# Representations, Higgs bundles and components: an overview

by André Oliveira*

## 1 Introduction

Compact oriented surfaces, without boundary, are among the most classical objects in geometry. Their topological classification has been completely achieved for quite a long time: such a surface must be equivalent to one, and precisely one, of the types $\Sigma_g$, where $\Sigma_0$ is the sphere, $\Sigma_1$ is the torus and, for $g \geq 2$, $\Sigma_g$ is the connected sum of $g$ copies of the torus $\Sigma_1$. The integer $g$ is called the *genus* of $\Sigma_g$. (See figure 1 for an example of $\Sigma_2$.)

Despite being a classical object, there are spaces naturally associated to $\Sigma_g$, whose geometry and topology is unknown and which lie on the edge of current mathematical research. One instance of such spaces are the so-called *character varieties* of $\Sigma_g$, also known as *spaces of surface group representations*. These are natural objects to consider, occurring in several areas of geometry, topology and even physics.

The study of character varieties is a motivation for introducing *Higgs bundles* over compact Riemann surfaces and their moduli spaces, the main subject of the present article. Higgs bundles and their moduli were introduced by Nigel Hitchin in the outstanding paper [17] almost thirty years ago. It is truly amazing the research that has been carried out based on that paper. However, the topology of moduli spaces of Higgs bundles on Riemann surfaces is far from being understood.

The aim of this article is to give an overview of the problem of studying the connected components of moduli spaces of Higgs bundles and of character varieties. Half of the article deals with the definition of character varieties, of Higgs bundles and with the relation between them. The goal is that the reader acquires a feeling of this exciting area of mathematics, also of the problems we address and (hopefully) of some techniques to handle them. The interested reader may find more details in the references.

Although we introduce Higgs bundles as a motivation for the study of character varieties, we stress the fact that they play a crucial role in many other different areas including gauge theory, Kähler and hyperkähler geometry, integrable systems, mirror symmetry, Langlands duality and more. We do not touch any of these topics.

## 2 Two moduli spaces

### 2.1 Representations and character varieties

Let us start with a fixed closed oriented surface $\Sigma_g$. Assume that $g \geq 2$. The fundamental group of $\Sigma_g$ is a finitely generated group, with $2g$ generators, such that the product of

* CMUP, Faculdade de Ciências–Univ. do Porto [andre.oliveira@fc.up.pt]
On leave from: Dep. de Matemática, Univ. de Trás-os-Montes e Alto Douro, UTAD [email: agoliv@utad.pt]

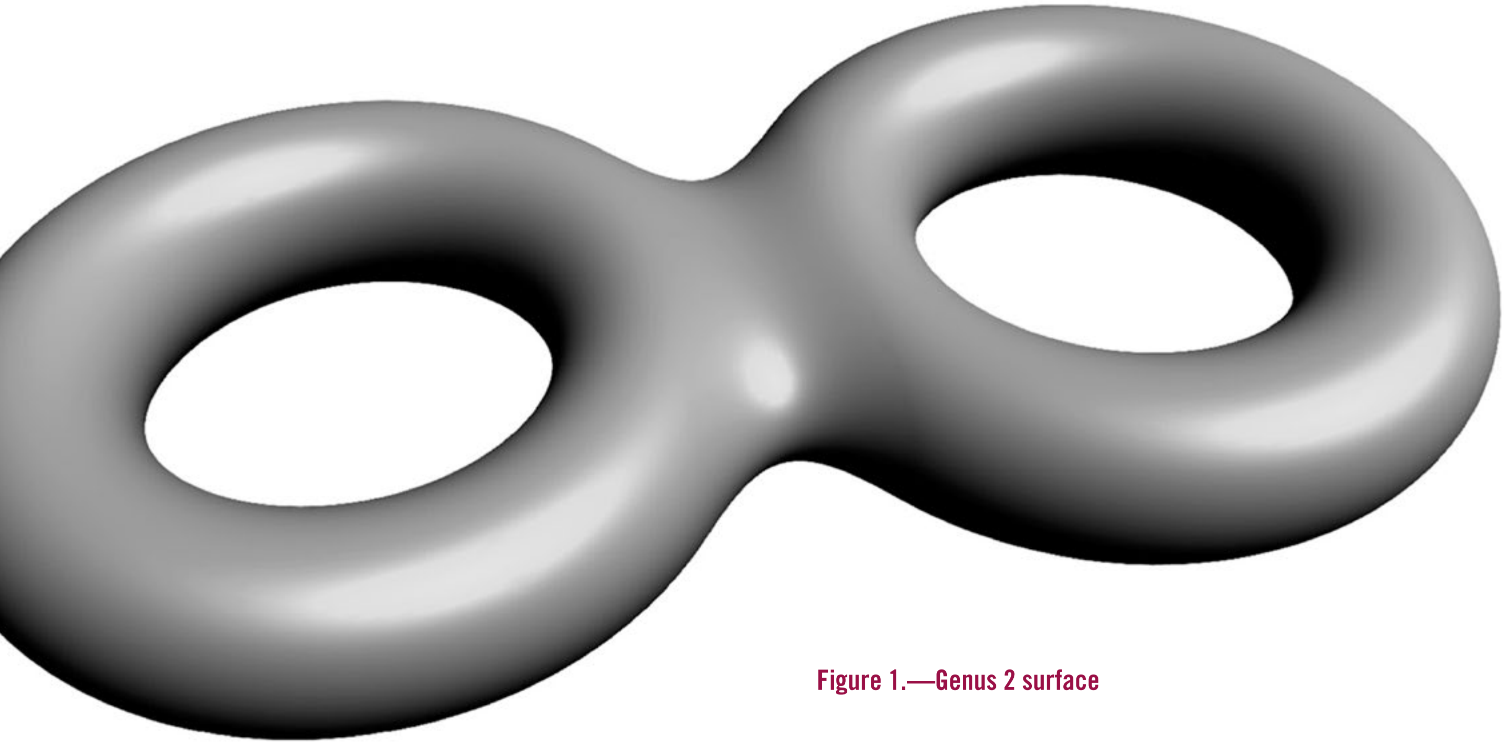

**Figure 1.—Genus 2 surface**

their commutators is trivial:

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, \dots, a_g, b_g \,:\, \prod_{i=1}^{g}[a_i, b_i] = 1 \right\rangle.$$

Let $G$ be a connected real semisimple Lie group, which we assume admits a complexification $G^{\mathbb{C}}$. Consider the set $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ of all group homomorphisms from $\pi_1(\Sigma_g)$ to $G$. Such a homomorphism $\rho : \pi_1(\Sigma_g) \to G$ is also called a *representation* of $\pi_1(\Sigma_g)$ in $G$. The name comes from the fact that $G$ is often a linear Lie group, acting naturally on a vector space, so yielding a representation of $\pi_1(\Sigma_g)$ on that vector space. Any representation is determined by its values on the $2g$ generators, so $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ is contained in $G^{2g}$ as the subset of those $2g$-tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the equation $\prod_{i=1}^{g}[A_i, B_i] = 1$. So we take the induced topology on $\mathrm{Hom}(\pi_1(\Sigma_g), G)$, which coincides with the compact-open topology, hence does not depend on the choice of the generators of $\pi_1(\Sigma_g)$.

It is natural to consider two representations equivalent when they lie in the same orbit of the $G$-action on $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ by conjugation: $g \cdot \rho = g\rho g^{-1}$. Indeed if $\rho$ and $\rho'$ are in the same orbit under this action, and if $G$ acts on a vector space $\mathbb{V}$ through a linear representation, say $\alpha$, then the difference between the representations $\alpha \circ \rho$ and $\alpha \circ \rho'$ of $\pi_1(\Sigma_g)$ in $\mathbb{V}$ is just given by a change of basis. Hence we are interested on the quotient space $\mathrm{Hom}(\pi_1(\Sigma_g), G)/G$. However, it may not be Hausdorff due to the existence of non-closed orbits whose closures intersect. A way to solve this is to take only *reductive* representations, meaning the ones that become a sum of irreducible representations when composed with the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$, i.e., with $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. In any case, reductive representations are dense in $\mathrm{Hom}(\pi_1(\Sigma_g), G)$. Denote the space of such representations by $\mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma_g), G)$.

Definition 1.— The *$G$-character variety* of $\Sigma_g$ is the quotient space

$$\mathcal{R}(G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma_g), G)/G.$$

One natural question is about the existence of discrete invariants of such classes of representations. Indeed we can define them in an easy way. Take a class $[\rho]$ in $\mathcal{R}(G)$ and choose a representative $\rho : \pi_1(\Sigma_g) \to G$. Let $(A_1, B_1, \dots, A_g, B_g) \in G^{2g}$ be the images through $\rho$ of the generators of $\pi_1(\Sigma_g)$. Consider the universal covering $p : \widetilde{G} \to G$, whose kernel is isomorphic to $\pi_1(G)$, the fundamental group of $G$. Choose a lift $(\tilde{A}_1, \tilde{B}_1, \dots, \tilde{A}_g, \tilde{B}_g) \in \widetilde{G}^{2g}$ of $(A_1, B_1, \dots, A_g, B_g)$ under $p$ and define the element

$$c([\rho]) = \prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] \in \pi_1(G). \tag{1}$$

This $c([\rho]) \in \pi_1(G)$ is an invariant of the class $[\rho]$. It does not depend on the choices made because $\pi_1(G)$ is contained in the centre of $\widetilde{G}$.

Given $c \in \pi_1(G)$, denote by $\mathcal{R}_c(G)$ the subspace of $\mathcal{R}(G)$ whose invariant defined by (1) is $c$. Each $\mathcal{R}_c(G)$ is a union of connected components of $\mathcal{R}$, and we have a disjoint union $\mathcal{R}(G) = \bigsqcup_{c \in \pi_1(G)} \mathcal{R}_c(G)$.

## 2.2 Higgs bundles and their moduli spaces

We defined character varieties in a purely topological context. In contrast, to introduce Higgs bundles we have to impose a Riemann surface structure on $\Sigma_g$. So let $X$ be a compact Riemann surface of genus $g \geq 2$, whose underlying 2-dimensional manifold is the surface $\Sigma_g$. Hence $X$ is a complex manifold of (complex) dimension 1. It will be fixed throughout.

The definition of Higgs bundles on $X$ requires some basic knowledge of holomorphic algebraic geometry, which we provide next. We emphasise that these are not the rigorous definitions (which can be found in [15, 23, 25]).

A *holomorphic vector bundle* $V$ *of rank* $n$ on $X$ is a family of vector spaces $V_x$, for each $x \in X$, varying holomorphically with the point $x$. Each of these vector spaces $V_x$, called the *fibre* of $V$ at $x$, is non-canonically biholomorphic to $\mathbb{C}^n$. Moreover, $V$ is required to be locally trivial, that is, for every point $x \in X$, there is an open neighbourhood $U_x$ of $x$ such that $V$ restricted to $U_x$ is biholomorphic to the product $U_x \times \mathbb{C}^n$. The *trivial vector bundle* over $X$ is just the cartesian product $X \times \mathbb{C}^n$ (hence the name "locally trivial" above). When $n = 1$ we say that $V$ is a *line bundle*.

Any operation over vector spaces, such as tensor product, direct sum or duality, can be extended to the vector bundle setting. In particular if we consider the wedge product and have a rank $n$ vector bundle $V$, we construct the line bundle $\bigwedge^n V$, whose fibres are biholomorphic to the top wedge power of the fibres of $V$. This is called the *determinant of* $V$.

A *section* of a vector bundle $V$ over $X$ is a holomorphic map $s : X \to V$ such that $s(x) \in V_x$. The vector space of sections of $V$ is denoted by $H^0(X, V)$.

If $G$ is a complex Lie group, a *holomorphic principal* $G$*-bundle* or, for short, $G$*-bundle*, $E$ is a family of (non-canonical) copies of the group $G_x \simeq G$, for each $x \in X$, varying holomorphically with the point $x$. Again, $E$ is required to be locally trivial, that is, every point $x \in X$ has an open neighbourhood $U_x$ such that $E$ restricted to $U_x$ is biholomorphic to the product $U_x \times G$. The *trivial* $G$*-bundle* on $X$ is again the product $X \times G$.

If $G$ acts on some vector space $\mathbb{V}$ and if $E$ is a $G$-bundle, then there is a canonical way to construct a vector bundle, with fibres isomorphic to $\mathbb{V}$, out of the action $G \to \mathrm{GL}(\mathbb{V})$ and of $E$. Denote this vector bundle by $E(\mathbb{V})$.

Now we pass to some definitions of Lie theory, which again are not given in complete detail. These may be found for instance in [6]. Let $H \subseteq G$ be a maximal compact subgroup of $G$ and $H^{\mathbb{C}} \subseteq G^{\mathbb{C}}$ be its complexification. If $\mathfrak{h}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ are the corresponding Lie algebras, there is a *Cartan decomposition*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}, \tag{2}$$

where $\mathfrak{m}^{\mathbb{C}}$ is a complex vector space. An example of (2) is the fact that any complex square matrix can be uniquely written as a sum of a symmetric and a skew-symmetric matrix. Now, the adjoint representation $\mathrm{Ad} : G^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{g}^{\mathbb{C}})$ induces a representation of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$. So, if $E$ is an $H^{\mathbb{C}}$-bundle over $X$, let $E(\mathfrak{m}^{\mathbb{C}})$ be the vector bundle associated to $E$ and to the action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$, as explained in the preceding paragraph.

Let $K = T^*X$ be the canonical line bundle of $X$. By definition this is the holomorphic cotangent bundle of $X$, that is, the dual of its tangent bundle.

**Definition 2.—** A *$G$-Higgs bundle* over $X$ is a pair $(E, \varphi)$ where $E$ is a (holomorphic) $H^{\mathbb{C}}$-bundle and $\varphi$ is a section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, called the *Higgs field*.

We now give some examples of $G$-Higgs bundles $(E, \varphi)$. Whenever $H^{\mathbb{C}}$ acts in $\mathbb{C}^n$ in a standard way, we take the corresponding vector bundle associated to $E$.

**Examples 1.—**

1. If $G$ is compact, a $G$-Higgs bundle is simply a (holomorphic) $G^{\mathbb{C}}$-bundle, hence $\varphi \equiv 0$.

2. If $G$ is complex with maximal compact $H$, then $H^{\mathbb{C}} = G$ and also $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}$. Thus a $G$-Higgs bundle is a pair $(E, \varphi)$ where $E$ is a $G$-bundle and $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$ where $E(\mathfrak{g})$ denotes the adjoint bundle of $E$, obtained from $E$ under the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. As an example, an $\mathrm{SL}(n, \mathrm{C})$-Higgs bundle is a pair $(V, \varphi)$ with $V$ a holomorphic rank $n$ vector bundle with trivial determinant and $\varphi \in H^0(X, \mathrm{End}_0(V) \otimes K)$, where $\mathrm{End}_0(V)$ denotes the bundle of traceless endomorphisms of $V$; so we can think of $\varphi$ as a map $\varphi : V \to V \otimes K$ (linear on each fibre) such that $\mathrm{tr}(\varphi) \equiv 0$. These are the "original" Higgs bundles, introduced by Nigel Hitchin in the seminal paper [17].

3. Let $G = \mathrm{Sp}(2n, \mathbb{R})$ — the group of automorphisms of $\mathbb{R}^{2n}$ preserving a symplectic form. An $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle is a triple $(V, \beta, \gamma)$ where $V$ is a holomorphic rank $n$ vector bundle, and the Higgs field splits as $\varphi = (\beta, \gamma)$, with $\beta : V^* \to V \otimes K$ such that $\beta^t \otimes \mathrm{Id}_K = \beta$ and likewise for $\gamma : V \to V^* \otimes K$.

There is a natural notion of isomorphism between two $G$-Higgs bundles over $X$. Further, these being complex algebraic objects, one can construct their *moduli spaces*; cf. [23]. Roughly speaking, these moduli spaces are algebraic varieties whose points parametrize isomorphism classes of $G$-Higgs bundles. Yet, in order to have a nice algebraic structure on these moduli, we cannot consider all $G$-Higgs bundles, but only the ones which are called *polystable*. Since this point is quite technical, we will not even define the meaning of this word. Just to give an example, a holomorphic vector bundle is polystable if it can be written as a direct sum of vector bundles (with certain conditions on their degrees) which are indecomposable, meaning that they cannot be further decomposed as direct sums. A polystable $G$-Higgs bundle behaves in a way which generalises the example of vector bundles. We can see here a certain parallelism between the notion of polystability and the notion of reductivity of a representation presented before Definition 1. Anyway, the reader just has to keep in mind that polystable $G$-Higgs bundles are the objects we have to consider in order to construct a moduli space of $G$-Higgs bundles on $X$, having the structure of an algebraic variety.

DEFINITION 3.— If $G$ is a semisimple Lie group, the *moduli space of $G$-Higgs bundles* over the compact Riemann surface $X$ is the variety whose points are given by isomorphism classes of polystable $G$-Higgs bundles over $X$. We denote it by $\mathcal{M}(G)$.

REMARK 1.— $\mathcal{M}(G)$ is a finite dimensional complex algebraic variety. It can be defined more generally for real reductive Lie groups. Although we are focusing our attention on semisimple Lie groups, all this theory generalises to real reductive Lie groups, with some slight modifications. In particular, when $G$ is complex reductive, the complex dimension of $\mathcal{M}(G)$ is

$$\dim_{\mathbb{C}}(\mathcal{M}(G)) = (2g-2)\dim_{\mathbb{C}}(G) + 2\dim_{\mathbb{C}}(Z(G)), \quad (3)$$

where $Z(G)$ is the centre of $G$. When $G$ is semisimple then $\dim_{\mathbb{C}}(Z(G)) = 0$.

As in the case of representations, we can define discrete invariants of (isomorphism classes of) $G$-Higgs bundles, which distinguish them in the $C^\infty$ category, but not in the holomorphic one. Given a $G$-Higgs bundle $(E,\varphi)$, we associate to it the topological invariant of the underlying $H^{\mathbb{C}}$-bundle $E$. This is well-known [21] to be given by an element

$$c(E) \in \pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G).$$

For an element $c \in \pi_1(G)$, let $\mathcal{M}_c(G)$ be the subspace of $\mathcal{M}(G)$ such that the corresponding $G$-Higgs bundles have topological invariant given by $c$. Again, we have a disjoint union $\mathcal{M}(G) = \bigsqcup_{c\in\pi_1(G)} \mathcal{M}_c(G)$, and each $\mathcal{M}_c(G)$ is a union of connected components.

## 2.3 The correspondence

Although apparently unrelated, the spaces $\mathcal{R}(G)$ and $\mathcal{M}(G)$ are tightly connected, by the following fundamental theorem.

THEOREM 4 ([17, 24, 8 ).—] If $G$ is semisimple, then there is a natural correspondence between $\mathcal{M}_c(G)$ and $\mathcal{R}_c(G)$, which induces a homeomorphism $\mathcal{M}_c(G) \cong \mathcal{R}_c(G)$, for any $c \in \pi_1(G)$. This correspondence comes from the fact that a $G$-Higgs bundle over $X$ is polystable if and only if it corresponds to a reductive representation of $\pi_1(\Sigma_g)$ in $G$.

REMARKS 1.—

1. This theorem is known as the *non-abelian Hodge correspondence*, since it generalises usual Hodge theory obtained when $G = \mathbb{C}^*$. In fact, the moduli space of $\mathbb{C}^*$-Higgs bundles (with fixed topological type $d \in \mathbb{Z}$) is isomorphic to the cotangent bundle to the Jacobian variety $\mathrm{Jac}(X)$ of $X$. This cotangent bundle is trivial, so the moduli space is $\mathrm{Jac}(X) \times \mathbb{C}^g$. The Jacobian variety of a compact Riemann surface is one of the most classical objects in algebraic geometry [15]. Topologically, it is a $2g$-dimensional real torus $(S^1)^{2g}$, hence

$$\mathcal{M}_d(\mathbb{C}^*) \cong \mathrm{Jac}(X) \times \mathbb{R}^{2g} \cong (S^1 \times \mathbb{R})^{2g} \cong (\mathbb{C}^*)^{2g}.$$

In particular, $\dim_{\mathbb{C}}(\mathcal{M}(\mathbb{C}^*)) = 2g$ as in formula 3. On the other hand, since $\mathbb{C}^*$ is abelian, $\mathcal{R}_d(\mathbb{C}^*) = (\mathbb{C}^*)^{2g}$, so we see here explicitly an homeomorphism $\mathcal{M}_d(\mathbb{C}^*) \cong \mathcal{R}_d(\mathbb{C}^*)$. Note that $\mathbb{C}^*$ is reductive but not semisimple. However, although Theorem 4 is stated for semisimple groups, there is a similar result which holds, more generally, for reductive groups.

2. Theorem 4 generalises the Narasimhan and Seshadri Theorem [19], which implies that $\mathcal{R}_0(\mathrm{SU}(n))$ is homeomorphic to $\mathcal{M}_0(\mathrm{SU}(n))$. This theorem was then generalised in [21] for any compact connected group. In these cases Higgs bundles are not really into the picture, because the groups in question are compact.

3. Recall also that in order to define Higgs bundles, we had to consider a Riemann surface structure $X$ on $\Sigma_g$. The structure of $\mathcal{M}_c(G)$ as an algebraic variety depends on this choice, but Theorem 4 shows that its topological structure is independent of it.

4. Although the spaces are homeomorphic, their geometric structures tend to be very different. For example $\mathcal{M}_c(G)$ has always the complex structure coming from the one of $X$ whereas, if $G$ is real, $\mathcal{R}_c(G)$ is not a complex variety.

## 2.4 The Hitchin equations and their relation with Higgs bundles and representations

There is a third space $\mathcal{H}_c(G)$, homeomorphic to $\mathcal{M}_c(G)$, very important on its own and also fundamental in the proof of the theorem. This space $\mathcal{H}_c(G)$ is the space of equivalence classes of solutions to the so-called *Hitchin equations*. These are partial differential equations on the infinite-dimensional space of connections on a fixed $C^\infty$ vector (or principal) bundle, coming from the Yang-Mills equations [1, 17]. We roughly explain it in a few lines, referring to [15, 25] for basic definitions of differential geometry, such as connection or curvature.

Given a $G$-Higgs bundle $(E, \varphi)$, denote the $C^\infty$-objects underlying $E$ and $\varphi$ by the same symbols. Then the Higgs field may be viewed as a $(1,0)$-form on $X$ with values in $E(\mathfrak{m}^\mathbb{C})$: $\varphi \in \Omega^{1,0}(X, E(\mathfrak{m}^\mathbb{C}))$. Let $H \subseteq G$ be a maximal compact subgroup, so that its Lie algebra $\mathfrak{h}$ is a compact form of $\mathfrak{g}$. Given a $C^\infty$ reduction of structure group $h$ of $E$ to $H$ (thus $h$ is a metric in $E$), let $F_h$ be the curvature of the unique $H$-connection compatible with the holomorphic structure on $E$. Let also $\tau_h$ be the involution on $\Omega^{1,0}(X, E(\mathfrak{m}^\mathbb{C}))$ given by the compact conjugation on $\mathfrak{g}^\mathbb{C}$ (which determines its compact form), and which is given fibrewise by the metric $h$. It is a fundamental result that $(E, \varphi)$ is polystable if and only if there is a metric $h$ of $E$ that satisfies the *Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = 0. \tag{4}$$

This so-called *Hitchin-Kobayshi correspondence* yields a homeomorphism $\mathscr{H}_c(G) \cong \mathscr{M}_c(G)$, first proved for $G = \mathrm{SL}(2, \mathbb{C})$ by Hitchin in [17]. A proof in full generality can be found in [8]. The Hitchin-Kobayshi correspondence comprises half of the proof of Theorem 4. The other half, also done in [17], yields a homeomorphism $\mathscr{R}_c(G) \cong \mathscr{H}_c(G)$ and relies on theorems of Donaldson and Corlette.

The proof of Theorem 4 involves deep existence results of solutions of partial differential equations on manifolds. Hence it is not at all clear which polystable $G$-Higgs bundle corresponds to a given reductive representation $\rho$ and vice-versa. It would be very interesting to find a way to see explicitly the correspondence of Theorem 4.

REMARK 2.— A natural question is to ask if the terminology *Higgs bundles* somehow reveals some relation with the Higgs boson or with the Higgs field in the standard model of particle physics (the name *Higgs* is in both cases after the theoretical physicist Peter Higgs). The author's lack of competence to answer this question in a satisfactory way, is solved by referring to Remark 7.1 in [26], where some indications are provided by E. Witten, using the Hitchin equations (4).

## 3 Their topology

The spaces $\mathscr{R}_c(G)$ and $\mathscr{M}_c(G)$ are hence topologically equal. In the next sections we give some ideas on how to study their topology. We do it on the side of $\mathscr{M}_c(G)$, since there we have the powerful tools of complex algebraic geometry at our disposal.

### 3.1 The Hitchin functional

It is known that the moduli space of $G$-Higgs bundles $\mathscr{M}_c(G)$ ought to have an extremely rich topology. However, its Poincaré polynomial — which encodes the dimensions of the compactly supported cohomology groups of $\mathscr{M}_c(G)$ — has been computed only for some low rank cases for the group $\mathrm{SL}(n, \mathbb{C})$ and with topological type $d \in \mathbb{Z}$ coprime with $n$, so that the moduli $\mathscr{M}_d(\mathrm{SL}(n, \mathbb{C}))$ is smooth; cf. [17, 13, 19]. The key tool is the following real functional, which we define here for linear groups:

DEFINITION 5.— The *Hitchin functional* on $\mathscr{M}_c(G)$ is the real function $f : \mathscr{M}_c(G) \to \mathbb{R}$ defined as

$$f(E, \varphi) = \|\varphi\|^2_{L^2} = \int_X \mathrm{tr}(\varphi\varphi^*)\omega,$$

where $\varphi^*$ is the adjoint of $\varphi$ with respect to $h$ (the metric that solves the Hitchin equations (4)) and $\omega$ is the volume form on $X$.

### 3.2 The smooth case and Morse theory

The functional $f$ is proper [17]. In fact, in the few cases where $\mathscr{M}_c(G)$ is smooth, $f$ is a perfect Morse-Bott function, so the Poincaré polynomial can, in theory, be computed using Morse theory and by studying the topology of the critical subvarieties of $f$. The identification of these subvarieties uses the crucial fact that the moduli spaces $\mathscr{M}_c(G)$ carry a non-trivial $\mathbb{C}^*$-action by multiplication on the Higgs field,

$$\lambda \cdot (E, \varphi) = (E, \lambda\varphi). \tag{5}$$

The critical subvarieties of $f$ coincide with the subvarieties of fixed points under this $\mathbb{C}^*$-action. The problem is that these subvarieties also have in general a very intricate topology, whose complete study is quite difficult. This is the basic reason why only a few low rank cases have been successfully addressed, even in the smooth case.

On the other hand, recent developments [10, 9, 22] were achieved on the study of $\mathscr{M}_c(\mathrm{SL}(n, \mathbb{C}))$, which seem to confirm some fascinating conjectures [16].

### 3.3 Connected components

In general, however, $\mathscr{M}_c(G)$ are singular spaces so the Morse theory picture breaks down. The topology of $\mathscr{M}_c(G)$ is hence basically unknown, with the honourable exception of the most basic topological invariant: the number of connected components.

Actually, the properness of $f$ is enough to draw conclusions on the components of these moduli spaces. Since $f$ is bounded below and proper, it attains a minimum on every component. The number of components of $\mathscr{M}_c(G)$ is thus bounded above by the number of components of the subvarieties $\mathcal{N}_c(G) \subset \mathcal{M}_c(G)$ of local minima of $f$.

The idea pioneered by Hitchin in [17, 18] to study the connected components of $\mathcal{M}_c(G)$ is to identify the minimum subvarieties $\mathcal{N}_c(G)$, among the fixed points of the $\mathbb{C}^*$-action (5), study the components of $\mathcal{N}_c(G)$ and then see what we can conclude about the components of $\mathcal{M}_c(G)$. This procedure has been extensively studied for many families of real semisimple Lie groups $G$ [14, 3, 4, 20, 7, 11, 5, 12]. In the next section we describe some results in this direction.

## 4 Components and real forms

### 4.1 Compact Lie groups

When the group $G$ is compact, we are really in the world of (holomorphic) $G^{\mathbb{C}}$-bundles. Then it is known for a long time that $\mathcal{M}_c(G)$ is non-empty and connected for any $c \in \pi_1(G)$ (see [21]).

### 4.2 Complex Lie groups

The same conclusion remains true for complex Lie groups. The most general form of the following connectedness theorem has been proved recently in [12].

Theorem 6 ([12 ).—] Let $G$ be a complex reductive connected Lie group. Then $\mathcal{M}_c(G)$ is non-empty and connected for every $c \in \pi_1(G)$.

This theorem is even valid for non-connected groups. There we proved that the only local minima of $f$ are the Higgs bundles with $\varphi \equiv 0$. Hence the subvarieties of local minima are moduli spaces of $H$-Higgs bundles, where $H \subset G$ is a maximal compact subgroup. Then Theorem 6 follows from the previous subsection.

These techniques generalise to the real case but become much more complicated. Indeed, if the group is real, then this story is completely different, as we will see in the remaining part of the article.

### 4.3 Split real forms

Assume that $G$ is a *split real form* of $G^{\mathbb{C}}$. Roughly, this means that $G$ is the *maximally non-compact* real form of $G^{\mathbb{C}}$ (cf. [6]). Examples are $G = \mathrm{SL}(n, \mathbb{R})$ and $G = \mathrm{Sp}(2n, \mathbb{R})$.

Theorem 7 ([18 ).—] For $G$ a split real form, there is at least one $c \in \pi_1(G)$ such that $\mathcal{M}_c(G)$ is disconnected and has a connected component of $\mathcal{M}_c(G)$ isomorphic to a vector space.

The contractible component of $\mathcal{M}_c(G)$ mentioned in the theorem is called the *Hitchin component*. It carries relevant information about geometric structures on the surface $X$ itself. For example, it is isomorphic to Teichmüller space when $G = \mathrm{SL}(2, \mathbb{R})$ [18]. Theorems 7 and 4 prove the existence of a Hitchin component in $\mathcal{R}_c(G)$, being an interesting question to characterise the representations in it.

### 4.4 Hermitian type groups

A different kind of phenomena occurs if $G$ is a real non-compact group of *hermitian type*. One possible definition of such groups is that they are characterised by the fact that the centre of their maximal compact subgroup contains a circle. For instance $\mathrm{Sp}(2n, \mathbb{R})$ is a group of hermitian type since a maximal compact is $\mathrm{U}(n)$, whose centre is $\mathrm{U}(1)$. The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is especially interesting because is the only one, up to finite covering, that is simultaneously split and hermitian.

In this hermitian case, there is a new phenomena concerning the non-emptiness of $\mathcal{M}_c(G)$. Indeed, the free part of $\pi_1(G)$ is isomorphic to $\mathbb{Z}$, giving rise to an integer $d$. There is a bound for $|d|$, called the *Milnor-Wood inequality*, above which $\mathcal{M}_d(G)$ is empty (see [4, 2]). Moreover, for some groups of hermitian type, $\mathcal{M}_d(G)$ is disconnected when $|d|$ is maximal. This is the case of $G = \mathrm{Sp}(2n, \mathbb{R})$, studied by García-Prada, Gothen and Mundet i Riera in [7]:

Theorem 8 ([7 ).—]The moduli space $\mathcal{M}_d(\mathrm{Sp}(2n, \mathbb{R}))$ is non-empty if and only if $|d| \leq n(g-1)$. Moreover, if $n \geq 3$, $\mathcal{M}_{n(g-1)}(\mathrm{Sp}(2n, \mathbb{R}))$ has $3 \times 2^{2g}$ non-empty connected components.

Recall from subsection 2.2 that $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles are given by a triple $(V, \beta, \gamma)$. The distinctive feature of the case $d = n(g-1)$ is that $\gamma : V \to V^* \otimes K$ is an isomorphism precisely for that value of $d$ (the case $d = -n(g-1)$ is similar but it is $\beta$ that becomes an isomorphism). This uncovers $2 \times 2^{2g}$ *hidden* components. A further analysis, using the Hitchin functional as before, proves the existence of the remaining $2^{2g}$ components, which are the Hitchin ones mentioned in the preceding subsection (recall that $\mathrm{Sp}(2n, \mathbb{R})$ is also split).

In general further difficulties arise for the study of components for non-maximal and non-zero $d$. In the known cases, the non-maximal subspaces are connected for each fixed topological type. We expect that the same holds true in general, but a potential proof of this conjecture seems out of reach at the moment.

### 4.5 Other real forms

The components of $\mathcal{M}_c(G)$, hence also of $\mathcal{R}_c(G)$, have been studied for many families of groups, not necessarily split or hermitian; see for instance [11]. Until recently, no examples were known of real groups, neither split nor hermitian, for which $\mathcal{M}_c(G)$ is disconnected. However, by considering the group $\mathrm{SO}_0(p, q)$ — the identity component of the group of automorphisms of $\mathbb{R}^{p+q}$ preserving an orthogonal structure with signature $(p,q)$ — we recently realised that there exist many different local minima of $f$. This panoply of local minima may potentially give rise to new components of $\mathcal{M}_c(\mathrm{SO}_0(p,q))$ whose geometric structure differs from all the known cases. This is still work in progress. Again, it would be very interesting to characterise the representations lying in these new components of $\mathcal{M}_c(\mathrm{SO}_0(p,q))$.

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