tau_0 : \mathcal{P}_0 \to Q/G$ IS a stratified fibration with respect to the secondary stratification of $\mathcal{P}_0$ and the isotropy stratification of Q/G.

- The frontier conditions: (**gluing cotangent bundles**):

$$
\begin{aligned}
T^*(Q_{(K)}/G) \subset \partial\, T^*(Q_{(H)}/G) &\Leftrightarrow (H) < (K) \\
T^*(Q_{(K)}/G) \subset \partial\, S_{K\to H} &\Leftrightarrow (H) < (K) \\
S_{K\to H} \subset \partial\, T^*(Q_{(H)}/G) &\Leftrightarrow (H) < (K) \\
S_{K'\to H} \subset \partial\, S_{K\to H} &\Leftrightarrow (H) < (K) < (K') \\
S_{K\to H'} \subset \partial\, S_{K\to H} &\Leftrightarrow (H) < (H') < (K)
\end{aligned}
$$

- The strata are coisotropic submaifolds of their respective symplectic strata.