CONVEX OPTIMIZATION VIA LINEARIZATION

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Notation

X denotes a l.c. Hausdorff t.v.s and X^* its topological dual endowed with the weak*-topology.

$\mathbb{R}_+^{(T)} = \left\{\lambda : T \to \mathbb{R}_+ \mid |\text{supp}\,\lambda| < +\infty\right\}$, with $\text{supp}\,\lambda = \{t \in T \mid \lambda_t \neq 0\}$.

The *asymptotic* (or *recession*) *cone* of $C \subset X$ is

$$
\begin{aligned}
C^\infty &= \{z \in X \mid C + z \subset C\} \\
&= \left\{ z \in X \;\middle|\; \begin{array}{l} \exists c \in C \text{ such that} \\ c + \lambda z \in C \;\; \forall \lambda \geq 0 \end{array} \right\} \\
&= \{z \in X \; c + \lambda z \in C \;\; \forall c \in C \text{ and } \forall \lambda \geq 0\}
\end{aligned}
$$

For a set $D \subset X$, the *normal cone* of D at x is

$$N_D(x) = \begin{cases} \{u \in X^* \mid u(y-x) \leq 0 \text{ for all } y \in D\}, & \text{if } x \in D, \\ N_D(x) = \varnothing, & \text{if } x \notin D. \end{cases}$$

The effective domain, the graph, and the epigraph of $h : X \to \mathbb{R} \cup \{+\infty\}$ are denoted by

$$\text{dom} h, \quad \text{gph}\, h \text{ and } \text{epi}\, h$$

The *subdifferential* of h at a point $x \in \text{dom} h$ is

$$\partial h(x) = \{u \in X^* \mid h(y) \geq h(x) + u(y-x) \quad \forall y \in X\}$$

The *conjugate* of h is

$$h^*(v) = \sup\{v(x) - h(x) \mid x \in \text{dom } h\}$$

h^* is also a proper l.s.c. convex function and its conjugate (*biconjugate* of h) is $h^{**} = h$.

In particular, if $f(x) = a'x + b$, then

$$f^*(u) = \sup_{x \in \mathbb{R}^n} \left\{(u-a)'x - b\right\} = \delta_{\{a\}}(u) - b,$$

i.e., $f^* = \delta_{\{a\}} - b$.

The *asymptotic function* of h is h^∞ such that

$$\operatorname{epi} h^\infty = (\operatorname{epi} h)^\infty.$$

The *indicator function* of $D \subset X$ is

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D \\ +\infty, & \text{if } x \notin D. \end{cases}$$

If $D \neq \emptyset$ is closed and convex, then δ_D is a proper l.s.c. convex function.

The *support function* of D is

$$\delta^*_D(u) = \delta^*_{\mathrm{cl}(\mathrm{conv} D)}(u) = \sup_{x \in D} u(x), \ \ u \in X^*.$$

In particular,

$$\delta^*_{\mathbb{R}^n}(u) = \sup_{x \in \mathbb{R}^n} u'x = \delta_{\{0_n\}}(u) \Rightarrow \delta^*_{\mathbb{R}^n} = \delta_{\{0_n\}}.$$

LINEARIZING CONVEX SYSTEMS

We consider

$$\sigma := \{f_t(x) \leq 0, t \in T;\ x \in C\},$$

where

♦ T is an arbitrary (possibly infinite) index set,

♦ C is a nonempty closed convex subset of X, and

♦ $f_t : X \to \mathbb{R} \cup \{+\infty\}$ is a proper l.s.c. convex function, $\forall\ t \in T$.

In many applications $C = X$, in which case we write

$$\sigma := \{f_t(x) \leq 0,\ t \in T\}$$

Given $t \in T$,

$$f_t(x) \leq 0$$

$$\begin{aligned}
&\iff f_t^{**}(x) \leq 0 \\
&\iff u_t(x) - f_t^*(u_t) \leq 0,\ \forall u_t \in \text{dom} f_t^* \\
&\iff u_t(x) \leq f_t^*(u_t),\ \forall u_t \in \text{dom} f_t^* \\
&\iff u_t(x) \leq f_t^*(u_t) + \alpha, \\
&\qquad\qquad\qquad \forall u_t \in \text{dom} f_t^*, \quad \forall \alpha \in \mathbb{R}_+
\end{aligned}$$

Analogously,

$$x \in C \Leftrightarrow \delta_C(x) \leq 0$$

$$\begin{aligned}
&\iff u(x) \leq \delta_C^*(u),\ \forall u \in \text{dom}\delta_C^* \\
&\iff u(x) \leq \delta_C^*(u) + \beta, \\
&\qquad\qquad\qquad \forall u \in \text{dom}\delta_C^*\ ,\ \forall \beta \in \mathbb{R}_+
\end{aligned}$$

Consequently, the following linear systems are equivalent to σ :

$$\left\{ \begin{array}{l} u_t(x) \leq f_t^*(u_t), \;\; u_t \in \text{dom} f_t^*, \;\; t \in T \\ u(x) \leq \delta_C^*(u), \;\; u \in \text{dom}\delta_C^* \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} u_t(x) \leq f_t^*(u_t) + \alpha, \;\; u_t \in \text{dom} f_t^*, \;\; t \in T, \;\; \alpha \in \mathbb{R}_+ \\ u(x) \leq \delta_C^*(u) + \beta, \;\; u \in \text{dom}\delta_C^*, \;\; \beta \in \mathbb{R}_+ \end{array} \right\}$$

EXISTENCE THEOREMS

For linear systems [(Chu, 1966), Goberna et al. (1995)]:

(i) $\{a_t(x) \leq b_t,\ t \in T\}$ is consistent

$\Updownarrow$

(ii) $(0, -1) \notin \text{cl cone}\{(a_t, b_t),\ t \in T\}$

$\Updownarrow$

(iii) $\text{cl cone}\{(a_t, b_t),\ t \in T;\ (0, 1)\}$
$\neq \text{cl cone}\{a_t,\ t \in T\} \times \mathbb{R}$.

Associating with σ the convex cones

$$
\begin{aligned}
M &= \text{cone}\left\{\bigcup_{t\in T} \text{dom}\, f_t^* \cup \text{dom}\, \delta_C^*\right\}\\
N &= \text{cone}\left\{\bigcup_{t\in T} \text{gph}\, f_t^* \cup \text{gph}\, \delta_C^*\right\}\\
K &= \text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}\\
P &= \text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* + \text{epi}\delta_C^*\right\}
\end{aligned}
$$

we get

(i) σ is consistent

$\Updownarrow$

(ii) $(0,-1) \notin$ cl K (cl N, cl P)

$\Updownarrow$

(iv) cl $K \neq$ cl $M \times \mathbb{R}$

1. Short history of these cones

K: Chu (1966), in LISs.

M, N and K: Charnes, Cooper & Kortanek (1965-1969), in LSIP.

P: Jeyakumar, Dinh & Lee (2004), in CP.

2. Closedness

$$\begin{array}{c} P \text{ is weak*-closed} \\ \Downarrow \\ K \text{ is weak*-closed} \\ \Uparrow \\ N \text{ is weak*-closed and } \sigma \text{ is consistent} \end{array}$$

The converse statements are not true and the consistency of σ is not superfluous.

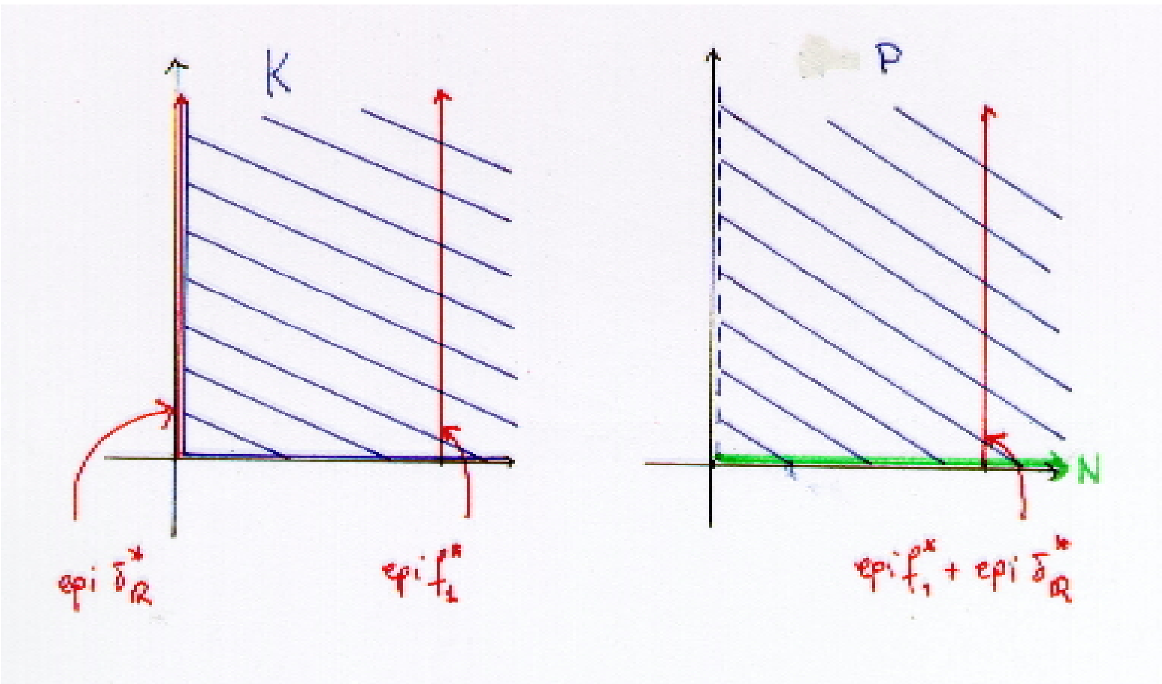
Example 1: $C = X = \mathbb{R}$ and $\sigma = \{f_1(x) = x \leq 0\}$.

Since $f_1^* = \delta_{\{1\}}$ and $\delta_{\mathbb{R}}^* = \delta_{\{0\}}$,

$$\text{epi}\,\delta_{\mathbb{R}}^* = \mathbb{R}_+(0,1)$$

and

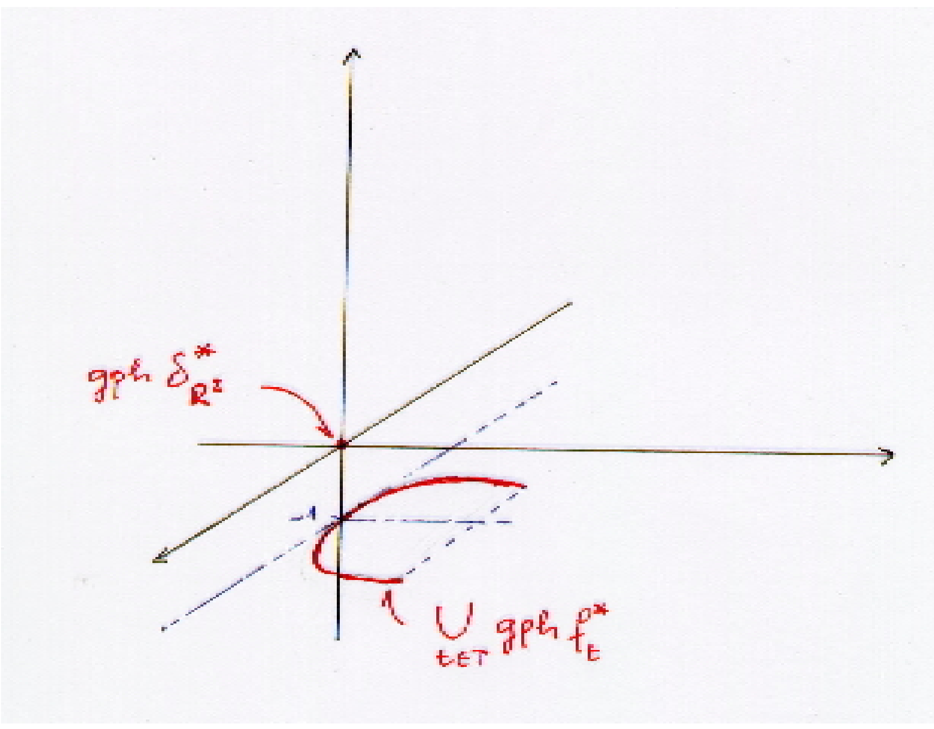
$$\text{epi}\,f_1^* = \text{epi}\,f_1^* + \text{epi}\,\delta_{\mathbb{R}}^* = (1,0) + \mathbb{R}_+(0,1).$$



Example 2: $C = X = \mathbb{R}^2$ and

$$\sigma = \left\{ f_t(x) = tx_1 + t^2 x_2 + 1 \leq 0, t \in [-1, 1] \right\}$$

Since $\operatorname{gph} \delta^*_{\mathbb{R}^2} = \{(0,0,0)\}$ and $\operatorname{gph} f_t^* = \left\{ \left(t, t^2, -1\right) \right\} \forall t$, $N = \operatorname{cone} \left\{ \bigcup_{t \in T} \operatorname{gph} f_t^* \right\}$ is closed whereas K is non-closed.



The following *recession condition* was introduced by Borwein (1981):

(RC)

$$C^{\infty} \cap \{x \in X \mid f_t^{\infty}(x) \leq 0,\ t \in T\} = \{0\}$$

Generalized Fan's theorem: if either

(a) K (N) is weak*-closed, or

(b) **(RC)** holds and K (N) is solid if X is infinite dimensional

then σ is consistent iff

$\forall \lambda \in \mathbb{R}_+^{(T)}$, $\exists x_\lambda \in C$ such that

$$\sum_{t \in T} \lambda_t f_t(x_\lambda) \leq 0$$

Some previous versions

Under closedness conditions

Bohnenblust, Karlin & Shapley (1950), with $X = \mathbb{R}^n$ and C compact.

Fan (1957), assuming that $f_t : X \to \mathbb{R}\ \forall t \in T$ and C is compact.

Shioji & Takahashi (1988), with C compact.

Under recession conditions

Rockafellar (1970), with $X = \mathbb{R}^n$.

ASYMPTOTIC FARKAS LEMMA

From now on we assume that σ is consistent with solution set $A \neq \emptyset$.

Given $v \in X^*$ and $\alpha \in \mathbb{R}$, then $v(x) \leq \alpha$ is a consequence of the consistent system $\{a_t(x) \leq b_t,\ t \in T\}$ iff

$$(v, \alpha) \in \operatorname{cl} \operatorname{cone} \{(a_t, b_t),\ t \in T;\ (0, 1)\}$$

Applying this result (Chu, 1966) to the linearization of σ we get the

Asymptotic Farkas Lemma for linear inequalities: given $v \in X^*$ and $\alpha \in \mathbb{R}$, $v(x) \leq \alpha$ is consequence of σ iff $(v, \alpha) \in \operatorname{cl} K$.

From now on $f : X \to \mathbb{R} \cup \{+\infty\}$ denotes a proper l.s.c. convex function.

Another consequence is the

Asymptotic Farkas Lemma for convex inequalities: $f(x) \leq \alpha$ is a consequence of σ iff $(0, \alpha) + \text{epi} f^* \in \text{cl}\, K$.

From here we get the following

Characterization of the set containment convex-convex: $A \subset \{x \in X \mid h_w(x) \leq 0, w \in W\}$ (h_w as f_t) iff

$$\bigcup_{w \in W} \text{epi} h_w^* \subset \text{cl}\, K$$

Precedents:

For Farkas' Lemma: see Jeyakumar (2001).

For set containment:

Goberna & López (1998), with $C = X = \mathbb{R}^n$ and f_t and h_w affine $\forall t \in T$, $\forall w \in W$.

Mangasarian (2002), with $C = X = \mathbb{R}^n$ and $|T| < \infty$ and $|W| < \infty$.

Jeyakumar (2003), with $C = X = \mathbb{R}^n$ and h_w affine $\forall w \in W$.

Goberna, Jeyakumar & Dinh (2006), with $C = X = \mathbb{R}^n$.

FARKAS-MINKOWSKI SYSTEMS

The following concept was introduced by Charnes, Cooper & Kortanek (1965), in LSIP:

σ is *FM* if K is weak*-closed.

Since $\operatorname{cl} K = \operatorname{epi}\delta_A^*$, $\{\delta_A(x) \leq 0\}$ is a FM representation of A.

If σ is FM, then every continuous linear consequence of σ is also consequence of a finite subsystem of σ. The converse statement holds if σ is linear (but not if σ is convex).

Example 3: Let $X = C = \mathbb{R}^n$ and $\sigma = \left\{ f_1(x) := \frac{1}{2} \|x\|^2 \leq 0 \right\}$.

Since $f_1^*(v) = \frac{1}{2} \|v\|^2$, $K = \left(\mathbb{R}^n \times \mathbb{R}_{++} \right) \cup \{0\}$ is not closed.

Non-asymptotic Farkas lemma for linear inequalities: let σ be FM, $v \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Then:

(i) $v(x) \geq \alpha$ is consequence of σ

$\Updownarrow$

(ii) $-(v, \alpha) \in K$

$\Updownarrow$

(iii) $\exists \overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$v(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq \alpha, \ \forall x \in C$$

Non-asymptotic Farkas Lemma for convex inequalities: if σ is FM, then $f(x) \leq \alpha$ is consequence of σ iff $(0, \alpha) + \text{epi} f^* \subset K$.

Asymptotic Farkas Lemma for reverse-convex inequalities: If σ is FM, then $f(x) \geq \alpha$ is consequence of σ iff

$$(0, -\alpha) \in \text{cl}\left(\text{epi} f^* + K\right).$$

From here we get the following **characterization of the set containment convex-reverse convex:** $A \subset \{x \in X \mid h_w(x) \geq 0, w \in W\}$ (h_w as f_t) iff

$$0 \in \bigcap_{w \in W} \text{cl}\left\{\text{epi} h_w^* + K\right\}$$

<u>Precedents:</u> Jeyakumar (2003) and Bot & Wanka (2005), in CSISs.

The following *closedness condition* was introduced by Burachik & Jeyakumar (2005):

(CC)

$$\operatorname{epi} f^* + \operatorname{cl} K \text{ is weak}^*\text{-closed.}$$

Each of the following conditions implies **(CC)**:

(i) $\operatorname{epi} f^* + K$ is weak*-closed.

(ii) σ is FM and f is linear.

(ii) σ is FM and f is continuous at some point of A.

Non-asymptotic Farkas Lemma for reverse-convex inequalities: if σ is FM, **(CC)** holds, and $\alpha \in \mathbb{R}$, then

(i) $f(x) \geq \alpha$ is consequence of σ

$\Updownarrow$

(ii) $(0, -\alpha) \in \text{epi} f^* + K$

$\Updownarrow$

(iii) $\exists \overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq \alpha, \ \forall x \in C$$

<u>Precedents:</u> Gwinner (1987) and Dinh, Jeyakumar & Lee (2005) under strong assumptions.

FM SYSTEMS IN CONVEX OPTIMIZATION

From now on we consider the CP problem

$$\begin{array}{lll} (\mathrm{P}) & \text{Minimize} & f(x) \\ & \text{s.t.} & f_t(x) \leq 0, \ t \in T, \\ & & x \in C. \end{array}$$

Solvability theorem: if $X = \mathbb{R}^n$ and σ satisfies **(RC)**, then (P) is solvable.

This is not true for reflexive Banach spaces, unless $f + \delta_A$ is coercive (Zalinescu, 2002).

Example 4: let $X = \ell^2$ (Hilbert space),

$$C := \left\{ x = \{\xi_n\} \in \ell^2 \mid |\xi_n| \le n \ \forall n \in \mathbb{N} \right\},$$

and $f(x) := \sum_{n=1}^{\infty} \frac{\xi_n}{n}$, with $f \in X'$.

C is a closed convex set which is not bounded (because $ne_n \in C$, for every $n \in \mathbb{N}$) and such that $C^\infty = \{0\}$. Thus **(RC)** holds.

Consider $c^k := (\gamma_n^k)_{n \ge 1,}\ k = 1, 2, ...,$

$$\gamma_n^k := \begin{cases} -n, & \text{if } n \le k, \\ 0, & \text{if } n > k. \end{cases}$$

We have $\left\{c^k\right\} \in C$ and $f(c^k) = -k$, $k \in \mathbb{N}$, so that f is not bounded from below on C and no minimizer exists.

KKT optimality theorem: assume that σ is FM, that **(CC)** holds, and let $a \in A \cap \operatorname{dom} f$. Then a is a minimizer of (P) iff

$\exists \lambda \in \mathbb{R}_+^{(T)}$ such that

(i) $\partial f_t(a) \neq \emptyset \; \forall t \in \operatorname{supp} \lambda$

(ii) $\lambda_t f_t(a) = 0$, $\forall t \in T$, and

(iii) $0 \in \partial f(a) + \sum_{t \in T} \lambda_t \partial f_t(a) + N_C(a)$

Precedent: without the FM property, the optimality condition is $0 \in \partial f(a) + N_A(a)$ (Burachik & Jeyakumar, 2005).

Now we consider the *parametric problem* (P_u), for $u \in \mathbb{R}^T$,

$$
\begin{array}{lll}
(\mathrm{P}_u) & \text{Minimize} & f(x) \\
 & \text{subject to} & f_t(x) \le u_t, \;\; t \in T, \\
 & & x \in C,
\end{array}
$$

with feasible set A_u.

Defining $\psi(x,u) := f(x) + \delta_{A_u}(x)$, $\psi : X \times \mathbb{R}^T \to \mathbb{R} \cup \{+\infty\}$, we can write

$$
(\mathrm{P}_u) \quad \text{Minimize} \quad \psi(x,u), x \in X.
$$

Then

$$(\text{P}) \equiv (\text{P}_0) \quad \text{Minimize} \quad \psi(x, 0), x \in X.$$

The *dual problem* of (P) is

$$(\text{D}) \quad \text{Maximize} \quad -\psi^*(0, \lambda), \lambda \in \mathbb{R}_+^{(T)}.$$

Duality theorem: if (P) is bounded, σ is FM, and **(CC)** holds, then v (D) = v (P) and (D) is solvable.

Precedents: Rockafellar (1974) and Bonnans & Shapiro (2000).

Consider the *Lagrange function* $L : X \times \mathbb{R}^{(T)} \to \mathbb{R} \cup \{+\infty\}$, where $L(x, \lambda)$ is

$$\begin{cases} f(x) + \sum_{t \in T} \lambda_t f_t(x), & \text{if } x \in C, \lambda \in \mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Lagrange optimality theorem: suppose that σ is FM and that **(CC)** holds. Then a point $a \in A$ is minimizer of (P) iff

$\exists \lambda_0 \in \mathbb{R}_+^{(T)}$ such that (a, λ_0) is a *saddle point* of the Lagrangian function L, i.e.,

$$L(a, \lambda) \leq L(a, \lambda_0) \leq L(x, \lambda_0), \; \forall \lambda \in \mathbb{R}_+^{(T)} \; \forall x \in C.$$

Then λ_0 is a maximizer of (D).

Denote by $v(u)$ the value of (P_u), so that v(P) $= v(0)$. The following stability concepts (Laurent, 1972) involve the *value function* $v : \mathbb{R}^T \to \overline{\mathbb{R}}$, whose directional derivative at 0 in the direction u is denoted by $v'(0, u)$.

(P) is called:

♦ *inf-stable* if $v(0) \in \mathbb{R}$ and v is l.s.c. at 0.

♦ *inf-dif-stable* if $v(0) \in \mathbb{R}$ and $\exists \lambda_0 \in \mathbb{R}^{(T)}$ such that

$$v'(0, u) \geq \lambda_0(u), \;\; \forall u \in \mathbb{R}^T.$$

These concepts are related as follows:

(P) inf-stable

$\Updownarrow$

$\mathsf{v}(\mathsf{D}) = \mathsf{v}(\mathsf{P}) \in \mathbb{R}$ (called *normality* in Zalinescu, 2002).

(P) inf-dif-stable

$\Updownarrow$

$\partial v(0) \neq \emptyset$ (called *calmness* in Clarke, 1976)

$\Updownarrow$

$\mathsf{v}(\mathsf{D}) = \mathsf{v}(\mathsf{P})$ and (D) is solvable.

Thus, (P) inf-dif-stable $\Longrightarrow$ (P) inf-stable

Stability Theorem: if (P) is bounded, σ is FM, and **(CC)** holds, then (P) is inf-dif-stable.

LOCALLY FARKAS-MINKOWSKI SYSTEMS

The following local c.q. was introduced by Puente & Vera de Serio (1999), in LSIP. It is also equivalent to the so-called basic c.q. in CP (Hiriart Urruty & Lemarechal, 1993) if $x \in \operatorname{int} A$ and $\sup_{t \in T} f_t$ is continuous at x :

σ is *LFM* at $x \in A$ if

$$N_A(x) \subseteq N_C(x) + \text{cone}\left(_{t \in T(x)} \partial f_t(x)\right),$$

where $T(x) := \{t \in T \mid f_t(x) = 0\}$.

σ is said to be *LFM* if it is LFM at every feasible point $x \in A$.

As a consequence of the optimality theorem,

$$\sigma \text{ FM} \Rightarrow \sigma \text{ LFM}$$

If σ is LFM at $x \in A$, then every continuous linear consequence of σ which is binding at x is also consequence of a finite subsystem of σ. The converse statement holds if σ is linear (but not if σ is convex).

σ is LFM at a iff the KKT optimality theorem holds for any l.s.c. convex function f such that $a \in \operatorname{dom} f$ and f is continuous at some point of A.

Precedent: Li & Ng (2005), with real-valued functions and basic c.q.

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