Great Moments in XXth Century Mathematics

by Marcelo Viana

In this issue we present the answer of Prof. Marcelo Viana to the question "If you had to mention one or two great moments in XXth century mathematics which one(s) would you pick?".

The first that comes to my mind are Gödel's theorems on the incompleteness of Arithmetics. These theorems changed our vision of human thought, Mathematics included, in much the same way as the great discoveries of Physics in the XXth century revolutionized our vision of the universe.

Mathematics had gone a long way towards establishing itself on a rigorous basis, from the times of Newton and Leibniz to those of Weierstrass and Hilbert. By the late XIXth century the axiomatic point of view seemed to allow every hope[2]*. To solve the crisis raised by Cantor's treatment of infinite sets, and the host of paradoxes unleashed by it, Hilbert proposed to establish the whole mathematical edifice on an axiomatic basis. A small number of postulates should be found, from which all other statements would be deduced through formal rules. Most important, one should prove that the postulates were consistent, that is, they would never lead to contradictory statements. Russell and Whitehead, Bourbaki, and others, set themselves to carry the gigantic task.*

Then, in 1931, Gödel proved that in any axiomatic system that is consistent and rich enough to contain Arithmetics, there are true statements that can not be proved nor disproved from the axioms. *In particular,* consistency of the axioms can not be proved within the system. *A fatal blow to Hilbert's program.*

The proof itself is a jewel of ingenuity. In a first step, Gödel shows how every formal statement, an admissible finite sequence of symbols, may be assigned a code, an integer number, in a constructive one-to-one fashion. Thence, assertions about formal statements may be interpreted in terms of integers, and properties like "provable from the axioms"may be expressed in the axiomatic system. The second step is to write down a special statement $\mathcal{S}$ whose interpretation is "N can not be proved from the axioms", where N is precisely the number of $\mathcal{S}$. If the axiomatic system is consistent, $\mathcal{S}$ can not be proved nor disproved, which implies that it is true!

Gödel's theorems have been used as an argument in favor of human over artificial intelligence: arguably, they show that humans are able to identify true statements that automatic machines, conditioned by formal rules, could never find. More certainly, these results prove that there are limitations to the axiomatic formulation of Mathematics, just as the uncertainty principle of quantum physics set fundamental limits to how much of reality can be aprehended experimentally.

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[2] Even the axiomatization of Physics, Hilbert's 6th problem!