

Concentrations and constrained Hamilton-Jacobi equations arising in adaptive dynamics

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Abstract

We consider an integral equation that exhibits a concentration phenomena. It describes a general birth and death process for a population structured by a quantitative trait and subject to small and frequent mutations. The population interacts through an environmental variable that is shared by the full population. This leads to the exponential growth of the population with the best adapted trait and to the exponential decay of the rest of the population.

A small parameter that measure the mutations is introduced and after time rescaling, the population density typically converges to a Dirac mass in the trait space with weight and position evolving according to the new time scale. Following a formalism introduced in [12], these parameters are described by a Hamilton-Jacobi equation with a constraint and a Lagrange multiplier.

The aim of the paper is to prove uniqueness for a class of such constrained H.-J. equations, naturally posed in the viscosity solution sense. Our method combines changes of variables and unknowns with a contraction principle.

1 Introduction

Hamilton-Jacobi equations are useful tools to describe several singular asymptotics. The upmost classical example is the WKB method for Helmholtz, Schrödinger or wave (and more generally dispersive) equations in the *high frequency* regimes. The ansatz $u_\varepsilon \approx a_\varepsilon e^{i\varphi_\varepsilon(t,x)/\varepsilon}$ is then used to produce the Hamilton-Jacobi (H.-J. in short) equation for φ_ε . This asymptotic holds true before caustics but should be understood as kinetic transport globally. The formalism is also a powerful tool to analyze concentrations for limiting cases of the Sobolev imbeddings, see [19].

Another class of examples, more related to our study here, arises in parabolic PDEs to describe *front propagation*. This is possible when the PDEs are scaled with small viscosity and large reaction terms, and the analysis was carried out more recently, see [14, 3, 29] for an introduction to the subject. Combustion and ecology are two examples where such models describe very well observations; a sharp transition front from the colonized (burnt) region $u_\varepsilon(t, x) \approx 1$ to the uncolonized (unburnt) region $u_\varepsilon(t, x) \approx 0$. In the simplest cases, the ansatz $u_\varepsilon = e^{\varphi_\varepsilon(t,x)/\varepsilon}$ now gives a H.-J. equation for the limit φ of φ_ε , with $\varphi = 0$ in the colonized region and $\varphi < 0$ in the uncolonized region. Typically, this is possible because the limiting H.-J. equation is a variational inequality. A striking feature of this theory is also that it holds globally, i.e., after the solution to the H.-J. equation has developed a

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singularity. This is because the concept of *viscosity solution* to H.-J. equations applies in this context ([1, 2, 9, 16, 13, 21, 17]) and gives a unique global solution. Notice also that diffusion always leads to a H.-J. equation with a quadratic nonlinearity, and one usually refers to it as the Eikonal equation. This is not the case in the present paper, our mutation kernels lead to a class of hamiltonians with exponential growth, just as in the case of front propagation in ecological situations where the motion occurs by jumps ([27]).

Additionally, this paper is devoted to another class of singular asymptotic behaviors for time evolution equations, namely *concentrations* as evolving Dirac masses. As we show it below, H.-J. equations again play a role here. An example is presented in [12] based on selection-mutation equations describing the adaptive dynamics of a quantitative trait under the influence of an ecological feedback loop ([23, 11, 10]); this is a classical subject of theoretical biology that can be viewed from various points of view, [6, 15, 18, 8].

Our aim here is to explain in this paper how the derivation of Dirac concentrations leads to a H.-J. equation with a constraint ($\max_{x \in \mathbb{R}} \varphi(t, x) = 0$ below) and a Lagrange multiplier (the quantity $\varrho(t)$ below),

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \varphi(t, x) = \frac{b(x)}{1+\varrho(t)} - (1 + \varrho(t))d(x) + \frac{1}{1+\varrho(t)}\beta(x)H\left(\frac{\partial}{\partial x}\varphi(t, x)\right), \\ \varphi(t = 0) = \varphi^0 \in \mathcal{L}ip(\mathbb{R}), \quad \max_{x \in \mathbb{R}} \varphi^0(x) = 0, \\ \max_{x \in \mathbb{R}} \varphi(t, x) = 0 \quad \forall t > 0. \end{array} \right.$$

We wish to make the derivation simpler by choosing a simpler framework than in [12], but also more general by the choice of the model. For the most general models that we present in Section 2, uniqueness and a priori estimates are not always available and our aim here is to prove partial uniqueness and existence results for the case of Lipschitz solutions (a natural class in this context). The general and formal derivation of the constrained H.-J. equation is performed in Section 3. The uniqueness results are presented in Section 4.

2 The selection-mutation model

Following [12], we depart from a growth-death model for a population structured by a quantitative trait $x \in \mathbb{R}$, i.e., a variable with physiological interpretation that influences the population capacity to use the environment (a nutrient say) that all the population shares. Here we consider a very simplified model but in section 5 a slightly more elaborate equation is proposed that shows additional difficulties that might arise then.

We use the simplifying assumptions (i) that birth gives rise to small but frequent mutations and we denote by ε the corresponding small parameter, (ii) that the environment reacts instantaneously to the population evolution, and (iii) we take simplified forms of the population-environment interactions through a linear term in the total population number, that increases the death rate $d(x)$ and decreases

the birth terms. Then, we arrive to the equation for the density $n(t, x)$ of individuals with trait x ,

$$\begin{cases} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(t, x) = \left[\frac{b(x)}{1+\varrho_\varepsilon(t)} - (1 + \varrho_\varepsilon(t))d(x) \right] n_\varepsilon(t, x) + \frac{1}{1+\varrho_\varepsilon(t)} \int \frac{1}{\varepsilon} K\left(\frac{x-y}{\varepsilon}\right) \beta(y) n_\varepsilon(t, y) dy, \\ \varrho_\varepsilon(t) = \int n_\varepsilon(t, x) dx, \\ n_\varepsilon(t=0, x) = n_\varepsilon^0(x) \geq 0, \quad n_\varepsilon^0 \in L^1 \cap L^\infty(\mathbb{R}). \end{cases} \quad (1)$$

The parameter ε in front of the time derivatives just indicates that we have rescaled time to consider the long time behavior on the scale ε . We also have used two different birth rates for neutral births (rate $b(x)$) and for the births arising with mutations ($\beta(x)$). Both are limited by a factor depending upon the total population size $\varrho_\varepsilon(t)$. The probability kernel $K(\cdot)$ describes the mutation rate and we assume that

$$K(\cdot) \geq 0, \quad \int K(z) dz = 1, \quad \int K(z) e^{|z|^2} dz < \infty. \quad (2)$$

We need that a population described by such a model does not explode neither get extinct. For that we assume

$$b, d, d' \text{ and } \beta \text{ are Lipschitz continuous.} \quad (3)$$

$$0 < b_m \leq b(x) + \beta(x) \leq b_M < \infty, \quad 0 < d_m \leq d(x) \leq d_M < \infty, \quad (4)$$

with the following assumption and notations

$$\varrho_m = \sqrt{\frac{b_m}{d_M}} - 1 > 0, \quad \varrho_M = \sqrt{\frac{b_M}{d_m}} - 1. \quad (5)$$

The existence of solutions is a classical matter (see [28] for instance). An example of existence result with uniform bounds is as follows (we drop the dependency of n and ϱ upon ε which is not relevant here)

Theorem 2.1 *With the assumptions (2)–(5), the system (1) has a unique nonnegative solution such that $n, \frac{\partial}{\partial t} n \in C(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}))$.*

Assume also that $\varrho^0 := \varrho(t=0)$, satisfies $\varrho_m \leq \varrho^0 \leq \varrho_M$, then we have for all $t \geq 0$

$$\varrho_m \leq \varrho(t) \leq \varrho_M. \quad (6)$$

Proof. We do not provide a complete proof of this theorem but just mention that the lower and upper bounds on $\varrho(t)$ are easily deduced, integrating in x the equation (1) which gives

$$\varepsilon \frac{\partial}{\partial t} \varrho(t) = \int \left[(1 + \varrho(t)) (b(x) + \beta(x)) - \frac{d(x)}{1 + \varrho(t)} \right] n(t, x) dx$$

and thus

$$\varepsilon \frac{\partial}{\partial t} \varrho(t) \leq \varrho(t) \left[(1 + \varrho(t)) b_M - \frac{d_m}{1 + \varrho(t)} \right],$$

and whenever $\varrho(t)$ attains the value ϱ_M then $\frac{\partial}{\partial t} \varrho(t)$ vanishes and thus this value is never overpassed.

A similar argument gives the lower bound. \square

Remark 2.2 *The usual chemostat model contains also an evolution equation for the substrate $S(t)$ (nutrient shared by the full population). With $\varepsilon = 1$ it reads*

$$\begin{cases} \frac{d}{dt}S(t) = S_0 - S(t)[1 + \int n(t, x)\eta(x)dx], \\ \frac{\partial}{\partial t}n(t, x) = [S(t)b(x) - 1]n(t, x) + S(t) \int K(x - y)\beta(y)n(t, y)dy, \\ n(t = 0, x) = n^0(x) \geq 0, \quad S(t = 0) = S^0(x) > 0. \end{cases}$$

Here $\eta(\cdot)$ represents the predation ability of individuals with trait x and $b + \beta = \eta$ to ensure the conservation law of some representative molecule in the biomass

$$\frac{d}{dt}[S(t) + \int n(t, x)dx] = S_0 - S(t) - \int n(t, x)dx.$$

The model (1) we have used here assumes that the substrate reaches its steady state faster than the living population, leading to

$$S(t) = \frac{S_0}{1 + \int n(t, x)\eta(x)dx}.$$

Then, n has been changed to ηn (and notations for b and β have been changed accordingly), and an additional death rate has been included.

3 Formal asymptotic

Let us turn to the study of the limit $\varepsilon \rightarrow 0$ in the equation (1). Since, in the weak measure sense $\frac{1}{\varepsilon}K(\frac{\cdot}{\varepsilon}) \rightarrow \delta(\cdot)$, the mutations are not a first-order effect (and act only in the present time scale) therefore we can think of the model as close to

$$\frac{\partial}{\partial t}n(t, x) = \left[\frac{b(x) + \beta(x)}{1 + \varrho(t)} - (1 + \varrho(t))d(x) \right] n(t, x), \quad \varrho(t) = \int n(t, x)dx$$

which concentrates to a Dirac mass for long times:

$$n(t, x) \rightarrow \varrho_l \delta(x = x_l) \quad \text{as } t \rightarrow \infty,$$

at least if $\sqrt{\frac{b(x) + \beta(x)}{d(x)}} - 1$ attains its maximum ϱ_l at a single point x_l .

For this reason one expects the same type of concentration for $n_\varepsilon(t, x)$ but with varying weights and positions

$$n_\varepsilon(t, x) \rightarrow \bar{\varrho}(t) \delta(x = \bar{x}(t)) \quad \text{as } t \rightarrow \infty, \quad (7)$$

The interesting point here is that the parameters $\bar{\varrho}(t)$ and $\bar{x}(t)$ cannot be described in a simple manner by opposition to the case of 'canonical equations' in [10].

A numerical simulation of this concentration phenomena is exhibited in Figures 1 and 2. The value $x \in (0, 1)$ has been discretized with 1500 points and $\varepsilon = 1/300$.

In order to describe these concentrations, and following again [12], an asymptotic calculation is needed which we now perform in a formal way. We consider the ansatz

$$n_\varepsilon(t, x) = e^{\varphi_\varepsilon(t, x)/\varepsilon},$$

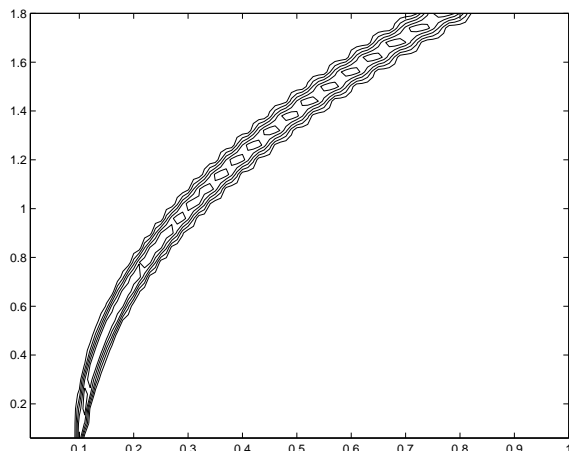


Figure 1: EVOLUTION OF THE TRAIT $x \in (0, 1)$ AS A FUNCTION OF TIME (y AXIS) FOR THE MODEL (1) WITH $b(x) = 1 + 3x$, $d = \beta = 1$. THE PARAMETE ε IS APPROXIMATELY $3 \cdot 10^{-3}$. THIS FIGURE SHOWS THE ISOVALUES OF $n(t, x)$ AND A PLOT OF n IS PRESENTED IN FIGURE 2.

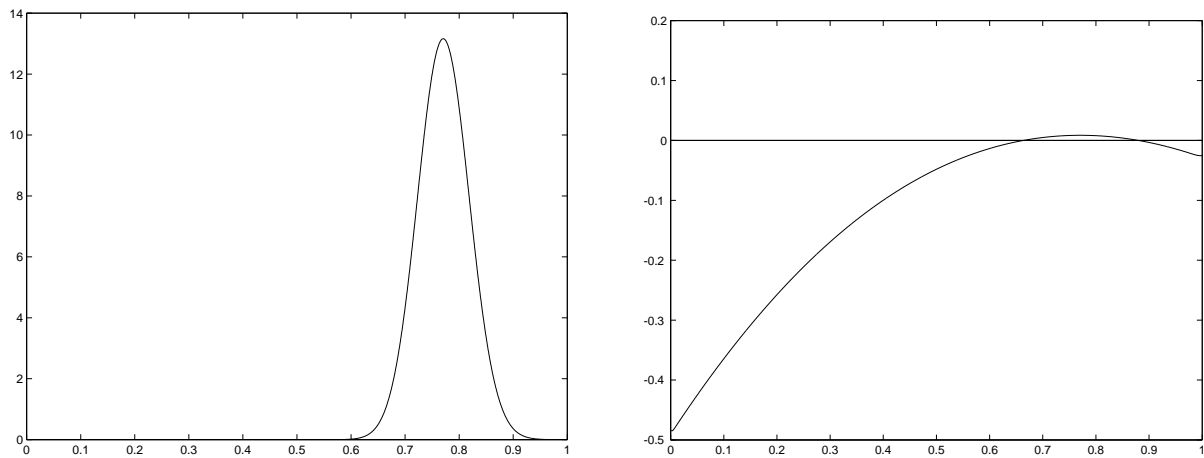


Figure 2: PLOT OF THE DENSITY $n(t, \cdot)$, $\varphi(t, \cdot)$ AT THE FINAL TIME OF THE SIMULATION. THIS IS THE SAME SIMULATION AS IN FIGURE 1.

and obtain from equation (1)

$$\frac{\partial}{\partial t} \varphi_\varepsilon(t, x) = \left[\frac{b(x)}{1 + \varrho_\varepsilon(t)} - (1 + \varrho_\varepsilon(t))d(x) \right] + \frac{1}{1 + \varrho_\varepsilon(t)} \int K(z)\beta(x + \varepsilon z)e^{(\varphi_\varepsilon(t, x + \varepsilon z) - \varphi_\varepsilon(t, x))/\varepsilon} dz,$$

after changing y to $z = (x - y)/\varepsilon$. The limit $\varepsilon \rightarrow 0$ is now clear. We set

$$H(p) := \int K(z)e^{zp} dz, \tag{8}$$

a convex Hamiltonian, and we obtain for the limit

$$\frac{\partial}{\partial t} \varphi(t, x) = \frac{b(x)}{1 + \varrho(t)} - (1 + \varrho(t))d(x) + \frac{1}{1 + \varrho(t)} \beta(x)H\left(\frac{\partial}{\partial x} \varphi(t, x)\right). \tag{9}$$

On the other hand, the a priori bound in Theorem 2.1 tells us that n_ε , in total mass, does not explode neither vanishes. Therefore $\varphi(t, x)$ has to be nonpositive (otherwise n_ε blows up), but cannot be everywhere negative (otherwise n_ε would get extinct). Therefore we arrive at the conclusion that

$$\max_{x \in \mathbb{R}} \varphi(t, x) = 0 \quad \forall t > 0. \quad (10)$$

If this maximum is attained at a single point, $\bar{x}(t)$ we arrive at the conclusion that (7) indeed holds true and the mass can be recovered by the fact that

$$\frac{\partial}{\partial t} \varphi(t, \bar{x}(t)) = \frac{\partial}{\partial q} \varphi(t, \bar{x}(t)) = 0,$$

which implies that the growth rate vanishes (this is the so-called pessimization principle, cf. [24]). This gives the characterization of the limiting parameters

$$\begin{cases} \varphi(t, \bar{x}(t)) = 0 & (\text{defines } \bar{x}(t)), \\ \frac{b(\bar{x}(t)) + \beta(\bar{x}(t))}{1 + \varrho(t)} = (1 + \varrho(t))d(\bar{x}(t)) & (\text{defines } \bar{\varrho}(t)). \end{cases}$$

From a PDE point of view the system (9), (10) is a constrained H.-J. equation which has to be solved for $\varphi(t, x)$ and $\varrho(t)$. the parameter $\varrho(t)$ is a Lagrange multiplier associated with the constraint (10). This makes the originality and difficulty of this problem.

Remark 3.1 *This formalism is well adapted to systems of population equations as well as in the case of front propagation ([3]). The classical example of juveniles and adults is treated in [7] where dimorphism is possible (even though not asymptotically stable).*

4 A Uniqueness Result through a fixed point argument

We now consider the limiting system derived in Section 3, that we rewrite as a Cauchy problem. This requires to scale appropriately the initial data $n_\varepsilon^0 = e^{\varphi_\varepsilon^0/\varepsilon}$ in such a way that $\varphi_\varepsilon^0 \rightarrow \varphi^0$ as ε vanishes (while keeping the finite mass requirement). For the reader convenience, we recall the constrained H.-J. equation

$$\begin{cases} \frac{\partial}{\partial t} \varphi(t, x) = \frac{b(x)}{1 + \varrho(t)} - (1 + \varrho(t))d(x) + \frac{1}{1 + \varrho(t)}\beta(x)H\left(\frac{\partial}{\partial x} \varphi(t, x)\right), \\ \varphi(t = 0) = \varphi^0 \in \mathcal{Lip}(\mathbb{R}), \quad \max_{x \in \mathbb{R}} \varphi^0(x) = 0, \\ \max_{x \in \mathbb{R}} \varphi(t, x) = 0 \quad \forall t > 0. \end{cases} \quad (11)$$

In this section we just use that the hamiltonian satisfies

$$H(\cdot) \text{ is smooth and satisfies } H(0) = 1. \quad (12)$$

We also recall that we know from the above derivation that

$$\varrho_m \leq \varrho(t) \leq \varrho_M. \quad (13)$$

Formally, these inequalities hold true because

$$\frac{b(x)}{1 + \varrho_m} - (1 + \varrho_m)d(x) + \frac{1}{1 + \varrho_m}\beta(x) \leq 0, \quad \forall x \in \mathbb{R},$$

$$\frac{b(x)}{1 + \varrho_M} - (1 + \varrho_M)d(x) + \frac{1}{1 + \varrho_M}\beta(x) \geq 0, \quad \forall x \in \mathbb{R},$$

and thus the Lagrange multiplier $\varrho(t)$ should satisfy (13) to be able to sustain the constraint $\max_{x \in \mathbb{R}} \varphi(t, x) = 0$. Indeed at the maximum point $\bar{x}(t)$ we have

$$\frac{\partial}{\partial x} \varphi(t, \bar{x}(t)) = 0, \quad \frac{\partial}{\partial t} \varphi(t, \bar{x}(t)) = 0,$$

and this means that

$$\frac{b(t, \bar{x}(t))}{1 + \varrho(t)} - (1 + \varrho(t))d(t, \bar{x}(t)) + \frac{1}{1 + \varrho(t)}\beta(t, \bar{x}(t)) = 0.$$

Especially, the a priori bounds (13) are not enough to assert that $\varrho(\cdot)$ is continuous, a fact that is wrong in general because the best adapted trait $\bar{x}(t)$ may jump, see the example in [12]. Since the most classical background for viscosity solutions is when coefficients are continuous, we should use a more specific theory. In order to give a sense to problem (11), we may use the theory of viscosity solutions for equations with a measurable dependence in time which was first studied by H. Ishii [20] and then by P.L. Lions and B. Perthame [22]. We refer also the reader to D. Nunziante [25, 26] for extensions to the second-order case and to M. Bourgoing [4, 5] for a more modern presentation of this notion. However there is simpler argument in our specific context.

In order to prove uniqueness for (11), we are going to change its formulation through a change of unknowns and of variable. We first do it formally and then we provide a justification.

We begin with changing the time variable setting

$$t = \tau(s), \quad \tau'(s) = \frac{1}{1 + \varrho(\tau(s))}, \quad \tau(0) = 0. \quad (14)$$

We arrive at a new equation for $\psi(s, x) = \varphi(\tau(s), x)$,

$$\frac{\partial}{\partial s} \psi(s, x) = [b(x) - (1 + \varrho(\tau(s)))^2 d(x)] + \beta(x) H\left(\frac{\partial}{\partial x} \psi(s, x)\right).$$

One of the difficulties here is that the only a priori estimate we have on $\varrho(t)$ is the above L^∞ bound while viscosity solutions are more convenient in a continuous context. Therefore we perform a change of unknown setting

$$\Phi(s, x) = \psi(s, x) + d(x)\Sigma(s), \quad \Sigma(s) := \int_0^s (1 + \varrho(\tau(s')))^2 ds'. \quad (15)$$

We finally arrive at the equation

$$\frac{\partial}{\partial s} \Phi(s, x) = b(x) + \beta(x) H\left(\frac{\partial}{\partial x} \Phi(s, x) - \Sigma(s) \frac{\partial}{\partial x} d(x)\right). \quad (16)$$

It is worth pointing out that, for this last equation, we may use the classical continuous definition of viscosity solutions ([1, 2, 9, 16, 13, 21]).

Moreover we still have the constraint on Φ

$$\sup_{x \in \mathbb{R}} (\Phi(s, x) - d(x)\Sigma(t)) = 0 \quad \text{for any } s > 0, \quad (17)$$

and the initial condition

$$\Phi(0, x) = \varphi^0 \quad \text{in } \mathbb{R}. \quad (18)$$

Our first result is the

Proposition 4.1 *Assume (3)–(5), (12). If (φ, ρ) is a solution of (11), then (Φ, Σ) is a solution of (16)–(18).*

Proof. In the above formal arguments, the only difficulty comes from the change of time and more precisely the fact that Σ is not smooth since the only information available is $\Sigma' \in L^\infty$. We first re-interpret the O.D.E. satisfied by τ (which has an unpleasant L^∞ nonlinearity) as

$$\int_0^{\tau(t)} (1 + \varrho(s)) ds = t \quad \text{for } t > 0,$$

which means that the O.D.E. is equivalent to $\psi(\tau(t)) = t$ where ψ is a strictly increasing Lipschitz continuous function. Therefore τ is well-defined but non-necessarily smooth.

To justify the time change in the equation, we consider a sequence $(\rho_n)_n$ of smooth functions, satisfying the same bounds as ρ and which converges to ρ in L^1 . We define τ_n with ρ_n as τ is defined with ρ . Since τ_n is smooth, the time change can be made with τ_n and, to conclude, we use the stability result for viscosity solutions of equations with measurable time dependence (see [20, 22, 4]) together with the facts that $\rho_n \rightarrow \rho$ in L^1 and $\tau_n \rightarrow \tau$ uniformly for bounded t . \square

We can now state our main uniqueness result for the constrained H.-J. equation

Theorem 4.2 *We assume (3)–(5). We also assume either that H is Lipschitz continuous or that (8), (2) hold and β is constant. Then, there exists a unique Lipschitz solution (Φ, Σ) to the constrained H.-J. equation (16)–(18). As a consequence, the system (11) has at most one solution.*

We would like to point out that the additional assumption that H is Lipschitz continuous is not really a restriction here since we expect in general Lipschitz continuous solutions to (11): if H is not Lipschitz continuous, it can therefore be replaced by a Lipschitz one. The condition that β is constant, for instance, enforces the Lipschitz continuity of the solution.

Proof. We are going to use a fixed point argument for a suitable contraction mapping. To do so, we denote by E the Banach space of continuous, real-valued functions Σ on $[0, T]$ such that $\Sigma(0) = 0$, equipped with the usual sup-norm and where $T > 0$ will be chosen later on.

For $\Sigma \in E$, we denote by $\Phi[\Sigma]$ the unique (bounded Lipschitz continuous) viscosity solution of

$$\begin{cases} \frac{\partial}{\partial s} \Phi(s, x) = b(x) + \beta(x)H\left(\frac{\partial}{\partial x} \Phi(s, x) - \Sigma(s)\frac{\partial}{\partial x} d(x)\right) & \text{in } (0, T) \times \mathbb{R}, \\ \Phi(0, x) = \varphi^0(x) & \text{in } \mathbb{R}. \end{cases} \quad (19)$$

Then we introduce the map $\chi : E \rightarrow E$ defined by, for $s \in [0, T]$

$$\chi(\Sigma)(s) = \Sigma(s) + \kappa \sup_{x \in \mathbb{R}} (\Phi[\Sigma](s, x) - d(x)\Sigma(s)),$$

where $\kappa > 0$ will be also chosen later.

The key lemma is

Lemma 4.3 *The map χ is well-defined and there exists $T, \kappa > 0$ depending only on the data of the problem such that χ is a strict contraction on E .*

Proof of Lemma 4.3. (i) The fact that χ is well-defined just comes from the definition of E and the fact that $\Phi[\Sigma]$ is a bounded uniformly continuous function and this, for any choice of T . The construction of a viscosity solution to (19) is thus a standard issue (see Remark 4.6 for the a priori Lipschitz bound).

(ii) The next step to prove that χ is a contraction consists in estimating $\Phi[\Sigma_1] - \Phi[\Sigma_2]$ and proving that, for $\Sigma_1, \Sigma_2 \in E$, we have

$$\|\Phi[\Sigma_1] - \Phi[\Sigma_2]\|_{L^\infty([0, T] \times \mathbb{R})} \leq KT \|\Sigma_1 - \Sigma_2\|_{L^\infty([0, T])}. \quad (20)$$

We just provide a (half) formal argument which can be easily justified. To do so, we remark that since H is Lipschitz continuous, $w := \Phi[\Sigma_1] - \Phi[\Sigma_2]$ is a subsolution of

$$\frac{\partial}{\partial s} w \leq K \left| \frac{\partial}{\partial x} w \right| + K |(\Sigma_1 - \Sigma_2)(s)| \quad \text{in } (0, T) \times \mathbb{R}.$$

where K depends on the Lipschitz constant of H and the L^∞ -norms of β and $\frac{\partial}{\partial x} d$. By standard comparison results, since $w(0, x) = 0$ in \mathbb{R} , we have, for $s \in [0, T]$

$$w(x, s) \leq K \int_0^s |(\Sigma_1 - \Sigma_2)(s)| ds.$$

Exchanging the role of $\Phi[\Sigma_1]$ and $\Phi[\Sigma_2]$, we finally deduce the inequality (20).

(iii) Now we consider the quantity $\|\chi(\Sigma_1) - \chi(\Sigma_2)\|_{L^\infty([0, T])}$. This maximum is achieved at a point $s_0 \in [0, T]$, $\|\chi(\Sigma_1) - \chi(\Sigma_2)\|_{L^\infty([0, T])} = |(\chi(\Sigma_1) - \chi(\Sigma_2))(s_0)|$ and we may assume that $(\chi(\Sigma_1) - \chi(\Sigma_2))(s_0) > 0$ (otherwise we exchange the role of these two functions). Then, we have

$$\begin{aligned} (\chi(\Sigma_1) - \chi(\Sigma_2))(s_0) &= \Sigma_1(s_0) - \Sigma_2(s_0) \\ &+ \kappa \left(\sup_{x \in \mathbb{R}} (\Phi[\Sigma_1](s_0, x) - d(x)\Sigma_1(s_0)) - \sup_{x \in \mathbb{R}} (\Phi[\Sigma_2](s_0, x) - d(x)\Sigma_2(s_0)) \right), \end{aligned}$$

and assuming that the second sup is achieved at some \bar{x} (otherwise we argue with approximate maximisers), we deduce

$$(\chi(\Sigma_1) - \chi(\Sigma_2))(s_0) = (1 - \kappa d(\bar{x})) (\Sigma_1(s_0) - \Sigma_2(s_0)) + \kappa (\Phi[\Sigma_1](s_0, \bar{x}) - \Phi[\Sigma_2](s_0, \bar{x})),$$

and finally

$$(\chi(\Sigma_1) - \chi(\Sigma_2))(s_0) \leq (1 - \kappa d(\bar{x})) (\Sigma_1(s_0) - \Sigma_2(s_0)) + \kappa KT \|\Sigma_1 - \Sigma_2\|_{L^\infty([0, T])}.$$

At that level, we choose κ in such a way that $0 < \kappa d_m < 1$ and therefore

$$0 < 1 - \kappa d(\bar{x}) \leq 1 - \kappa d_m < 1.$$

Therefore,

$$(\chi(\Sigma_1) - \chi(\Sigma_2))(s_0) \leq (1 - \kappa d_m + \kappa KT) \|\Sigma_1 - \Sigma_2\|_{L^\infty([0, T])}.$$

Choosing finally $KT = d_m/2$, this inequality yields

$$\|\chi(\Sigma_1) - \chi(\Sigma_2)\|_{L^\infty([0, T])} \leq (1 - \kappa d_m/2) \|\Sigma_1 - \Sigma_2\|_{L^\infty([0, T])},$$

which is the announced contraction property and Lemma 4.3 is proved. \square

Now we return to the proof of Theorem 4.2. It is clear from the very definition of χ that the fixed points of χ are the Σ 's for which (Φ, Σ) is a solution of (16)-(17)-(18) on the time interval $(0, T)$. Since χ is a contraction, it has a unique fixed point and therefore such a solution is unique on $(0, T)$. But since T depends only on the data of the problem, we can repeat the same arguments successively on the time interval $(T, 2T)$, $(2T, 3T)$,... and therefore we obtain the existence and uniqueness of the solution on $(0, +\infty)$.

Finally to deal with the original formalution (11), in the time t for the solution φ , it suffices to remark that, since

$$\tau'(t) = \frac{1}{(\Sigma'(t))^{1/2}},$$

we automatically deduce that the change of time is uniquely determined and then the uniqueness of φ follows. \square

Remark 4.4 *We point out that, in the above arguments, the one-dimensional aspect plays no role; the same result holds true if the equation is set in higher dimension. In the same way, b , β and d may depend on t with easy adaptations of the assumptions.*

Remark 4.5 *Also the Lipschitz continuity of the initial data can be relaxed. If we assume it is uniformly continuous, then we arrive to a uniformly continuous solution. Indeed, once we have a Lipschitz theory, we can use the contraction property to obtain that two solutions also satisfy*

$$\|\Phi_1 - \Phi_2\|_{L^\infty([0, T] \times \mathbb{R})} \leq C \|\varphi_1 - \varphi_2\|_{L^\infty(\mathbb{R})}.$$

Remark 4.6 *The difficulty to handle the case where the hamiltonian is given through (8) with the assumption (2) and β non-constant comes from the Lipschitz regularity of the solution to (19). Indeed, in order to estimate the space derivative, we set $p(t, x) = \frac{\partial}{\partial x} \Phi(t, x)$ and compute for a solution to (19),*

$$\frac{\partial}{\partial t} p(t, x) = \frac{b'(x)}{1 + \varrho(t)} - (1 + \varrho(t)) d'(x) + \frac{1}{1 + \varrho(t)} \beta H' \left(\frac{\partial}{\partial x} \varphi(t, x) \right) \frac{\partial p(t, x)}{\partial x}.$$

This is a transport equation with source, and thus we have indeed the a priori bound

$$\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|p^0(\cdot)\|_{L^\infty(\mathbb{R})} + Mt,$$

with

$$M = \frac{\|b'(x)\|_{L^\infty(\mathbb{R})}}{1 + \varrho_m} + (1 + \varrho_M) \|d'(x)\|_{L^\infty(\mathbb{R})}.$$

Then we can now also obtain an estimate on the time derivative because the right hand side of (19) is bounded. If β is not constant, that creates a superlinear extra term that allows for blow-up in the equation on p .

5 An open question

One of the question that is not solved precisely by the H.-J. equation is to precise the limit, in the weak-measure sense, of n_ε in the equation (1). As long as $\max_{x \in \mathbb{R}} \varphi(t, x)$ is attained at a unique point $\bar{x}(t)$, we have

$$n_\varepsilon \rightarrow n = \varrho(t)\delta(x - \bar{x}(t)). \quad (21)$$

Such a population is called *monomorphic*.

A more general statement for the limit would be useful to analyze the following model which generalizes the equation (1),

$$\left\{ \begin{array}{l} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(t, x) = \left[\frac{b(x)}{1+\varrho_\varepsilon^{(1)}(t)} - (1 + \varrho^{(2)}r_\varepsilon(t))d(x) \right] n_\varepsilon(t, x) + \frac{1}{1+\varrho_\varepsilon^{(1)}(t)} \int \frac{1}{\varepsilon} K\left(\frac{x-y}{\varepsilon}\right) \beta(y) n_\varepsilon(t, y) dy, \\ \varrho_\varepsilon^{(i)}(t) = \int \eta^{(i)}(x) n_\varepsilon(t, x) dx, \quad i = 1, 2, \\ n_\varepsilon(t = 0, x) = n_\varepsilon^0(x) \geq 0, \quad n_\varepsilon^0 \in L^1 \cap L^\infty(\mathbb{R}). \end{array} \right. \quad (22)$$

This system represents a case where the total population pressure is different on birth, on death (and one could go further and distinguish the pressure on neutral birth and mutations). The different weights $\eta^{(i)} \geq 0$ are given and measure these different pressures.

In the limit, the three Lagrange multipliers are associated with the same constraint. Indeed, arguing as in Section 3, we obtain the constrained H.-J. Eq.

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \varphi(t, x) = \frac{b(x)}{1+\varrho^{(1)}(t)} - (1 + \varrho^{(2)}(t))d(x) + \frac{\beta(x)}{1+\varrho^{(1)}(t)} H\left(\frac{\partial}{\partial x} \varphi(t, x)\right), \\ \varphi(t = 0) = \varphi^0 \in \mathcal{Lip}(\mathbb{R}), \quad \max_{x \in \mathbb{R}} \varphi^0(x) = 0, \\ \max_{x \in \mathbb{R}} \varphi(t, x) = 0 \quad \forall t > 0. \end{array} \right. \quad (23)$$

Of course this system has several solutions. For instance, one of them consists in $\varrho^{(1)}(t) = \varrho^{(2)}(t)$, but it is not the most reasonable here because it is not compatible with the interpretation of the limit n_ε . To be consistent with the definitions of the $\varrho_\varepsilon^{(i)}(t)$ and with the monomorphic interpretation (22) and (21), one would like to add the relation

$$\exists \varrho(t) \quad \text{such that} \quad \varrho^{(i)}(t) = \eta^{(i)}(\bar{x}(t))\varrho(t), \quad i = 1, 2.$$

This system does not enter our theory because it uses the knowledge of $\bar{x}(t)$ which has not been defined. Nevertheless we can hope for uniqueness in this case (once stated in the appropriate weak form).

But more deeply for non-uniqueness, one could face, as in [12] and [7], the problem of *dimorphic* situations. This means that (21) does not hold true anymore and we rather have

$$n_\varepsilon \rightarrow n = \varrho_1(t)\delta(x - \bar{x}_1(t)) + \varrho_2(t)\delta(x - \bar{x}_2(t)). \quad (24)$$

Then, these two independent weights can be derived from two independent Lagrange multipliers $\varrho^{(1)}$, $\varrho^{(2)}$, and this is the main difference with the model (1). The main open question is to discover which condition can be added to the system (22) to handle the transitions between monomorphic and dimorphic situations.

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