Optimization in Finance: Portfolio selection models

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Abstract

- Portfolio selection problem deals with how to form a satisfying portfolio, taking into account the uncertainty involved in the behavior of the financial markets.

- Markowitz (1952) established the relationship between the mean and variance of the investment in the framework of risk-return trade-off. Since then a variety of enlarged and improved models have been developed in several directions.

- Some models of portfolio management combines probability theory and optimization theory to represent the behavior of the economic agents. These representations of return and risk have permitted to apply different optimization tools to the portfolio management.

Abstract

- In this talk we provide some new models for portfolio selection in which the returns on the securities are considered as fuzzy numbers rather than random variables.
- In order to find the portfolio that minimizes the risk in achieving a given level of return we introduce different approaches. In some of them the expected total return is considered otherwise is the fuzzy total return.
- The return on each asset and their membership functions are described using historical data and the risk of the investment is approximated by using interval-valued means which evaluate the downside risk for a given portfolio.
- In order to illustrate the performance of our methods we have used weekly returns corresponding to a selection of assets from the Spanish Stock Market.

Outline

- Portfolio selection models
- Portfolio selection with fuzzy returns
    - Fuzzy background
    - Fuzzy downside risk models
- Portfolio selection with linear programs
- Portfolio selection with semi-infinite optimization
- Numerical examples

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Portfolio selection models

- Modern portfolio selection theory usually deals with two opposite concepts: risk aversion and maximization of returns. The main point of the modelling of this problem is how the risk and assets profitability are defined and measured.
- Classic models consider an asset return as a random variable and its profitability is defined as the mathematical expectation of that random variable, while the risk of the portfolio is measured by means of the variance.
- Since the formulation of the mean-variance model a variety of enlarged and improved models have been developed in several directions. One dealt with alternative portfolio selection models, for instance, a mean-semi-variance model, a mean-absolute deviation model or mean-downside risk models. Another approach concerned the modelling of uncertainty and the knowledge of the experts provided by fuzzy set theory.

Portfolio selection models

In a standard formulation of Markowitz model (Mean-Variance) we have the following quadratic programming problem, for a given expected return ρ:

$$
\begin{array}{rl}
\text{Min} & \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \\
\text{s.t.} & \sum_{j=1}^{n} x_j E(R_j) \geq \rho \\
& \sum_{j=1}^{n} x_j = 1 \\
& l_j \leqslant x_j \leqslant u_j \qquad j = 1 \ldots n
\end{array}
$$

- x_j is the percentage of investment in the asset j^{th}
- R_j is the random variable representing the return of asset j^{th}
- σ_{ij} is the covariance between R_i and R_j
- l_j, u_j represent the maximum and minimum amount of the total fund which can be invested in the asset j^{th}

Portfolio selection models

According to Konno and Yamazaki (1991), the (MV) problem is equivalent to the following model (MAD), which minimizes the sum of absolute deviations from the averages associated with the x_j choices, when the assets are multivariate normally distributed:

$$\begin{array}{ll} \text{Min} & E(|\sum_{j=1}^{n} R_j x_j - E(\sum_{j=1}^{n} R_j x_j)|) \\ \text{s.t.} & \sum_{j=1}^{n} x_j E(R_j) \geq \rho \\ & \sum_{j=1}^{n} x_j = 1 \\ & l_j \leqslant x_j \leqslant u_j \qquad j = 1 \ldots n \end{array}$$

- This approach permits to avoid one of the main drawbacks associated to the solution of the Mean-Variance (MV) model: the input problem of estimating $2n + n(n-1)/2$ parameters.

Portfolio selection models

- The (MAD) problem can be converted in a finite LP problem replacing its objective function by:

$$
\begin{array}{lll}
\text{Min} & (1/T)\sum_{k=1}^{T} y_k & \\
\text{s.t.} & y_k + \sum_{j=1}^{n}(r_{jk} - E(R_j))x_j \geq 0 & k = 1 \dots T \\
 & y_k - \sum_{j=1}^{n}(r_{jk} - E(R_j))x_j \geq 0 & k = 1 \dots T
\end{array} \tag{1}
$$

  where the observation of the assets returns over T periods are given and r_{jk} denotes the return of the j^{th} asset at the time k.

- That linear model (LMAD) gives portfolios which involves fewer non-zero components and hence reduces the numerous small transactions that are likely to appear in the (MV) model.

Portfolio selection models

- Simaan (1997) stated that although the minimization of mean-absolute deviation is close to the (MV) formulation they lead to two different efficient sets. The divergence between the two models is due to the fact that each model utilizes different sample statistics and consequently relies on a different set drawn from the sample.
- If the return is normally distributed the minimization of (LMAD) provides similar results as the (MV) formulation but the normality assumption rarely is verified in practice, then different portfolios can be obtained. Júdice *et al* (2003) studied the stability of the portfolios, their expected return and the computational time using both models in real-life capital market and conclude that none model is superior to the other.
- Both models have also been compared with out-of-sample data from shares traded in the Stockholm Stock Exchange (Papahristodoulou and Dotzauer, 2004) and the (MV) model yields higher utility levels and higher degrees of risk aversion in very similar computing times.

Portfolio selection models

The dissatisfaction with the traditional notion of variance as a measure of risk is due to the fact that it makes no distinction between gains and loses. In particular, Markowitz (1959) also proposed to use the semi-variance

$$\mathrm{w}(x) = \mathrm{E}((\mathrm{máx}\{0, \mathrm{E}(\sum_{j=1}^{n} R_j x_j) - \sum_{j=1}^{n} R_j x_j\})^2).$$

From then, several optimization models which consider only the downside risk of a portfolio have been introduced. If the risk is measured by means of the mean-absolute semi-deviation, as proposed Speranza (1993), we have the following downside risk function

$$\mathrm{w}(x) = \mathrm{E}(|\,\mathrm{mín}\{0, \sum_{j=1}^{n} R_j x_j - \mathrm{E}(\sum_{j=1}^{n} R_j x_j)\}|).$$

which can be easily evaluated in contrast with the complexity of computing the semi-variance of a given portfolio.

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Portfolio selection with fuzzy returns

Different elements can be fuzzified for managing the portfolio selection problem.

- Some authors use possibility distributions to model the uncertainty on the returns (Tanaka y Guo, 1999; Carlsson *et al*, 2002; Huang *et al*, 2006).
- Fuzzy numbers can also represent the decision maker aspiration levels for the expected return and risk (Watada, 1997; Arenas *et al*, 2000).
- The imperfect knowledge of the reality can be introduced by means of fuzzy quantities and/or fuzzy constraints (Ortí *et al*, 2002; Leon *et al*, 2002).

In our approach we propose some fuzzy models for portfolio selection based on two issues:

- the approximation of the rates of return on securities by means of fuzzy numbers, and
- the perception that the downside risk is a more realistic description of the preferences of the investor, because this risk function only penalizes the non-desired deviations.

Portfolio selection with fuzzy returns

- We consider a capital market with n risk assets offering uncertain returns.
- x_j is the proportion of the total investment fund devoted to asset j^{th}, the uncertainty on the return of asset j^{th} is modelled by means of fuzzy quantities $\tilde{R}_j = (a_{lj}, a_{uj}, c_j, d_j)_{LR}$ for $j = 1, 2, \ldots, n$ and R_f is the rate of return on the risk-less asset.
- The portfolio may be denoted by $P(x) = \{x_1, x_2, \ldots, x_n\}$ and the total return on the fuzzy portfolio is a convex linear combination of the individual asset returns, as follows:

$$\tilde{R}_P(x) = \sum_{j=1}^{n} x_j \tilde{R}_j,$$

- Different definitions of the average of a fuzzy number can be used to evaluate both the expected return and the risk of a given portfolio $P(x)$.

Portfolio selection with fuzzy returns

- Dubois and Prade (1987) introduce the mean interval of a fuzzy number as a closed interval bounded by the expectations calculated from its lower and upper probability mean values.
- Alternatively, Carlsson and Fullér (2001) define an interval-valued possibilistic mean of fuzzy numbers, their definition being consistent with the extension principle and also based on the set of level-cuts.
- Concerning the measure of investment risk, we will use a fuzzy downside risk function introduced in Leon *et al.* (2004), which evaluates the mean-absolute semi-deviation with respect the total return:

$$\mathrm{w_E}(P(x)) = \mathrm{E}(\mathrm{máx}\{0, \mathrm{E}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}).$$

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Fuzzy background

- A function $L, R : [0, +\infty) \to [0, 1]$ is said to be a reference function of a fuzzy number $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ if it satisfies the following conditions:
  (1) $L(x) = L(-x)$, $R(x) = R(-x)$,
  (2) $L(0) = 1$, $R(0) = 1$,
  (3) $L(x)$ and $R(x)$ are strictly decreasing and upper semi-continuous on $\mathrm{supp}(\tilde{A}) = \{x : \mu_{\tilde{A}}(x) > 0\}$.
- The membership function of an *LR*-fuzzy number $\tilde{A}_i = (a_{li}, a_{ui}, c_i, d_i)_{L_i R_i}$ has the following form:

$$\mu_{\tilde{A}_i}(x) = \begin{cases} L_i(\frac{a_{li}-x}{c_i}) & \text{if } x \leqslant a_{li}, \\ 1 & \text{if } a_{li} \leqslant x \leqslant a_{ui}, \\ R_i(\frac{x-a_{ui}}{d_i}) & \text{if } x \geqslant a_{ui}, \end{cases}$$

  where c_i (respectively d_i) is the left (right) spread and L_i and R_i are the reference functions defining the left and right shapes of $\tilde{A}_i$.

Fuzzy background

- The aggregation of positive linear combinations of *LR*-fuzzy numbers when their reference functions have the same shape, for all L_i and R_i respectively, using Zadeh's extension principle, provides an *LR*-fuzzy number with the corresponding reference function. But this is not the case for differently shaped fuzzy numbers, where this aggregation is defined with respect to the α-level sets of a fuzzy number $\tilde{A}$, i.e. $[\tilde{A}]^\alpha = \{t : \mu_{\tilde{A}}(t) \geqslant \alpha\}$.

- The calculation of the fuzzy expected return and risk of a given portfolio depends both on the characteristics of the *LR*-fuzzy numbers which represent the individual returns and the definition of the average of a fuzzy number.

- We work with membership functions for the individual returns on the assets from the power family and we evaluate all the shape parameters by means of the reverse rating procedure using historical data (Leon and Vercher, 2004).

Fuzzy background

- **Proposition 1** (Dubois and Prade, 1988). Let us assume that $\tilde{A}_i$, for $i = 1 \ldots n$, are *LR*-fuzzy numbers with different shapes and that the addition is based on the minimum operator, then the sum $\tilde{A} = \sum_{i=1}^{n} x_i \tilde{A}_i$, for $x_i \geqslant 0$, fulfills $[\tilde{A}]^\alpha = \sum_{i=1}^{n} [x_i \tilde{A}_i]^\alpha$ for any $\alpha \in [0, 1]$.

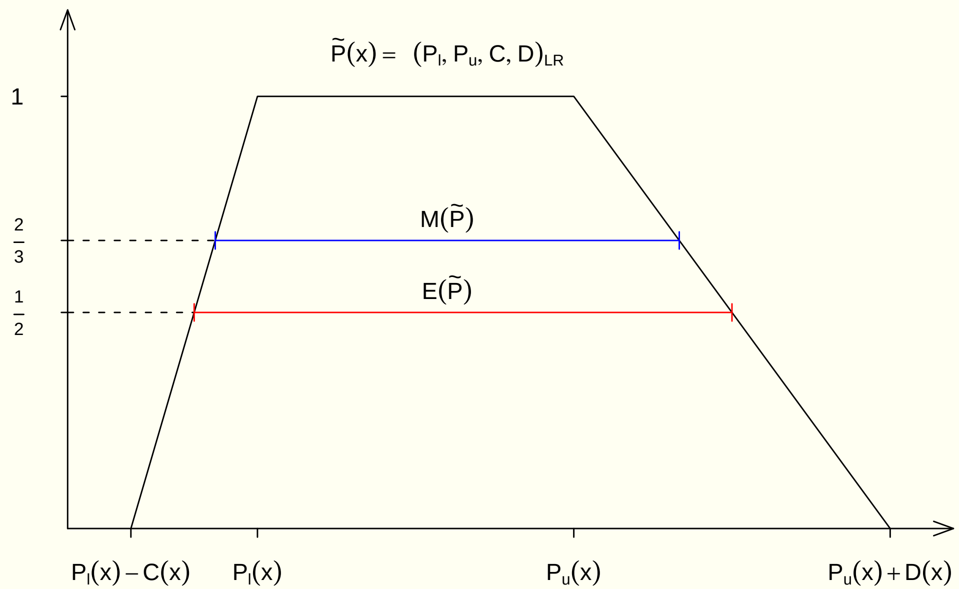
- **Definition** (Dubois and Prade, 1987). The interval-valued expectation of a fuzzy quantity $\tilde{A}$ is the interval $E(\tilde{A}) = [E_*(\tilde{A}), E^*(\tilde{A})]$ whose endpoints are: $E_*(\tilde{A}) = \int_0^1 (\inf \tilde{A}_\alpha)\, d\alpha$ and $E^*(\tilde{A}) = \int_0^1 (\sup \tilde{A}_\alpha)\, d\alpha$.

- **Definition** (Carlsson and Fullér, 2001). The interval-valued possibilistic mean of a fuzzy quantity $\tilde{A}$ is the interval $M(\tilde{A}) = [M_*(\tilde{A}), M^*(\tilde{A})]$ whose endpoints are: $M_*(\tilde{A}) = 2\int_0^1 \alpha(\inf \tilde{A}_\alpha)\, d\alpha$ and $M^*(\tilde{A}) = 2\int_0^1 \alpha(\sup \tilde{A}_\alpha)\, d\alpha$.

Fuzzy background

For instance, if the returns are modelled as trapezoidal fuzzy numbers the fuzzy total return is also a number of this type and the relationship between the above interval-valued mean definitions is the following:

- Dubois and Prade (1987): $E(\tilde{R}_P(x)) = [P_l - \frac{1}{2}\, C, P_u + \frac{1}{2}\, D]$
- Carlsson and Fullér (2001): $M(\tilde{R}_P(x)) = [P_l - \frac{1}{3}\, C, P_u + \frac{1}{3}\, D]$



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Fuzzy downside risk models

The fuzzy portfolio selection problem with downside risk can be formulated under different assumptions, for instance:

$$\begin{array}{rll} \text{(I) Min} & \mathrm{w_E}(P(x)) & \\ \text{s.t.} & \mathrm{E}(\sum_{j=1}^{n} x_j \tilde{R}_j) \geq \rho_0 & \\ & \sum_{j=1}^{n} x_j = 1 & \\ & l_j \leqslant x_j \leqslant u_j & j = 1 \dots n \end{array}$$

$$\begin{array}{rll} \text{(II) Min} & \mathrm{w_E}(P(x)) & \\ \text{s.t.} & \sum_{j=1}^{n} x_j \tilde{R}_j \gtrsim \tilde{R}_f & \\ & \sum_{j=1}^{n} x_j = 1 & \\ & l_j \leqslant x_j \leqslant u_j & j = 1 \dots n \end{array}$$

Fuzzy downside risk models

- Objective function: minimizing the fuzzy downside risk
  - $\mathrm{w_E}(P(x)) \stackrel{defuz}{\longrightarrow}$ linear function
  - $\mathrm{w_M}(P(x)) \stackrel{defuz}{\longrightarrow}$ linear function
- Constraint: achieving a given return
  - $\mathrm{E}(\tilde{R}_P(x)) \geq \rho \stackrel{defuz}{\longrightarrow}$ linear constraint
  - $\mathrm{M}(\tilde{R}_P(x)) \geq \rho \stackrel{defuz}{\longrightarrow}$ linear constraint
  - $\tilde{R}_P(x) \gtrsim \tilde{R}_f$
    - fuzzy returns of the same shape $\stackrel{defuz}{\longrightarrow}$ linear constraints
    - fuzzy returns of different shape $\stackrel{defuz}{\longrightarrow}$ linear semi-infinite constraints

Fuzzy downside risk models: expected return

- Concerning the expected return we will use either the mean interval of a fuzzy number:

$$E(\tilde{A}) = [E_*(\tilde{A}), E^*(\tilde{A})] = [\int_0^1 (\inf \tilde{A}_\alpha)\, d\alpha, \int_0^1 (\sup \tilde{A}_\alpha)\, d\alpha]$$

or the interval-valued possibilistic mean:

$$M(\tilde{A}) = [M_*(\tilde{A}), M^*(\tilde{A})] = [2\int_0^1 \alpha(\inf \tilde{A}_\alpha)\, d\alpha, 2\int_0^1 \alpha(\sup \tilde{A}_\alpha)\, d\alpha]$$

- The measure of investment risk evaluates the mean absolute semi-deviation with respect the total return:

$$\mathrm{w_E}(P(x)) = \mathrm{E}(\mathrm{máx}\{0, \mathrm{E}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}).$$

Fuzzy downside risk models: expected return

For comparison of the interval-valued means we will use the following ordering relations. Let $A = [a_l, a_r]$ a closed interval, $m(A) = \frac{1}{2}(a_l + a_r)$ and $hw(A) = \frac{1}{2}(a_r - a_l)$, its midpoint and half-width, respectively.

- Ishibuchi and Tanaka (1990): $A \leq_{lr} B$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$
- Ishibuchi and Tanaka (1990): $A \leq_{mw} B$ if $m(A) \leq m(B)$ and $hw(A) \geq hw(B)$
- Sengupta and Pal (2000): if $m(A) \leq m(B)$ it is said that $A \prec B$ with the acceptability grade $\frac{m(B)-m(A)}{hw(B)+hw(A)}$

Proposition 2 (Vercher *et al*, 2006). Let $\tilde{R}_P(x) = (P_l, P_u, C, D)_{LR}$ be a LR-fuzzy number with continuous strictly decreasing reference functions, such that $L = R$.

- If $C \geq D$ then $E(\tilde{R}_P(x)) \leq_{mw} M(\tilde{R}_P(x))$
- If $C < D$ then $M(\tilde{R}_P(x)) \prec E(\tilde{R}_P(x))$ with a grade of acceptability in (0,1).

Fuzzy downside risk models: expected return

Example in Carlsson *et al.* (2002): Three assets with trapezoidal form

$$\tilde{R}_1 = (-10.5, 70, 4, 100)_{LR};\ \ \tilde{R}_2 = (-8.1, 35, 4.4, 54)_{LR};\ \ \tilde{R}_3 = (-5, 28, 11, 85)_{LR}$$

- Applying an algorithm for finding the exact optimal solution in the sense of maximizing a given utility score, the authors find the following portfolios:

	share			total return
	x_1	x_2	x_3	$\tilde{R}_P(x)$
P_1	0.124	0.373	0.503	(-6.84, 35.82, 7.67, 75.30)$_{LR}$
P_2	0.163	0.837	0.000	(-8.50, 40.71, 4.33, 61.50)$_{LR}$
P_3	0.103	0.000	0.897	(-5.57, 32.33, 10.28, 86.55)$_{LR}$
P_4	0.000	0.000	1.000	(-5.00, 28.00, 11.00, 85.00)$_{LR}$

Being P_1 the optimal portfolio.

Fuzzy downside risk models: expected return

- Comparison of the expected returns for the four feasible portfolios:

	$E(\tilde{R}_P(x))$	$m(E(\tilde{R}_P))$	$hw(E(\tilde{R}_P))$	$M(\tilde{R}_P(x))$	$m(M(\tilde{R}_P))$	$hw(M(\tilde{R}_P))$
P_1	[-10.67,73.47]	31.4	42.07	[-9.40,60.92]	25.76	35.16
P_2	[-10.66,71.45]	30.4	41.06	[-9.44,61.20]	25.63	35.57
P_3	[-10.71,75.60]	32.4	43.16	[-8.99,61.17]	26.09	35.08
P_4	[-10.50,70.50]	30.0	40.50	[-8.67,56.33]	23.83	32.50

  - Using $E(\tilde{R}_P(x))$: $E(\tilde{R}_{P4}) \prec E(\tilde{R}_{P2}) \prec E(\tilde{R}_{P1}) \prec E(\tilde{R}_{P3})$.
  - Using $M(\tilde{R}_P(x))$: $M(\tilde{R}_{P1}) \leq_{lr} M(\tilde{R}_{P3})$, $M(\tilde{R}_{P2}) \leq_{mw} M(\tilde{R}_{P1})$, $M(\tilde{R}_{P2}) \leq_{mw} M(\tilde{R}_{P3})$ and $M(\tilde{R}_{P4}) \prec M(\tilde{R}_{P3})$.
  - Where $M(\tilde{R}_{Pi}) \prec E(\tilde{R}_{Pi})$ for $i = 1, 2, \ldots, 4$
- Then the portfolio with the best expected return is P_3, which contradicts the results in Carlsson *et al.* (2002).

Fuzzy downside risk models: Objective function

On the other hand, concerning the evaluation of the fuzzy downside risk for returns modelled by means of trapezoidal fuzzy numbers we have that

- interval-valued mean

$$\mathrm{w_E}(P(x)) = \mathrm{E}(\mathrm{máx}\{0, \mathrm{E}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}) = \left[0, P_u - P_l + \frac{1}{2}(C + D)\right]$$

- interval-valued possibilistic mean

$$\mathrm{w_M}(P(x)) = \mathrm{M}(\mathrm{máx}\{0, \mathrm{M}(\tilde{R}_P(x)) - \tilde{R}_P(x)\}) = \left[0, P_u - P_l + \frac{1}{3}(C + D)\right]$$

Fuzzy downside risk models: Objective function

But, if we have individual assets whose returns are modelled by means of *LR*-fuzzy numbers with different shapes we obtain new formulas for the downside risk. For instance:

Proposition 3 (Vercher *et al*, 2006). Let us denote by $\tilde{R}_j = (a_{lj}, a_{uj}, c_j, d_j)_{L_j R_j}$ the fuzzy return on the j^{th} asset, where L_j (respectively R_j) are power reference functions with a different parameter p_j (respectively q_j), for $j = 1, 2, \ldots, n$. Let $\tilde{R}_P(x) = \sum_{j=1}^{n} x_j \tilde{R}_j$, for $x_j \geq 0$, then:

$$\text{a) } E(\tilde{R}_P(x)) = [\sum_{j=1}^{n}(a_{lj} - c_j \frac{p_j}{p_j+1})x_j, \sum_{j=1}^{n}(a_{uj} + d_j \frac{q_j}{q_j+1})x_j]$$

$$\text{b) } \mathrm{w_E}(P(x)) = [0, \sum_{j=1}^{n}(a_{uj} - a_{lj} + c_j \frac{p_j}{p_j+1} + d_j \frac{q_j}{q_j+1})x_j].$$

Fuzzy downside risk models: Objective function

- Note that in all cases the interval representation of the fuzzy downside risk corresponds to an interval whose lower endpoint is zero. Then, we decide to select as objective function minimizing the width of the interval of the mean-absolute semi-deviation. That is to minimize the upper limit of $w_E(P(x))$ which also permits us to ensure that the non-desired deviations on the expected return are minimal.

- Concerning the defuzzification of the interval-valued mean of the expected return of the portfolio in the fuzzy constraint we decide to select as left hand-side of the constraint the midpoint of the mean interval.

- In fact, we are looking for a portfolio whose expected return is an interval with minimum width and midpoint greater than the achieved return (ρ).

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Portfolio selection with linear programs

$$
\begin{aligned}
\text{(P1) min} \quad & \sum_{i=1}^{n} \left(a_{ui} - a_{li} + \frac{1}{2}(c_i + d_i) \right) x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \left(\frac{1}{2}(a_{ui} + a_{li}) + \frac{1}{4}(d_i - c_i) \right) x_i \geq \rho
\end{aligned}
$$

$$
\begin{aligned}
\text{(P2) min} \quad & \sum_{i=1}^{n} \left(a_{ui} - a_{li} + \frac{1}{3}(c_i + d_i) \right) x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \left(\frac{1}{2}(a_{ui} + a_{li}) + \frac{1}{6}(d_i - c_i) \right) x_i \geq \rho \\
& \sum_{i=1}^{n} x_i = 1 \\
& l_i \leq x_i \leq u_i, \quad i = 1, \ldots, n
\end{aligned}
$$

Portfolio selection with linear programs

- Example in Carlsson *et al.* (2002) revisited. For $\rho = 30$, $l_1 = l_2 = l_3 = 0$ and $u_1 = u_2 = u_3 =$ BOUND, for different BOUND values.
- Using the definition of expected return due to Dubois and Prade (1987).

BOUND	0.4	0.5	0.6	0.8	1.0
x_1	0.20	0.07	0.06	0.03	0.00
x_2	0.40	0.43	0.34	0.17	0.00
x_3	0.40	0.50	0.60	0.80	1.00
$\mathrm{m}(\mathrm{E}(\tilde{R}_P(x)))$	33.1	30.0	30.0	30.0	30.0
$\mathrm{w_E}(P(x))$	87.8	81.1	81.1	81.1	81.0
$\mathrm{m}(\mathrm{M}(\tilde{R}_P(x)))$					
$\mathrm{w_M}(P(x))$					

So, the procedure minimizes the width of the interval $\mathrm{w_E}(P(x))$, s.t the constraint $\mathrm{m}(\mathrm{E}(\tilde{R}_P(x))) \geq \rho$, that is (P1).

Portfolio selection with linear programs

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x_1	0.20	0.07	0.06	0.03	0.00
x_2	0.40	0.43	0.34	0.17	0.00
x_3	0.40	0.50	0.60	0.80	1.00
$\mathrm{m}(\mathrm{E}(\tilde{R}_P(x)))$	33.1	30.0	30.0	30.0	30.0
$\mathrm{w}_\mathrm{E}(P(x))$	87.8	81.1	81.1	81.1	81.0
$\mathrm{m}(\mathrm{M}(\tilde{R}_P(x)))$	27.4	24.6	24.4	24.1	23.8
$\mathrm{w}_\mathrm{M}(P(x))$	74.1	67.7	67.2	66.1	65.0

So, the procedure minimizes the width of the interval $\mathrm{w}_\mathrm{E}(P(x))$, s.t the constraint $\mathrm{m}(\mathrm{E}(\tilde{R}_P(x))) \geq \rho$, that is (P1).

Portfolio selection with linear programs

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- Using the definition of expected return due to Carlsson and Fullér (2001).

BOUND	0.4	0.5	0.6	0.8	1.0
x_1	0.31	0.30	0.29	0.28	0.28
x_2	0.29	0.20	0.11	0.00	0.00
x_3	0.40	0.50	0.60	0.72	0.72
$\mathrm{m}(\mathrm{E}(\tilde{R}_P(x)))$					
$\mathrm{w_E}(P(x))$					
$\mathrm{m}(\mathrm{M}(\tilde{R}_P(x)))$	30.0	30.0	30.0	30.0	30.0
$\mathrm{w_M}(P(x))$	79.8	79.6	79.4	79.1	79.1

So, the procedure minimizes the width of the interval $\mathrm{w_M}(P(x))$, s.t the constraint $\mathrm{m}(\mathrm{M}(\tilde{R}_P(x))) \geq \rho$, that is (P2).

Portfolio selection with linear programs

- Example in Carlsson *et al.* (2002) revisited. For $\rho = 30$, $l_1 = l_2 = l_3 = 0$ and $u_1 = u_2 = u_3 =$ BOUND, for different BOUND values.
- Using the definition of expected return due to Carlsson and Fullér (2001).

BOUND	0.4	0.5	0.6	0.8	1.0
x_1	0.31	0.30	0.29	0.28	0.28
x_2	0.29	0.20	0.11	0.00	0.00
x_3	0.40	0.50	0.60	0.72	0.72
$\mathrm{m}(\mathrm{E}(\tilde{R}_P(x)))$	36.1	36.3	36.5	36.7	36.7
$\mathrm{w_E}(P(x))$	94.4	94.8	95.1	95.5	95.5
$\mathrm{m}(\mathrm{M}(\tilde{R}_P(x)))$	30.0	30.0	30.0	30.0	30.0
$\mathrm{w_M}(P(x))$	79.8	79.6	79.4	79.1	79.1

So, the procedure minimizes the width of the interval $\mathrm{w_M}(P(x))$, s.t the constraint $\mathrm{m}(\mathrm{M}(\tilde{R}_P(x))) \geq \rho$, that is (P2).

Outline

- Portfolio selection models
- Portfolio selection with fuzzy returns
  - Fuzzy background
  - Fuzzy downside risk models
- Portfolio selection with linear programs
- Portfolio selection with semi-infinite optimization
- Numerical examples

Portfolio selection with semi-infinite optimization

The objective function may be any of the two mean-valued definitions E or M, and w.r.t. the achieving return:

$$\begin{aligned} (\mathrm{P_I})\ \mathrm{Min} \quad & \mathrm{w_E}(P(x)) \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} x_j \tilde{R}_j \gtrsim \tilde{R}_f \\ & \textstyle\sum_{j=1}^{n} x_j = 1 \\ & l_j \leqslant x_j \leqslant u_j \qquad j = 1 \ldots n \end{aligned}$$

$$\begin{aligned} (\mathrm{P_{II}})\ \mathrm{Min} \quad & \mathrm{w_M}(P(x)) \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} x_j \tilde{R}_j \gtrsim \tilde{R}_f \\ & \textstyle\sum_{j=1}^{n} x_j = 1 \\ & l_j \leqslant x_j \leqslant u_j \qquad j = 1 \ldots n \end{aligned}$$

Portfolio selection with semi-infinite optimization

Consider n risk assets with returns modelled by means of fuzzy quantities and a risk-free asset $\tilde{R}_f$

$$\begin{array}{ll} \text{Min} & \mathrm{w_E}(P(x)) \\ \text{s.t.} & \sum_{j=1}^n x_j \tilde{R}_j \gtrsim \tilde{R}_f \\ & \sum_{j=1}^n x_j = 1 \\ & l_j \leqslant x_j \leqslant u_j \qquad j = 1 \dots n \end{array}$$

- The reference functions belong to the power family with parameters p_j and q_j: $L_j(r) = max\{0, 1 - |r|^{p_j}\}$, $p_j > 0$ and $R_j(r) = max\{0, 1 - |r|^{q_j}\}$, $q_j > 0$.
- The α-level sets of the total return of the fuzzy portfolio for α in [0,1] are:

$$[\tilde{R}_P(x)]^\alpha = [\sum_{j=1}^n (a_{lj} - c_j(1-\alpha)^{\frac{1}{p_j}}))x_j, \sum_{j=1}^n (a_{uj} + d_j(1-\alpha)^{\frac{1}{q_j}}))x_j]$$

Portfolio selection with semi-infinite optimization

For comparison of fuzzy quantities we will use the following ordering relation.

Definition (Tanaka *et al*, 1984). Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy numbers and h a real number in $[0, 1]$. Then, $\tilde{M} \gtrsim^h \tilde{N}$ if and only if for all $\alpha \in [h, 1]$ the following statements hold:

$$\inf\{s : \mu_{\tilde{M}}(s) \geq \alpha\} \geq \inf\{t : \mu_{\tilde{N}}(t) \geq \alpha\},$$

$$\sup\{s : \mu_{\tilde{M}}(s) \geq \alpha\} \geq \sup\{t : \mu_{\tilde{N}}(t) \geq \alpha\}.$$

- In fuzzy linear programs to use this ordering relation implies that each fuzzy constraint is replaced by two ordinary constraints indexed on a closed interval in the real line.
- For fuzzy numbers of the same shape the accomplishment of the above constraints can be tested with a finite number of points.

Portfolio selection with semi-infinite optimization

- Consider a trapezoidal fuzzy number for the risk-free value: $[\tilde{R}_f]^\alpha = [R_{lf} - c_0(1-\alpha), R_{uf} + d_0(1-\alpha)]$
- The fuzzy constraint $\sum_{j=1}^n x_j \tilde{R}_j \gtrsim^h \tilde{R}_f$ becomes two linear semi-infinite constraints:

$$\sum_{j=1}^{n} (a_{lj} - c_j(1-\alpha)^{\frac{1}{p_j}})x_j \geq R_{lf} - c_0(1-\alpha), \quad \alpha \in [h,1],$$

and

$$\sum_{j=1}^{n} (a_{uj} + d_j(1-\alpha)^{\frac{1}{q_j}})x_j \geq R_{uf} + d_0(1-\alpha), \quad \alpha \in [h,1].$$

- This ordering relation permits us to consider different possibility levels $h \in [0,1]$, which facilitates the incorporation of the investor's opinion with respect to the accomplishment of the fuzzy constraint.

Portfolio selection with semi-infinite optimization

Definition (Vercher, 2006). Let $\tilde{R}_P(x) = \sum_{j=1}^{n} x_j \tilde{R}_j$, for $x_j \geq 0$ the total fuzzy return of the portfolio and $\tilde{R}_f = (R_{lf}, R_{uf}, c_0, d_0)_{LR}$ the trapezoidal fuzzy number which represents the returns on the risk-free asset, then $\tilde{R}_P(x) \succeq_\gamma \tilde{R}_f$ for a threshold $\gamma \in [0,1]$ if the following statements hold for all $\alpha \in [0,1]$:

$$(i)\ \sum_{j=1}^{n} (a_{uj} + d_j (1-\alpha)^{\frac{1}{q_j}}) x_j \geq R_{uf} + d_0(1-\alpha),$$

$$(ii)\ \mathcal{A}_{\prec}([\tilde{R}_P(x)]^\alpha, [\tilde{R}_f]^\alpha) \leq \gamma.$$

The acceptability index $\mathcal{A}_{\prec}$ is (Sengupta and Pal, 2000):

$$\mathcal{A}_{\prec}(A,B) = \frac{m(B) - m(A)}{hw(A) + hw(B)}$$

where $hw(A) + hw(B) \neq 0$ and $m(.)$ and $hw(.)$ evaluate the midpoint and the half-width of the interval, respectively.

Portfolio selection with semi-infinite optimization

- Let us present the semi-infinite representation of condition

  $\mathcal{A}_{\prec}([\tilde{R}_P(x)]^\alpha, [\tilde{R}_f]^\alpha) \leq \gamma$

  in terms of the α-cuts:

$$(1+\gamma)\sum_{j=1}^{n} f_{uj}(\alpha)x_j + (1-\gamma)\sum_{j=1}^{n} f_{lj}(\alpha)x_j \geq (1-\gamma)b_u(\alpha) + (1+\gamma)b_l(\alpha),$$

  for all α in $[0,1]$.

  Where
  $f_{lj}(\alpha) = a_{lj} - c_j(1-\alpha)^{\frac{1}{p_j}}$, $f_{uj}(\alpha) = a_{uj} + d_j(1-\alpha)^{\frac{1}{q_j}}$,
  $b_l(\alpha) = R_{lf} - c_0(1-\alpha)$ and $b_u(\alpha) = R_{uf} + d_0(1-\alpha)$.

Portfolio selection with semi-infinite optimization

$$\begin{array}{rll} \text{(LSIP) Min} & \sum_{j=1}^{n} w_j x_j & \\ \text{s.t.} & z_i(x,\alpha) \geq 0, & \alpha \in [h,1]\ i \in \mathbf{q} \\ & \sum_{j=1}^{n} x_j = 1 & \\ & l_j \leqslant x_j \leqslant u_j & j = 1 \ldots n \end{array}$$

where the set of SIP constraints is either $\mathbf{q} = \{l, u\}$ or $\mathbf{q} = \{u, \gamma\}$.

$$z_l(x,\alpha) = \sum_{j=1}^{n} f_{lj}(\alpha) x_j - b_l(\alpha) \geq 0,$$

$$z_u(x,\alpha) = \sum_{j=1}^{n} f_{uj}(\alpha) x_j - b_u(\alpha) \geq 0,$$

$$z_\gamma(x,\alpha) = (1+\gamma) \sum_{j=1}^{n} f_{uj}(\alpha) x_j + (1-\gamma) \sum_{j=1}^{n} f_{lj}(\alpha) x_j - b_\gamma(\alpha) \geq 0.$$

Portfolio selection with semi-infinite optimization

- The slack constraints $z_l(x,\alpha)$ and $z_u(x,\alpha)$ have a finite representation if the return on each security has been modeled by means of LR-fuzzy number of the same shape (triangular, trapezoidal, power with identical parameter value, etc.).

- We have established that the new condition $z_\gamma(x,\alpha)$ has also a finite representation if the LR-fuzzy numbers which model the returns on all the assets have a trapezoidal form.

  Proposition 4 (Vercher, 2006). *Let* $\tilde{M} = (m_l, m_u, c_1, d_1)_{LR}$ *and* $\tilde{N} = (n_l, n_u, c_2, d_2)$ *be two trapezoidal fuzzy numbers. Then* $\mathcal{A}_{\prec}([\tilde{N}]^\alpha, [\tilde{M}]^\alpha) \leq \gamma$ *for all* $\alpha \in [0,1]$ *if and only if the inequality holds for* $\alpha \in \{0,1\}$*.*

- Hence, in some cases the linear semi-infinite programming problem can be reduced to a finite one, but not always.

Portfolio selection with semi-infinite optimization

- We assume that (LSIP) is a consistent problem for a given $h \in [0, 1]$ and denote its feasible set by F.

- In contrast to finite linear programs, semi-infinite problems do not necessarily have the strong duality property unless additional constraint qualifications hold (Goberna and Lopez, 1998).

- The Slater‘s constraint qualification for (LSIP) assumes the existence of a strictly feasible primal solution: "There are some $\hat{x} \in F$ such that $l_j < \hat{x} < u_j$ for $j = 1, \ldots n$ and $z_i(\hat{x}, \alpha) > 0$ for all α in $[h, 1]$, $i \in$ **q**."

- Karush-Kuhn-Tucker type optimality conditions for (LSIP) are established along the same lines as in standard mathematical programming, both for **q** $= \{l, u\}$ and **q** $= \{u, \gamma\}$.

Portfolio selection with semi-infinite optimization

Concerning the criterion used to determine either the optimality of the current iterate or the search direction, we will follow the next result:

Proposition 5 (Vercher, 2006). Let $x \in F$ and suppose that the Slater constraint qualification holds. Then x is optimal for the (LSIP) problem if and only if $v(A(x)) = 0$, where $v(.)$ is the objective value of the following auxiliary linear program:

$$
\begin{array}{rccl}
A(x)\ \mathrm{Min} & \sum_{j=1}^{n} w_j d_j & & \\
\mathrm{s.t.} & \sum_{j=1}^{n} f_{ij}(\alpha) d_j \geq 0, & & \alpha \in Z_i(x),\ i \in \mathbf{q} \\
 & e^T d = 0 & & \\
 & d_j \geq 0 & & j \in J_l(x) \\
 & d_j \leq 0 & & j \in J_u(x) \\
 & -1 \leq d_j \leq 1 & & j = 1 \ldots n
\end{array}
$$

Portfolio selection with semi-infinite optimization

- There are many semi-infinite programming algorithms available for solving (LSIP) problems. We will follow the hybrid method developed in Leon and Vercher (2004).

- Our hybrid method, which is a primal one, alternates purification steps and feasible-direction descent steps. The purification phase is used to proceed from a feasible solution to an improved extreme point. The descent rules may be used for every feasible solution, applying the above optimality criterion to generate a descent direction, or to stop.

- Concerning the particularities of these semi-infinite programs, which correspond to fuzzy portfolio selection problems, the more important one is the boundedness of the feasible set, which implies that there are no recession directions of F.

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Numerical examples

Example 1.a

- We consider the weekly returns on 6 assets of the IBEX35, the most popular index of the Spanish Stock Market. We have taken the observations of the Wednesday prices as an estimate of the weekly prices, thus the return on asset j^{th} during the k^{th} week is defined as follows:

$$r_{kj} = \frac{p_{(k+1)j} - p_{kj}}{p_{kj}}$$

  where p_{kj} is the price of the asset j^{th} on the Wednesday of the k^{th} week.

- We use the sample percentiles P_k to approximate the core, the spreads and the parameter value p of the fuzzy returns on the assets. We set the core of the fuzzy return as the interval $[P_{50}, P_{60}]$ and the quantities $P_{50} - P_{10}$ and $P_{95} - P_{60}$ as the left and right spreads, respectively. We have considered P_{30} and P_{75} as the values with a fifty-fifty possibility of being realistic in order to evaluate p_j and q_j, by means of the reverse rating procedure.

Numerical examples

Specifications of the fuzzy weekly returns on the assets.

returns	a_{lj}	a_{uj}	c_j	d_j	p_j	q_j	w_j
$\tilde{R}_1$	$0{,}0065$	$0{,}0190$	$0{,}0640$	$0{,}0858$	$0{,}72$	$0{,}41$	$0{,}064203$
$\tilde{R}_2$	$0{,}0063$	$0{,}0214$	$0{,}0782$	$0{,}1150$	$0{,}56$	$0{,}53$	$0{,}083069$
$\tilde{R}_3$	$0{,}0029$	$0{,}0112$	$0{,}0437$	$0{,}0548$	$0{,}72$	$0{,}62$	$0{,}047520$
$\tilde{R}_4$	$0{,}0092$	$0{,}0286$	$0{,}0582$	$0{,}0684$	$0{,}84$	$0{,}61$	$0{,}071897$
$\tilde{R}_5$	$0{,}0092$	$0{,}0181$	$0{,}0546$	$0{,}0602$	$0{,}64$	$0{,}51$	$0{,}050516$
$\tilde{R}_6$	$0{,}0020$	$0{,}0110$	$0{,}0561$	$0{,}0922$	$0{,}83$	$0{,}62$	$0{,}069737$

- We assume that the investor decided to impose both an expected profit by means of $\tilde{R}_f = (0, 0{,}007, 0{,}005, 0{,}0005)_{LR}$ and a specified portfolio diversification defined by a set of given bounds.

Numerical examples

- First, let us set the bounds $u_j = 0{,}7$ and $l_j = 0$, for all j.

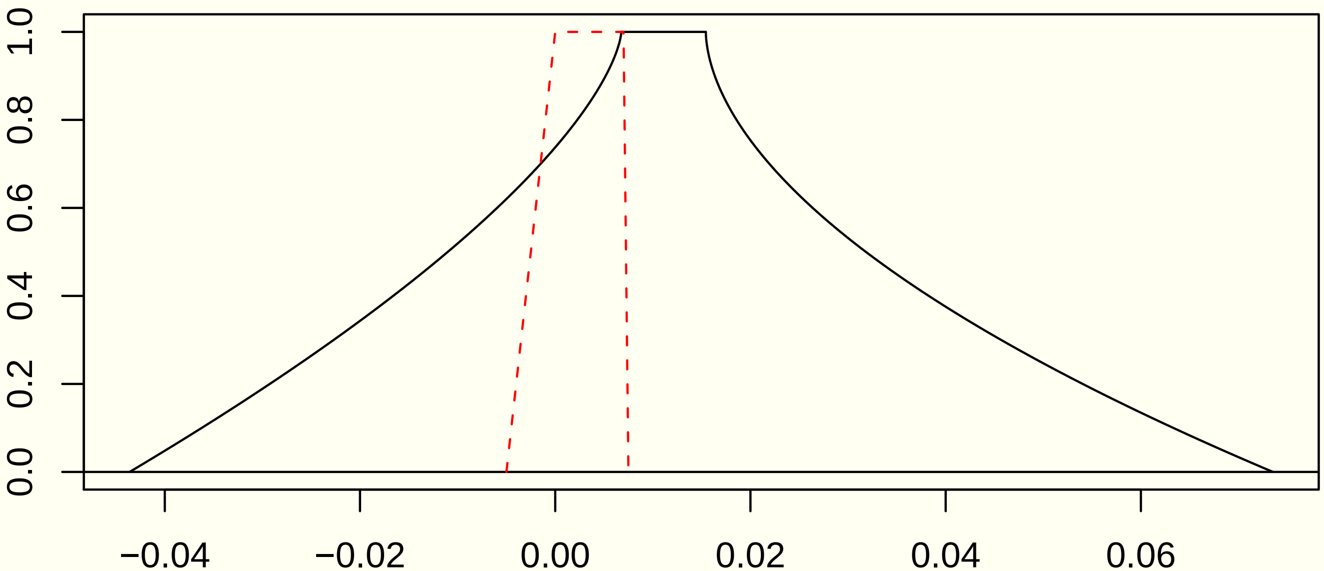
Optimal solutions for different possibility grades

h	x_1	x_2	x_3	x_4	x_5	x_6	obj. value
$0{,}8$	0	0	$0{,}700$	0	$0{,}300$	0	$0{,}048419$
$0{,}75$	0	0	$0{,}653$	0	$0{,}347$	0	$0{,}048560$
$0{,}72$	0	0	$0{,}497$	0	$0{,}503$	0	$0{,}049027$
$0{,}7$	0	0	$0{,}384$	0	$0{,}616$	0	$0{,}049367$

- Here we are using the ranking relation that implies to order the lower and upper limits of all the α-level cuts.

Numerical examples

- Figure 1 shows the relative position of the fuzzy numbers $\tilde{R}_f$ and $\tilde{R}_P(x^*)$, where the optimal portfolio is that obtained for the possibility grade $h = 0{,}7$.
- The core of the optimal portfolio is $[0{,}0068, 0{,}0154]$, where the left and right spreads are $0{,}0504$ and $0{,}0580$, respectively.



Total return on the fuzzy portfolio for $h = 0{,}7$

Numerical examples

Example 1.b

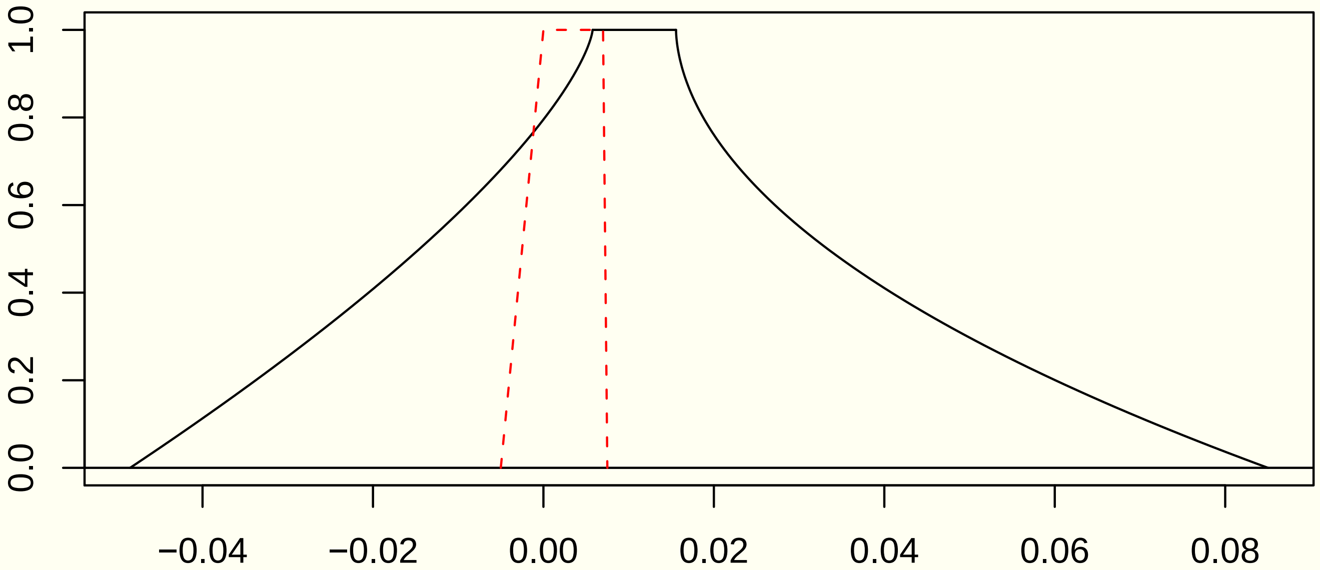
- This example emphasizes the importance of obtaining portfolios that overcome the upper limits of $\tilde{R}_f$ without considering its lower limits.
- Table 2 shows the results for the portfolio selection problem when the new ordering relation with $\gamma = 0{,}001$ and $h = 0$.

Optimal solutions for different upper bounds

u_j	x_1	x_2	x_3	x_4	x_5	x_6	obj. value
$0{,}7$	0	0	$0{,}7$	0	$0{,}3$	0	$0{,}048419$
$0{,}6$	0	0	$0{,}6$	0	$0{,}4$	0	$0{,}048718$
$0{,}5$	0	0	$0{,}5$	0	$0{,}5$	0	$0{,}049018$
$0{,}4$	$0{,}2$	0	$0{,}4$	0	$0{,}4$	0	$0{,}052055$
$0{,}3$	$0{,}3$	0	$0{,}3$	0	$0{,}3$	$0{,}1$	$0{,}055645$
$0{,}2$	$0{,}2$	0	$0{,}2$	$0{,}2$	$0{,}2$	$0{,}2$	$0{,}060775$

Numerical examples

- Notice that every solution verifies $\tilde{R}_P(x^*) \succeq_\gamma \tilde{R}_f$ for $\alpha \in [0,1]$. The graph shows the grade of unfulfillment of the lower semi-infinite constraint for the optimal solution obtained with $u_j = 0{,}3$
- The core of the optimal portfolio $P(x^*)$ is $[0{,}0058, 0{,}0156]$, where the left and right spreads are $0{,}0543$ and $0{,}0695$, respectively.



Total return on the fuzzy portfolio for $u_j = 0{,}3$

Conclusions

- First, taking the uncertainty of returns on assets in a financial market as LR-fuzzy numbers we generalize the mean-absolute semi-deviation using both interval-valued probabilistic and possibilistic means. Then based on a fuzzy downside risk measure we formulate new portfolio selection problems which can be solved using linear programming problems.
- Secondly we develop a linear semi-infinite programming approach to the fuzzy portfolio selection problem, where the returns on assets are modeled by nonlinear LR-fuzzy numbers and the investment risk is evaluated by means of a fuzzy downside risk function.
- We present some results and the explicit formulation of the semi-infinite program for fuzzy numbers belonging to the power family. Our semi-infinite approach could be a good alternative in those situations in which the description of the data set is made with LR-fuzzy numbers of different shapes. In all these situations semi-infinite optimization could be a useful methodology to find suitable portfolios.

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