Stability of Leaves

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Seminars of the CIM Scientific Council Meeting 2008

Goals

- Establish stability results for symplectic leaves of Poisson manifolds;
- Understand the relationship between (apparently) distinct stability results in different geometric settings;

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Outline

Flows: Stability of periodic orbits

Definition

A periodic orbit of a vector field $X \in \mathfrak{X}(M)$ is called stable if every nearby vector field also has a nearby periodic orbit.

- Basic Fact: Stability is controled by the Poincaré return map $h : T \to T$.
- Assumptions on $\mathrm{d}_{x_0} h$ can also guarantee stability.

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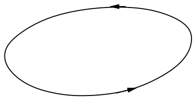

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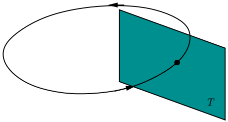

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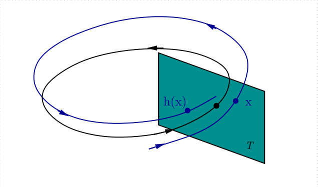

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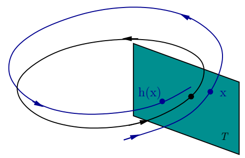

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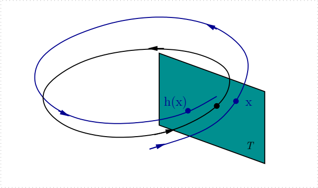

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Foliations: stability of leaves

Fix a manifold M ($\dim M = n$) and a foliation $\mathcal{F}$ ($\operatorname{codim}(\mathcal{F}) = q$).

- $\mathcal{F}$ is given by a foliation atlas $(U_i, \varphi_i)_{i \in I}$

$$\varphi_i : U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q, \quad \varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$$

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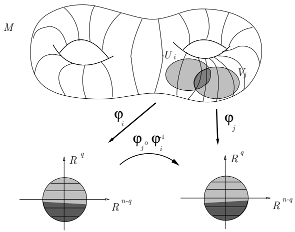

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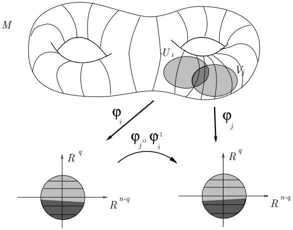

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Theorem (Frobenius)

$$\mathrm{Fol}_q(M) \longleftrightarrow \{D : M \to \mathrm{Gr}_q(TM) | D \text{ is involutive}\}$$

$$\mathcal{F} \longmapsto D := T\mathcal{F}$$

$\Longrightarrow \mathrm{Fol}_q(M)$ has a natural C^r compact-open topology

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A leaf L of a foliation $\mathcal{F} \in \mathrm{Fol}_k(M)$ is called stable if every nearby foliation in $\mathrm{Fol}_k(M)$ has a nearby leaf diffeomorphic to L.

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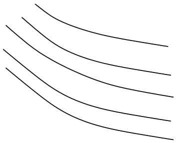

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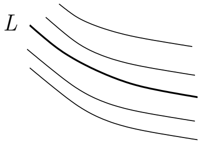

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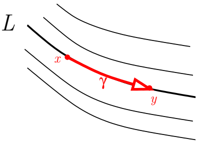

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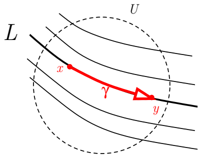

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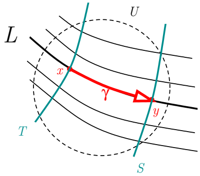

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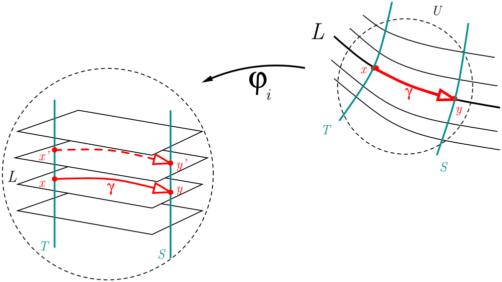

Construct diffeomorphism $f : T \to S$ that satisfies $f(x) = y$ and $y' = f(x')$ iff x' and y' are in same plaque. Then:

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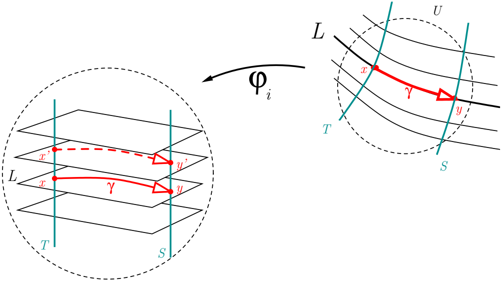

Construct diffeomorphism $f : T \to S$ that satisfies $f(x) = y$ and $y' = f(x')$ iff x' and y' are in same plaque. Then:

$$\mathrm{Hol}^{T,S}(\gamma) := \mathrm{germ}_x(f) : (T, x) \to (S, y).$$

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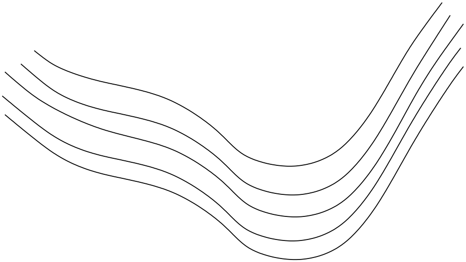

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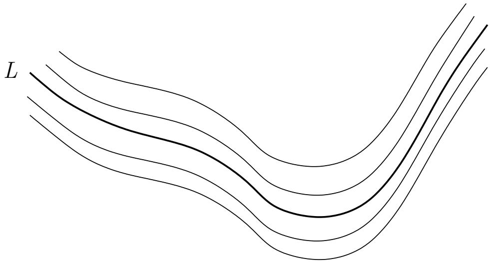

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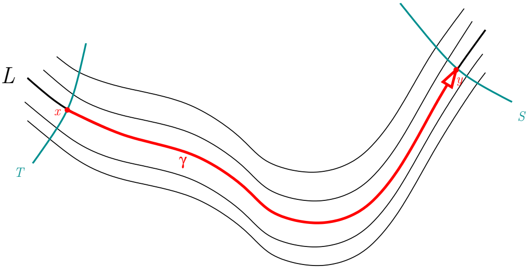

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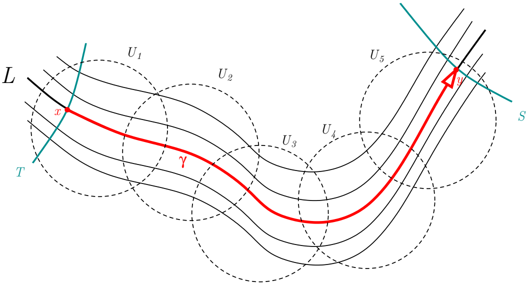

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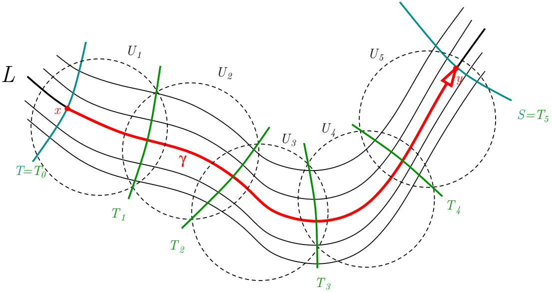

$$\mathrm{Hol}^{T,S}(\gamma) := \mathrm{Hol}^{T_k,T_{k-1}}(\gamma) \circ \cdots \circ \mathrm{Hol}^{T_2,T_1}(\gamma) \circ \mathrm{Hol}^{T_1,T_0}(\gamma)$$

Foliations: stability of leaves

Facts:

- Taking germs makes construction independent of choices;
- If γ, η are curves in L with $\gamma(0) = \eta(1)$ then: $\mathrm{Hol}^{T,S}(\gamma \cdot \eta) = \mathrm{Hol}^{T,S}(\gamma) \circ \mathrm{Hol}^{S,R}(\eta)$;
- If γ and γ' are homotopic curves in L, then: $\mathrm{Hol}^{T,S}(\gamma) = \mathrm{Hol}^{T,S}(\gamma')$;

Hence, if we fix $x \in L$ we obtain the holonomy homomorphism:

$$\mathrm{Hol} := \mathrm{Hol}^{T,T} : \pi_1(L, x) \to \mathrm{Diff}_x(T).$$

Note: The Poincaré return map is a special case of this construction.

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Differentiating gives the linear holonomy representation:

$$\rho : \pi_1(L, x) \to GL(\nu(L)_x), \quad \rho := \mathrm{d}_x \circ \mathrm{Hol}$$

Denote by $H^1(\pi_1(L, x), \nu(L)_x)$ the first group cohomology.

Theorem (Reeb, Thurston, Langevin & Rosenberg)

Let L be a compact leaf and assume that

$$H^1(\pi_1(L, x), \nu(L)_x) = 0.$$

Then L is stable.

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Group actions: stability of orbits

- Fix a manifold M and a Lie group M
- $\alpha(g, x) := g \cdot x$ - an action of G on M
- Action $\alpha : G \times M \to M \Leftrightarrow$ homomorphism $\alpha : G \to \mathrm{Diff}(M)$

$$\mathrm{Act}(G; M) \subset \mathrm{Maps}(G; \mathrm{Diff}(M))$$

$\Longrightarrow$ $\mathrm{Act}(G; M)$ has a natural C^r compact-open topology

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An orbit $\mathcal{O}$ of $\alpha \in \mathrm{Act}(G; M)$ is called stable if every nearby action in $\mathrm{Act}(G; M)$ has a nearby orbit diffeomorphic to $\mathcal{O}$.

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- $G_x := \{g \in G : g \cdot x = x\}$ isotropy group at $x \in \mathcal{O}$.
- $g \in G_x$ induces a map $\alpha_g : M \to M$, $y \mapsto g \cdot y$ that fixes x.
- $d_x\alpha_g : T_xM \to T_xM$

  $\Rightarrow \quad \rho(g) : \nu(\mathcal{O})_x \to \nu(\mathcal{O})_x \quad$ where $\nu(\mathcal{O})_x = T_xM/T_x\mathcal{O}$.

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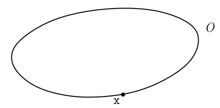

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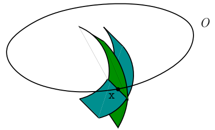

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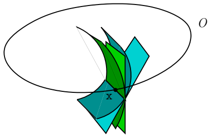

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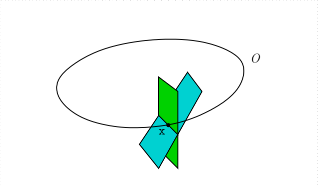

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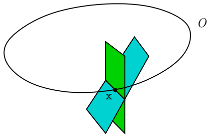

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Denote by $H^1(G_x, \nu(\mathcal{O})_x)$ the first group cohomology.

Theorem (Hirsch,Stowe)

Let $\mathcal{O}$ be a compact orbit and assume that

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- In general, the two theorems are quite different (e.g., dimension of orbits can vary).
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Hamilton's Equations

- $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$
- Classical Poisson bracket:

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial f_1}{\partial p_i} \right)$$

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$$\begin{cases} \dot{q}_i = \frac{\partial h}{\partial p_i} \\ \dot{p}_i = -\frac{\partial h}{\partial q_i} \end{cases} \quad (i = 1, \ldots, n) \qquad \Leftrightarrow \qquad \dot{x}_a = \{x_a, h\} \quad (a = 1, \ldots, 2n)$$

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A manifold M furnished with a Poisson bracket is called a Poisson manifold.

Basic examples

- Any symplectic manifold (M, ω) is a Poisson manifold:

$$\{f, g\} = -\omega(X_f, X_g).$$

(X_f is the unique vector field such that $\iota_{X_f}\omega = \mathrm{d}f$.)

- The dual of a Lie algebra $M = \mathfrak{g}^*$ is a Poisson manifold:

$$\{f, g\}(\xi) = \langle \xi, [\mathrm{d}_\xi f, \mathrm{d}_\xi g]\rangle.$$

- Any skew-symmetric matrix (a_{ij}) defines a quadratic Poisson bracket on $\mathbb{R}^n$:

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Symplectic foliation

Definition

On $(M, \{\ ,\ \})$, the hamitonian vector field determined by $h \in C^\infty(M)$ is the vector field $X_h \in \mathfrak{X}(M)$ given by:

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Write $x \sim y$ if there exists a piecewise smooth curve joining x to y made of integral curves of hamiltonian vector fields:

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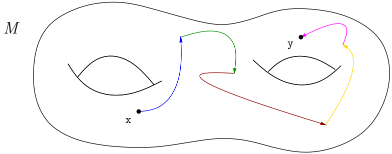

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Theorem (Weinstein)

The decomposition of $(M, \{\ ,\ \})$ into equivalence classes of $\sim$:

$$M = \bigsqcup_{\alpha \in A} S_\alpha.$$

satisfies:

(i) *Each S_α is a (immersed) submanifold of M;*

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Example

- $M = \mathfrak{sl}^*(2,\mathbb{R}) \simeq \mathbb{R}^3$: $\{x,z\} = y; \quad \{x,y\} = z; \quad \{z,y\} = x.$
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- Foliation is singular (dimension of leaves varies; e.g., cone $x^2 + y^2 = z^2$)

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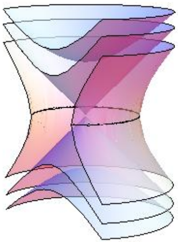

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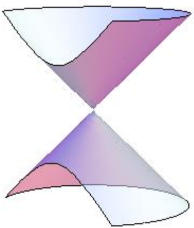

Poisson bivector

Given Poisson bracket $\{\ ,\ \}$ define the Poisson bivector:

$$\pi(\mathrm{d}f, \mathrm{d}g) := \{f, g\}.$$

- $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ is a skew-symmetric contravariant tensor;
- In local coordinates $(x^1, \dots, x^n)$:

$$\pi = \sum_{i<j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

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Stability of symplectic leaves

$$\text{Poiss}(M) \quad \longleftrightarrow \quad \{\pi : M \to \wedge^2(TM) |\ [\pi, \pi] = 0\}.$$

$\Longrightarrow$ Poiss(M) has a natural C^r compact-open topology

Definition

A symplectic leaf S of $\pi \in$ Poiss(M) is called stable if every nearby Poisson structure in Poiss(M) has a nearby leaf diffeomorphic to S.

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Main Theorem

Theorem (Crainic & RLF)

Let (M, π) be a Poisson structure and $S \subset M$ a compact symplectic leaf such that:

$$H^2_\pi(M, S) = 0.$$

Then S is stable.

- Again, this result is quite different from the previous ones;
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Stability of symplectic leaves

Example

$M = \mathfrak{su}^*(3) \simeq \mathfrak{su}(3)$ (via the Killing form) with linear Poisson structure:

- Symplectic leaves are the conjugacy classes of $SU(3)$:

$$A \sim \begin{pmatrix} i\lambda_1 & 0 & 0 \\ 0 & i\lambda_2 & 0 \\ 0 & 0 & i\lambda_3 \end{pmatrix} \qquad (\lambda_1+\lambda_2+\lambda_3 = 0,\ 0 \leq \lambda_1 \leq \lambda_2)$$

- Leaves have:
  - (i) Dimension 6 (flag);
  - (ii) Dimension 4 ($\mathbb{C}P(2)$);
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Example

- All leaves of $\mathfrak{su}^*(3)$ (including singular leaves) are stable;
- The same result applies for $\mathfrak{g}^*$, where $\mathfrak{g}$ is any semi-simple Lie algebra of compact type;
- This is related to (and explains!) a famous linearization theorem of Conn;
- If $\mathfrak{g}$ is semi-simple and non-compact stability, in general, does not hold (e.g., $\mathfrak{sl}(2, \mathbb{R})$).

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Ordinary Geometry

- Differential forms: $\Omega^k(M) = \Gamma(\wedge^k T^*M)$;
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Geometric interpretations of $H^\bullet_\pi(M)$ in low degrees:

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- Is there a general setup to deal with these kind of stability problems?

A positive answer to this question should lead to:

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Lie algebroids

Definition

A Lie algebroid is a vector bundle $A \to M$ with:

(i) a Lie bracket $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;

(ii) a bundle map $\rho : A \to TM$ (the anchor);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

$\operatorname{Im} \rho \subset TM$ is a integrable (singular) distribution

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Lie algebroids have a characteristic foliation

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Examples

- Flows. For $X \in \mathfrak{X}(M)$, the associated Lie algebroid is:
  $$A = M \times \mathbb{R}, \quad [f, g]_A := fX(g) - gX(f), \quad \rho(f) = fX.$$
  Leaves of A are the orbits of X.

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Stability Theorem

- For a fixed vector bundle A there is a natural compact-open topology on the set **Algbrd**(A) of Lie algebroid structures on A.
- A leaf L of A is called stable if every nearby Lie algebroid structure in **Algbrd**(A) has a nearby leaf diffeomorphic to L.
- There are natural A-cohomology theories. For a leaf L, one can define the relative A-cohomology with coefficients in the normal bundle $\nu(L)$, denoted $H^\bullet(A|_L; \nu(L))$.

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Theorem (Crainic & RLF)

Let L be a compact leaf of a Lie algebroid A, and assume that $H^1(A, L; \nu(L)) = 0$. Then L is stable.

- The theorem says: infinitesimal stability $\Rightarrow$ stability. Likewise, the proof is a "infinite dimensional transversality argument".
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Moral: There is a general framework to deal with stability of "leaf-type" problems.