FEATURE ARTICLE

Towards categorical behaviour of groups

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Abstract

We present a brief introduction to the study of homological categories, which encompass many algebraically-like categories, namely the category of groups, and also categories of topological algebras.

1. Introduction

Category Theory started more than 60 years ago, when Eilenberg and Mac Lane wrote a paper on natural transformations [11], having as role model a well-known example of natural equivalence:

For any vector space V over a field K, consider its dual V^* (i.e. the vector space of linear functionals from V to K). If V is finite-dimensional, then V^* has the same dimension as V, and so we can conclude that they are isomorphic, although there is no *natural* way of defining such isomorphism. This contrasts with the case of V^{**}, the dual of V^*. There is a (injective) linear transformation $\phi_V : V \to V^{**}$, which assigns to each $x \in V$ the linear transformation $\hat{x} : V^* \to K$, $f \mapsto \hat{x}(f) := f(x)$. Moreover, if V is finite-dimensional, ϕ_V turns out to be an isomorphism, defining this way a *natural equivalence*. That is, ϕ_V is not a mere equivalence between V and V^{**} but it is part of a collection $\phi = (\phi_V)_V$ of equivalences. To make this idea precise, in [11] Eilenberg and Mac Lane defined *categories*, *functors* (between categories) and then *natural transformations* (between functors).

Shortly after, the use of Category Theory proved to be useful in several areas of Mathematics. The notion of *abelian category* – encompassing abelian groups and, more generally, modules – became prominent. Quoting Mac Lane [20, Notes on Abelian Categories, page 209]:

"Shortly after the discovery of categories, Eilenberg and Steenrod [12] showed how the language of categories and functors could be used to give an axiomatic description of the homology and cohomology of a topological space. This, in turn, suggested the problem of describing those categories in which the values of such a homology theory could lie. After discussions with Eilenberg, this was done by Mac Lane [18, 19]. His notion of an "abelian bicategory" was clumsy, and the subject languished until Buchsbaum's axiomatic study [10] and the discovery by Grothendieck [15] that categories of sheaves (of abelian groups) over a topological space were abelian categories but not categories of modules, and that homological algebra in these categories was needed for a complete treatment of sheaf cohomology (Godement [14]). With this impetus, abelian categories joined the establishment."

Moreover, quoting now Borceux [2]:

An elementary introduction to the theory of abelian categories culminates generally with the proof of the basic diagram lemmas of homological algebra: the five lemma, the nine lemma, the snake lemma, and so on. This gives evidence of the power of the theory, but leaves the reader with the misleading impression that abelian categories constitute the most natural and general context where these results hold. This is indeed misleading, since all those lemmas are valid as well – for example – in the category of all groups, which is highly non-abelian.

However, in contrast with the smooth genesis of abelian categories, besides several attempts to identify relevant categorical features of the category of groups (cf. [16] for an account on the subject), it took a few decades until the right ingredient was identified. This was due to Bourn [6], who defined *protomodular category* and showed that a simple categorical condition (see condition (2) of Theorem 1) could become a key tool to handle with short exact sequences. In 1990, he presented his notion at the International Category Theory Meeting, CT90, but it took almost a decade until the mathematical community understood the potential of protomodularity. Indeed, the proposal, at CT99 (in Coimbra), of studying *semi-abelian categories* – which are in particular protomodular categories –, due to Janelidze, Márki and Tholen [16], was the main step for the recognition of the role of protomodularity in Categorical Algebra. The XXI century began with an explosion of results on protomodular and semi-abelian categories, being Bourn the main contributor. The monograph by Borceux and Bourn [3] contains the main achievements on the subject. While writing this monograph, the work of Borceux (together with the author of this article) on topological semi-abelian algebras [4, 5] led him to a new proposal for capturing the essential properties of group-like categories, which became known under the name *homological category*, basically because it turned out to be the right setting to develop Homological Algebra.

Throughout we will present a brief survey on these contributions.

2. Protomodularity

One of the key tools for Homological Algebra is the Short Five Lemma, which holds in abelian categories:

Short Five Lemma. *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{u} & X & \xrightarrow{p} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & K' & \xrightarrow{v} & X' & \xrightarrow{q} & Y' & \longrightarrow & 0 \end{array}$$

with exact rows (i.e. p, q are regular epimorphisms and $u = \ker p$, $v = \ker q$), if a and c are isomorphisms, b is an isomorphism as well.

This result is still valid in the category $\mathscr{G}rp$ of groups and homomorphisms, hence it is not exclusive of abelian categories. The notion of protomodular category is based on a weaker form of this result, the Split Short Five Lemma, stated below. This statement makes sense only in *pointed categories*, that is categories with a zero object.

Definition. In a pointed category $\mathscr{C}$, the *Split Short Five Lemma* holds if, for any given commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{u} & X & \underset{p}{\overset{s}{\leftrightarrows}} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & K' & \xrightarrow{v} & X' & \underset{q}{\overset{t}{\leftrightarrows}} & Y' & \longrightarrow & 0 \end{array}$$

in the sense that $b \cdot u = v \cdot a$, $c \cdot p = q \cdot b$ and $b \cdot s = t \cdot c$, with p, q split epimorphisms, $p \cdot s = 1_Y$ and $q \cdot t = 1_{Y'}$, and $u = \ker p$, $v = \ker q$, if a and c are isomorphisms, b is an isomorphism as well.

Bourn observed that the Split Short Five Lemma holds in a pointed category $\mathscr{C}$ with pullbacks of split epimorphisms if and only if the *kernel functor*

$$\begin{array}{rcl} K : Pt\mathscr{C} & \longrightarrow & \mathscr{C}/0 \times \mathscr{C} \\ (\, X \underset{s}{\overset{f}{\rightleftarrows}} Y \,) & \longmapsto & (\mathrm{Ker}\, f \to 0,\ X) \end{array}$$

is *conservative*, i.e. reflects isomorphisms. Here $Pt\mathscr{C}$ is the category of split epimorphisms – or *pointed objects* – of $\mathscr{C}$, i.e. pairs (f, s) with $f \cdot s = 1$; a morphism $(f, s) \to (f', s')$ in $Pt\mathscr{C}$ is a pair of morphisms of $\mathscr{C}$ (h, k) making the following diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ Y & \xrightarrow[k]{} & Y' \end{array}$$

commute (that is $k \cdot f = f' \cdot h$ and $s' \cdot k = h \cdot s$).

To avoid the assumption of $\mathscr{C}$ being pointed, one can focus on the second component of this functor, that is, on the functor which assigns to each object (f, s) of $Pt\mathscr{C}$ the codomain of f (=domain of s):

$$\begin{array}{rcl} p : Pt\mathscr{C} & \longrightarrow & \mathscr{C} \\ (f, s) & \longmapsto & \mathrm{cod}\, f, \end{array}$$

which is a fibration, the so-called *fibration of pointed objects* of $\mathscr{C}$.

If $\mathscr{C}$ has split pullbacks, every morphism $v : X \to Y$ in $\mathscr{C}$ induces, via pullback, the change-of-base functor

$$v^* : Pt_Y\mathscr{C} \longrightarrow Pt_X\mathscr{C}.$$

Proposition 1. [6] *Let $\mathscr{C}$ be a category with split pullbacks. If $\mathscr{C}$ has split pushouts (i.e. admits pushouts of split monomorphisms), then the change-of-base functors of the fibration p have left adjoints (i.e. p is also a cofibration). Conversely, if p is a cofibration and $\mathscr{C}$ admits finite products, then $\mathscr{C}$ has split pushouts.*

The remarkable novelty of Bourn's protomodularity is the recognition of the role played by these functors:

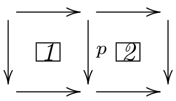
Definition. [6] A category $\mathscr{C}$ with split pullbacks is *protomodular* if the change-of-base functors of the fibration $p : Pt\mathscr{C} \to \mathscr{C}$ are conservative.

Protomodularity can be stated alternatively as a very simple condition on pullbacks.

Theorem 1. [6] *A category $\mathscr{C}$ is protomodular if and only if:*

(1) *$\mathscr{C}$ has split pullbacks;*

(2) *If in the commutative diagram*



the down-arrows are split epimorphisms and $\boxed{1}$ *and* $\boxed{1}\boxed{2}$ *are pullbacks, then* $\boxed{2}$ *is also a pullback.*

Moreover, in (2) one can assume only that p is a split epimorphism, provided that $\mathscr{C}$ has pullbacks.

Theorem 2. *Let $\mathscr{C}$ be a pointed category with split pullbacks. The following conditions are equivalent:*

(i) *$\mathscr{C}$ is protomodular;*

(ii) *The Split Short Five Lemma holds in $\mathscr{C}$.*

Examples. ([3])

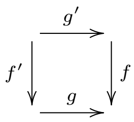
1. Every additive category with finite limits is protomodular, hence every abelian category is protomodular.
2. $\mathscr{G}rp$ is protomodular.
3. The dual category of an elementary topos is protomodular; in particular, $\mathscr{S}et^{\mathrm{op}}$ is protomodular.
4. If $\mathscr{C}$ is protomodular and $X \in \mathscr{C}$, then the slice category $\mathscr{C}/X$ and the coslice category $X\backslash\mathscr{C}$ are protomodular.
5. If $\mathscr{C}$ is protomodular, all the fibres $Pt_Y(\mathscr{C}) = p^{-1}(Y)$ of the fibration of points of $\mathscr{C}$ are protomodular.
6. If $\mathscr{C}$ is protomodular and finitely complete, then its category $\mathscr{G}rd(\mathscr{C})$ of internal groupoids is protomodular as well.

In Section 4 we will present a characterization of the protomodular varieties, i.e. of the varieties of universal algebras which are protomodular, as categories.

As for $\mathscr{G}rp$, monomorphisms in pointed protomodular categories do not need to be kernels. They behave however quite nicely. We select here some of the properties of monomorphisms and regular epimorphisms in protomodular categories.

Proposition 2. *Let $\mathscr{C}$ be a finitely complete protomodular category.*

(1) *Pulling back reflects monomorphisms, i.e. given a pullback*



f is a monomorphism provided that f' is.

(2) *If $\mathscr{C}$ is pointed, then:*

(a) *f is a monomorphism* $\Leftrightarrow$ $\mathrm{Ker}\, f = 0$*;*

(b) *f is a regular epi* $\Leftrightarrow$ $f = \mathrm{coker}(\ker f)$.

There is an interesting approach to normal subobjects in general protomodular categories that we will not describe here (cf. [3, 7]).

3. Semi-abelian categories

An *abelian category* is an additive category, with kernels and cokernels, and such that every monomorphism is a kernel and every epimorphism is a cokernel. We recall that an *additive category* is a *pointed* category *with biproducts* (i.e. finite products are biproducts, hence also coproducts) and with an additive abelian group structure in each hom-set so that composition of arrows is bilinear with respect to this addition (cf. [20, 13]).

Alternatively, an abelian category can be defined by the following two axioms:

(1) $\mathscr{C}$ has finite products, and a zero object,

(2) $\mathscr{C}$ has (normal epi, normal mono)-factorizations, i.e. every morphism factors into a cokernel followed by a kernel.

Observing that these two conditions imply that $\mathscr{C}$ is finitely complete and finitely cocomplete, so that condition (1) could be stated self-dually, and that condition (2) is obviously self-dual, it is clear that the notion of abelian category is self-dual, that is:

$$\mathscr{C} \text{ is abelian } \Leftrightarrow \mathscr{C}^{\mathrm{op}} \text{ is abelian.}$$

Roughly speaking, to define semi-abelianess Janelidze, Márki and Tholen replaced additivity by protomodularity, in a convenient way. The bridge they used was *Barr-exactness.*

We recall that a category $\mathscr{C}$ is *Barr-exact* [1] if

(1) $\mathscr{C}$ has finite products,

(2) $\mathscr{C}$ has pullback-stable (regular epi, mono)-factorizations,

(3) Every equivalence relation is effective (i.e. the kernel pair of some morphism).

If $\mathscr{C}$ satisfies (1) and (2) it is called *regular*.

A category is abelian if, and only if, it is additive and Barr-exact. Barr-exactness by itself is not restrictive enough to capture essential properties of group-like categories. It includes, for instance, pointed sets and monoids.

The authors of [16] observed that categories which are both Barr-exact and protomodular – including of course $\mathscr{G}rp$ – are very well-behaved. Namely, for a Barr-exact category with split pushouts, protomodularity is equivalent to the existence of semi-direct products (cf. [8]). They proposed the following:

Definition. A category $\mathscr{C}$ is *semi-abelian* if it is pointed, protomodular and Barr-exact.

It is interesting to notice that, although to be semi-abelian is not self-dual,

$$\mathscr{C} \text{ is abelian } \Leftrightarrow \mathscr{C} \text{ and } \mathscr{C}^{\mathrm{op}} \text{ are semi-abelian.}$$

This result follows essentially from the following

Proposition 3. *If $\mathscr{C}$ is pointed and both $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are protomodular, then $\mathscr{C}$ has biproducts.*

In semi-abelian categories many algebraic results are valid, specially results involving the behaviour of exact sequences (see Section 5). We refer to [16] for a very interesting incursion into this subject, and to [3, 2] for a thorough study of the properties of semi-abelian categories.

4. Semi-abelian varieties and topological algebras

The varieties of groups, of loops (or more generally semi-loops), of cartesian closed (distributive) lattices, of locally boolean distributive lattices, are varieties of universal algebras which are semi-abelian categories. In fact, for a variety, to be pointed protomodular is equivalent to be semi-abelian, since it fulfils always the exactness condition.

In 2003 Bourn and Janelidze [9] characterized semi-abelian (in fact protomodular) varieties as those having a finite family of generalized "subtractions" and a generalized "addition", as stated below.

Theorem 3. *A variety $\mathscr{V}$ of universal algebras is protomodular if and only if, for a given $n \in \mathbb{N}$, it has:*

(1) *n 0-ary terms $e_1, \cdots, e_n$;*

(2) *n binary terms $\alpha_1, \cdots, \alpha_n$ with $\alpha_i(x, x) = e_i$ for all $i = 1, \cdots, n$;*

(3) *one $(n+1)$-ary term θ satisfying*

$$\theta(\alpha_1(x, y), \cdots, \alpha_n(x, y), y) = x.$$

A variety $\mathscr{V}$ is semi-abelian if, and only if, it fulfils conditions (1)-(3) with $e_1 = \cdots = e_n = 0$.

In case $\mathscr{V}$ is the variety of groups, in the Theorem we put $n = 1$, $\alpha(x, y) = x - y$ and $\theta(x, y) = x + y$. It is clear that any variety which contains a unique constant and a group operation is semi-abelian. This is in particular the case of groups, abelian groups, Ω-groups, modules on a ring, rings or algebras without units, Lie algebras, Jordan algebras. Any semi-abelian variety has a Mal'cev operation p, defined as

$$p(x, y, z) = \theta(x, \alpha_1(y, z), \cdots, \alpha_n(y, z)).$$

(For more examples, see [4].) It is particularly interesting to study the corresponding topological algebras.

Let $\mathscr{C}$ be the category of topological algebras for a given semi-abelian variety $\mathscr{V}$. That is, objects of $\mathscr{C}$ are elements of $\mathscr{V}$ equipped with a topology making the operations continuous, and morphisms of $\mathscr{C}$ are continuous homomorphisms. Our basic example is of course the category $\mathscr{T}op\mathscr{G}rp$ of topological groups and continuous group homomorphisms. The main ingredient in the study of classical properties of topological groups is the existence of the homeomorphisms

$$G \xrightarrow{(-)+x} G$$

($x \in G$) which, although not living in $\mathscr{T}op\mathscr{G}rp$ (they are not homomorphisms), show that – topologically – G is homogeneous, i.e. its local properties do not depend on the point x considered. For topological semi-abelian algebras one replaces this set of homeomorphisms by a set of sections and retractions, as follows.

Let $A \in \mathscr{C}$. Condition (3) of the Theorem asserts that, for each $a \in A$, the continuous maps

$$\begin{array}{rcl} \iota_a : A & \longrightarrow & A^n \\ x & \longmapsto & (\alpha_1(x, a), \cdots, \alpha_n(x, a)) \end{array}$$

and

$$\begin{array}{rcl} \theta_A : A^n & \longrightarrow & A \\ (x_1, \cdots, x_n) & \longmapsto & \theta(x_1, \cdots, x_n, a) \end{array}$$

satisfy $\theta_a \cdot \iota_a = 1_A$, hence present A as a topological retract of A^n. Condition (2) says that $\iota_a(a) = (0, \cdots, 0)$, which allows the comparison between local properties at a and at 0. Indeed, from these properties one may conclude that, *for any $a \in A$, each of the sets*

$$\{\iota_a^{-1}(U \times \cdots \times U) \mid U \text{ open neighbourhood of } 0\}$$

and

$$\{\theta_a(U \times \cdots \times U) \mid U \text{ neighbourhood of } 0\}$$

is a fundamental system of neighbourhoods of a, the former one consisting of open neighbourhoods.

A convenient use of these properties guides us straightforward to the establishment of most of the classical topological properties known for topological groups. (For details see [4, 5].) Here we would like to mention one important property, which in fact follows directly from the existence of a Mal'cev operation:

In $\mathscr{C}$ a morphism is a regular epimorphism if and only if it is an open surjection.

We remark that a well-known important property of topological groups, namely that the profinite topological groups – i.e. projective limits of finite discrete topological groups – are exactly the compact and totally disconnected groups, in general is not true for semi-abelian topological varieties. For instance, the result fails for topological Ω-groups. It remains an open problem to characterize those varieties for which this equality holds. (Cf. Johnstone [17, Chapter 6] for more results on the subject.)

Analyzing now the categorical behaviour of such categories of topological semi-abelian algebras it is easy to check that *protomodularity is inherited from protomodularity of $\mathscr{V}$, but not exactness, hence they are not semi-abelian.* Indeed, the kernel pair of a continuous homomorphism $f : G \to H$ between, say, topological groups, is constructed like in $\mathscr{G}rp$ and it inherits the subspace topology of the product topology on $G \times G$. Hence, any equivalence relation on G provided with a topology which is strictly finer than the subspace topology of $G \times G$ is not a kernel pair, hence equivalence relations are not effective.

However, regularity is guaranteed, since the (regular epi, mono)-factorization of a morphism $f : A \to B$ is obtained via the (regular epi, mono)-factorization in $\mathscr{V}$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle e}\searrow \qquad \nearrow{\scriptstyle m} & \\ & M & \end{array}$$

equipping M with the quotient topology, which makes e necessarily an open map. Since open maps are pullback-stable, the factorization is pullback-stable as claimed.

At this stage one can raise the question: Are pointed regular protomodular categories interesting? The answer is definitely yes. These are the *homological categories* we will consider in the next section.

5. Homological categories

A category $\mathscr{C}$ is said to be *homological* if it is pointed, regular and protomodular.

Every semi-abelian category is homological, but there are interesting homological categories which are not semi-abelian, like $\mathscr{T}op\mathscr{G}rp$, and, more generally, any category of topological semi-abelian algebras.

As in any pointed category, in a homological category a sequence of morphisms

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0$$

is a *short exact sequence* if $k = \ker f$ and $f = \operatorname{coker} k$. Since in a pointed protomodular category every regular epimorphism is the cokernel of its kernel, in a homological category $0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0$ *is a short exact sequence if, and only if, $k = \ker f$ and f is a regular epimorphism.*

Furthermore, in a homological category the (regular epi, mono)-factorization of a morphism is obtained like in abelian categories, i.e. *if $f = m \cdot e$ is the (regular epi, mono)-factorization of f, then $e = \operatorname{coker}(\ker f)$.* Hence *every kernel has a cokernel* and, moreover, *every kernel is the kernel of its cokernel.*

Using (regular epi, mono)-factorizations, one can define exact sequences as follows.

Definitions. (1) In a homological category a *sequence of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is *exact* if, in the (regular epi, mono)-factorizations of f and g, $m = \ker e'$:

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle m} & & {\scriptstyle e'}\searrow \quad \nearrow{\scriptstyle m'} & \\ & M & & M' & \end{array}$$

(2) A *long exact sequence* of composable morphisms is *exact* if each pair of consecutive morphisms forms an exact sequence.

In a homological category a morphism $f : X \to Y$ can be part of an exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ only if, in its (regular epi, mono)-factorization $f = m \cdot e$, m is a kernel. (Such morphisms are called *proper*.) Still, in a homological category exact sequences identify monomorphisms and regular epimorphisms as follows.

Proposition 4. [3] *If $\mathscr{C}$ is a homological category and $f : X \to Y$ is a morphism in $\mathscr{C}$, then:*

(1) *f is monic if and only if the sequence* $0 \longrightarrow X \xrightarrow{f} Y$ *is exact;*

(2) *$k = \ker f$ if and only if the sequence* $0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y$ *is exact;*

(3) *f is a regular epimorphism if and only if the sequence* $X \xrightarrow{f} Y \longrightarrow 0$ *is exact;*

(4) *for a proper morphism f, $q = \operatorname{coker} f$ if and only if* $X \xrightarrow{f} Y \xrightarrow{q} Q \longrightarrow 0$ *is exact.*

Finally we would like to stress that the key results on short exact sequences are valid in this setting (cf. [3]):

Theorem 4. **(Short Five Lemma)** *For a pointed regular category $\mathscr{C}$, the following conditions are equivalent:*

(i) *$\mathscr{C}$ is homological.*

(ii) *The Short Five Lemma holds, that is, given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{u} & X & \xrightarrow{p} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & K' & \xrightarrow{v} & X' & \xrightarrow{q} & Y' & \longrightarrow & 0 \end{array}$$

with exact rows, if a and c are isomorphisms, b is also an isomorphism.

Theorem 5. **(3×3 Lemma)** *Let $\mathscr{C}$ be a homological category. Consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K'' & \xrightarrow{k''} & X'' & \xrightarrow{f''} & Y'' & \longrightarrow & 0 \\ & & {\scriptstyle u'}\downarrow & & {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle w'} & & \\ 0 & \longrightarrow & K' & \xrightarrow{k'} & X' & \xrightarrow{f'} & Y' & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K & \xrightarrow{k} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \end{array}$$

where the horizontal lines are short exact sequences and $v \cdot v' = 0$. Then, if two of the columns are short exact sequences, the third one is also a short exact sequence.

The *Noether Isomorphisms Theorems* are still valid in homological categories, as well as the *Snake Lemma* (for exact formulations of these results see [3]). Furthermore, one can associate to each short exact sequence of chain complexes the *long exact homology sequence*, provided that the chain complexes are proper.

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