Reaction-diffusion equations under Perturbations of the Domain

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General problem

Let us cosider the following evolution problem in space X_0

$$\begin{cases} x' + A_0 x = F_0(x), & t > 0 \\ x(0) = x_0 \in X_0 \end{cases}$$

Let us assume that appropriate conditions are satisfied so that we have global existence of solutions and contiuous dependence with respect to initial data. Hence, the equation generates a dynamical system in X_0 (phase space):

$$\begin{aligned} T_0(t) : X_0 &\to X_0 \\ x_0 &\to x(t, x_0) \end{aligned}$$

Under certain conditions on the equation we guarantee that T_0 is dissipative and asymptotically compact

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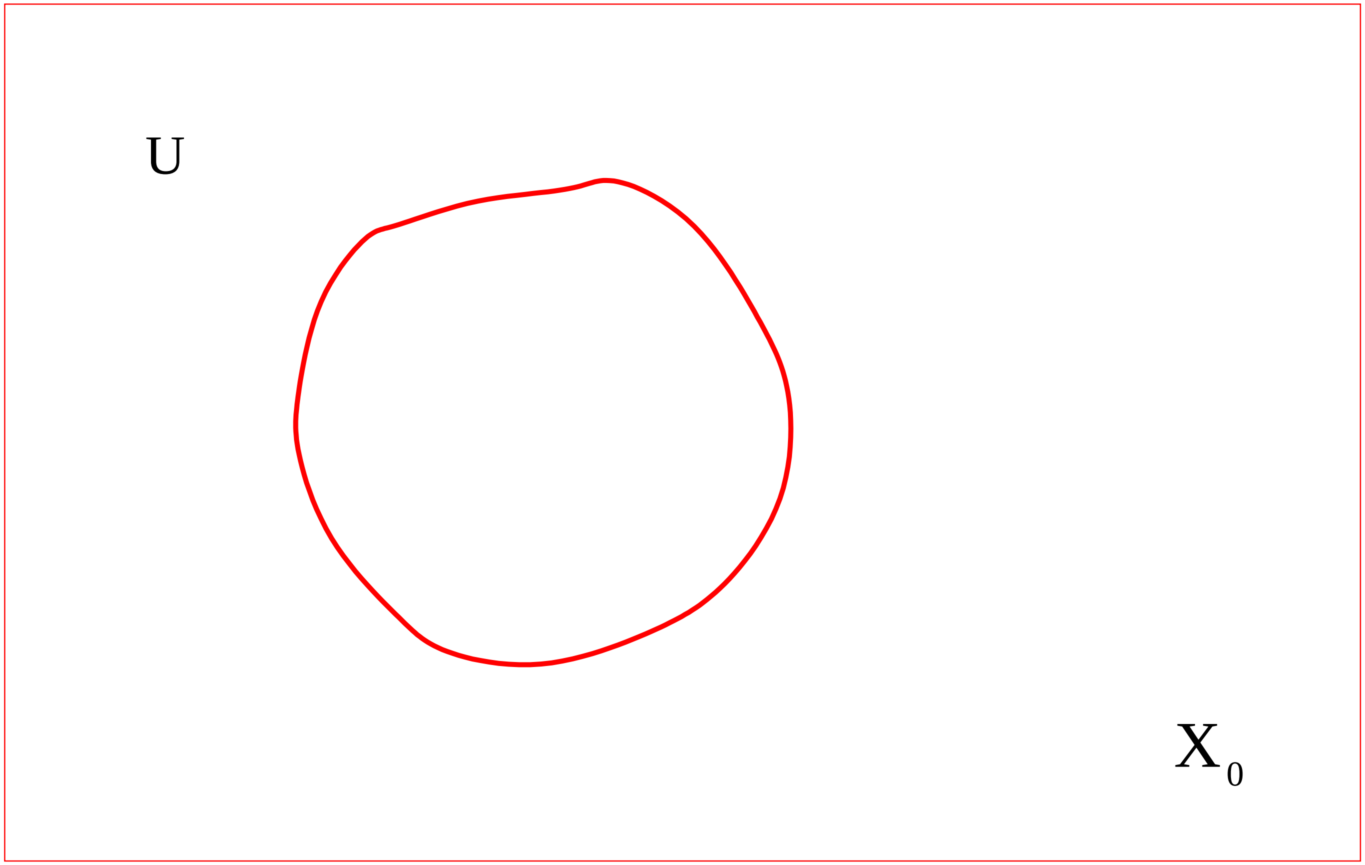
These two conditions guarantee the existence of the attractor of the equation, $\mathcal{A}_0 \subset X_0$.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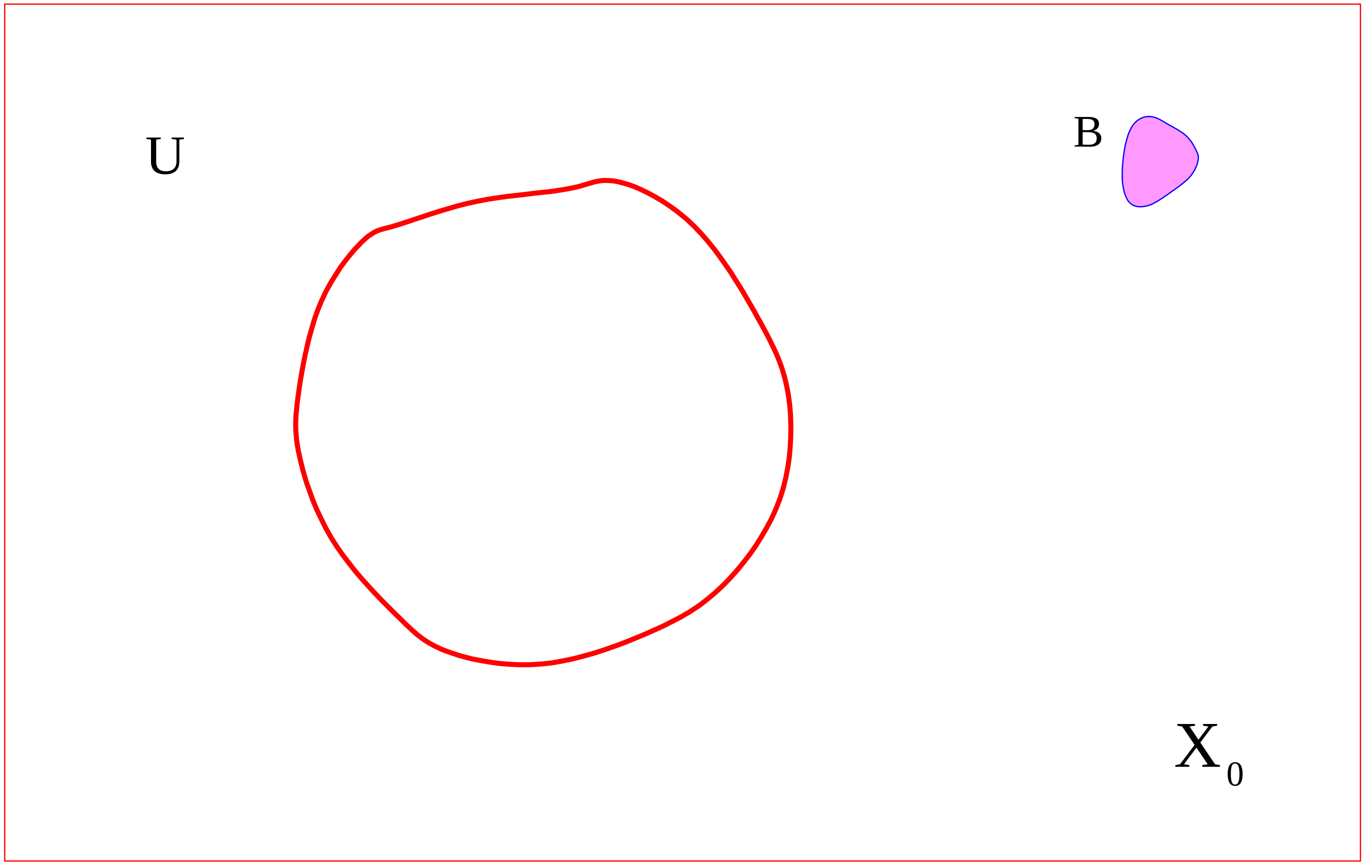
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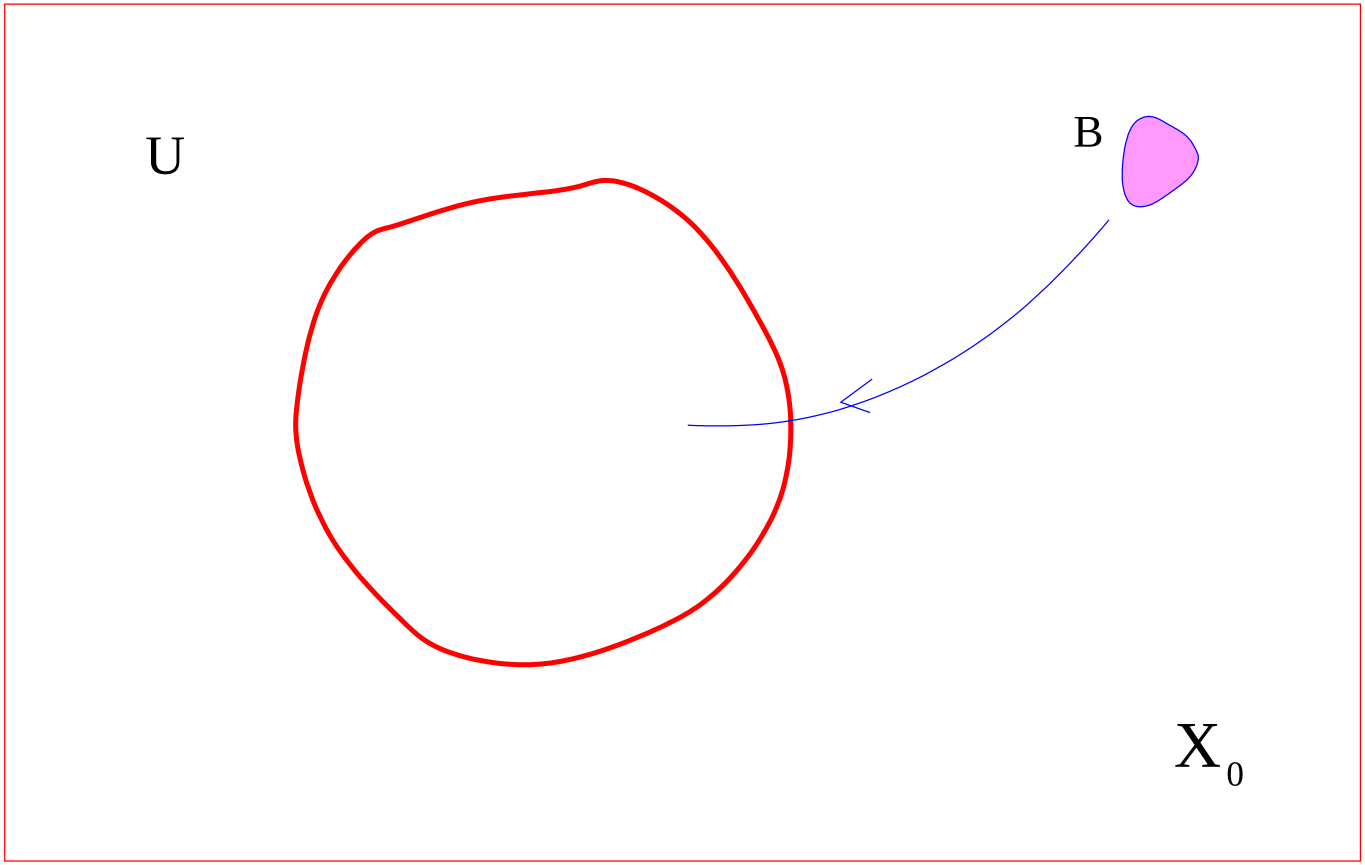
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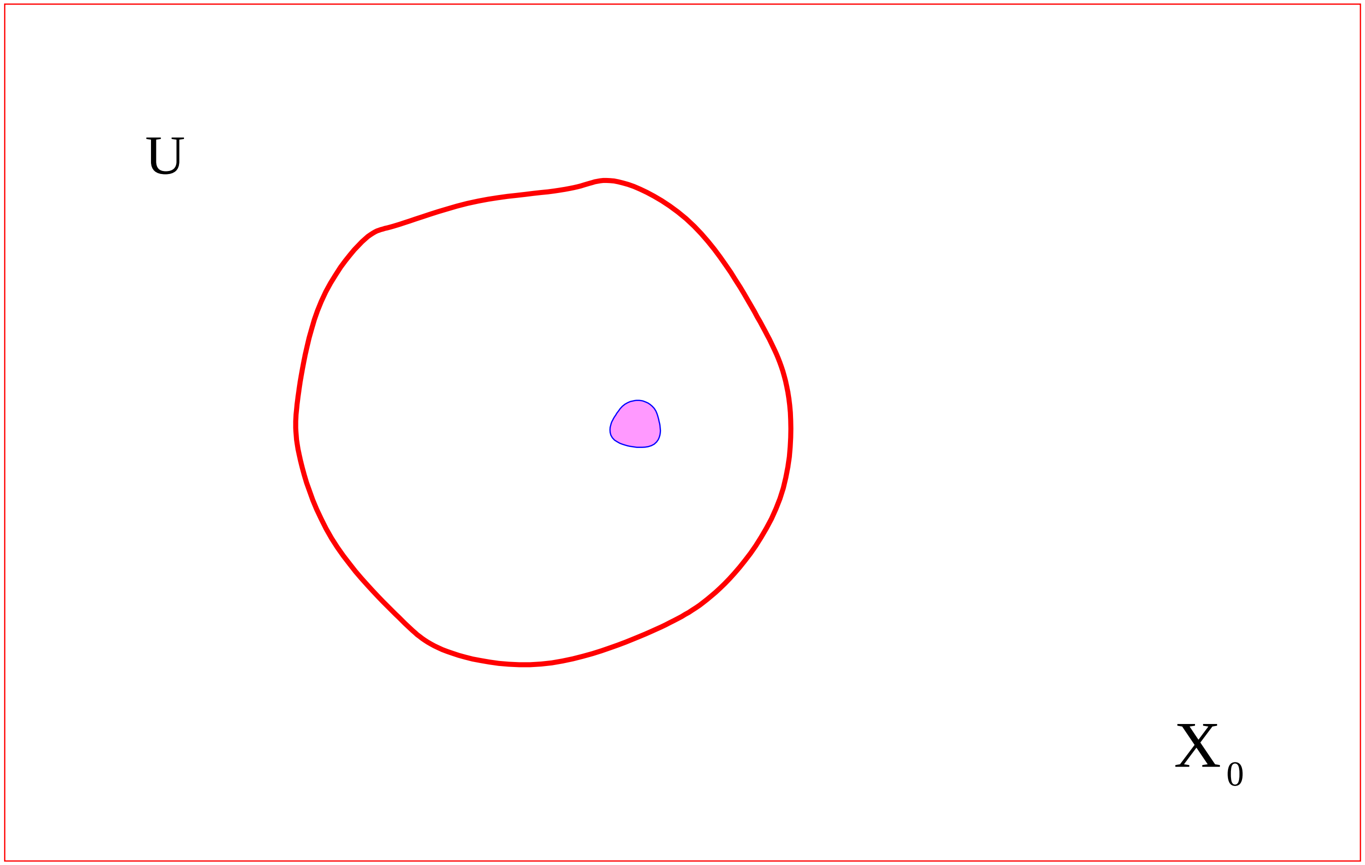
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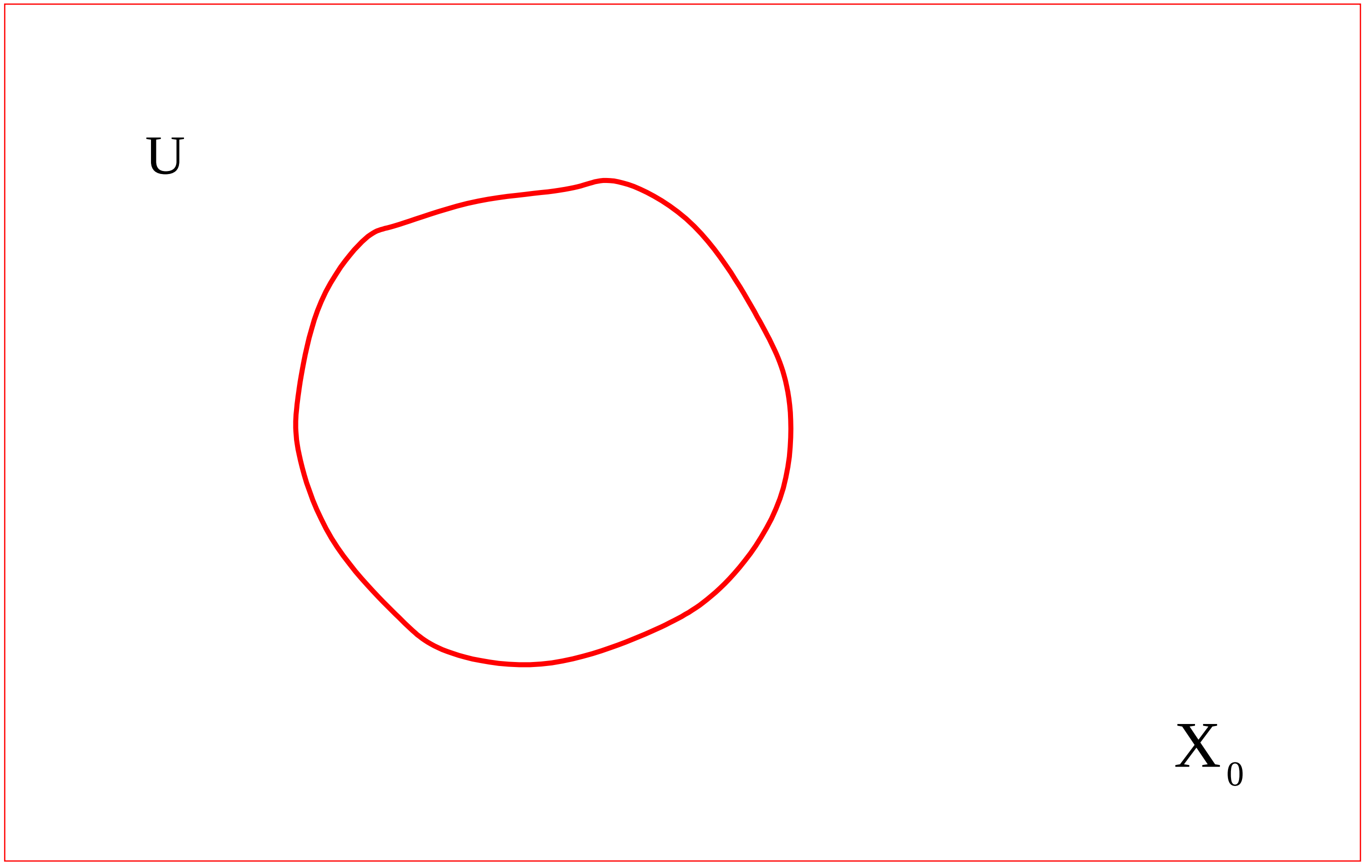
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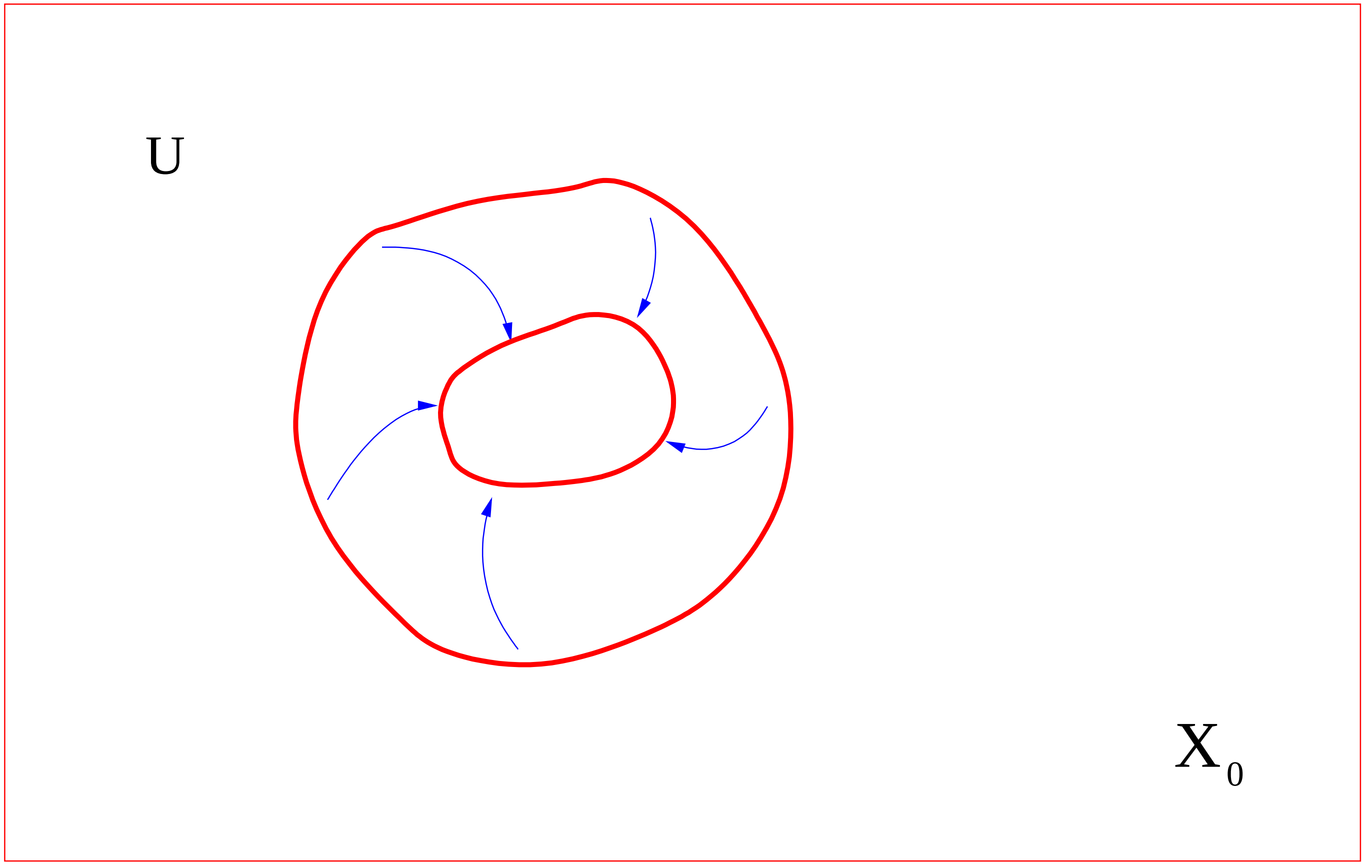
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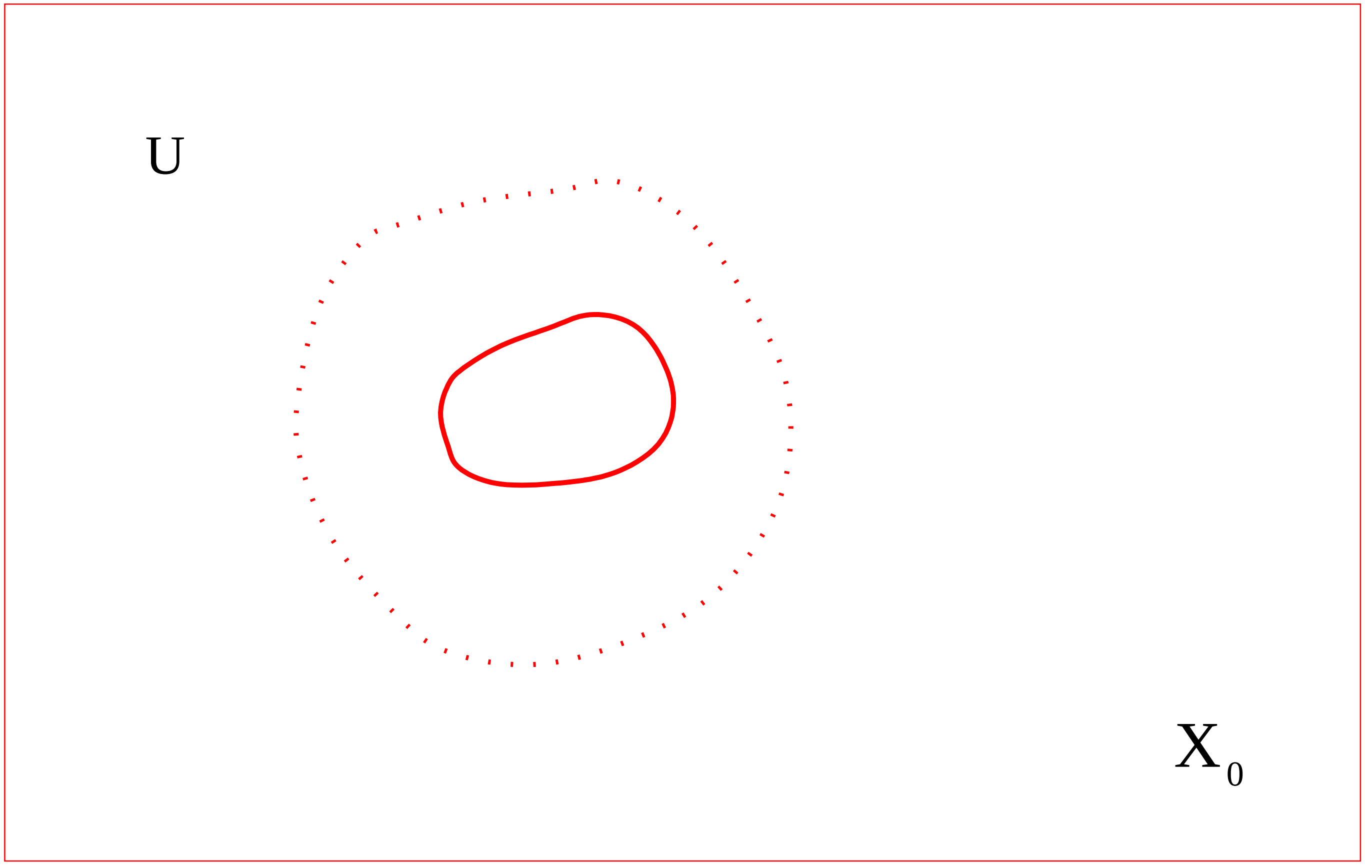
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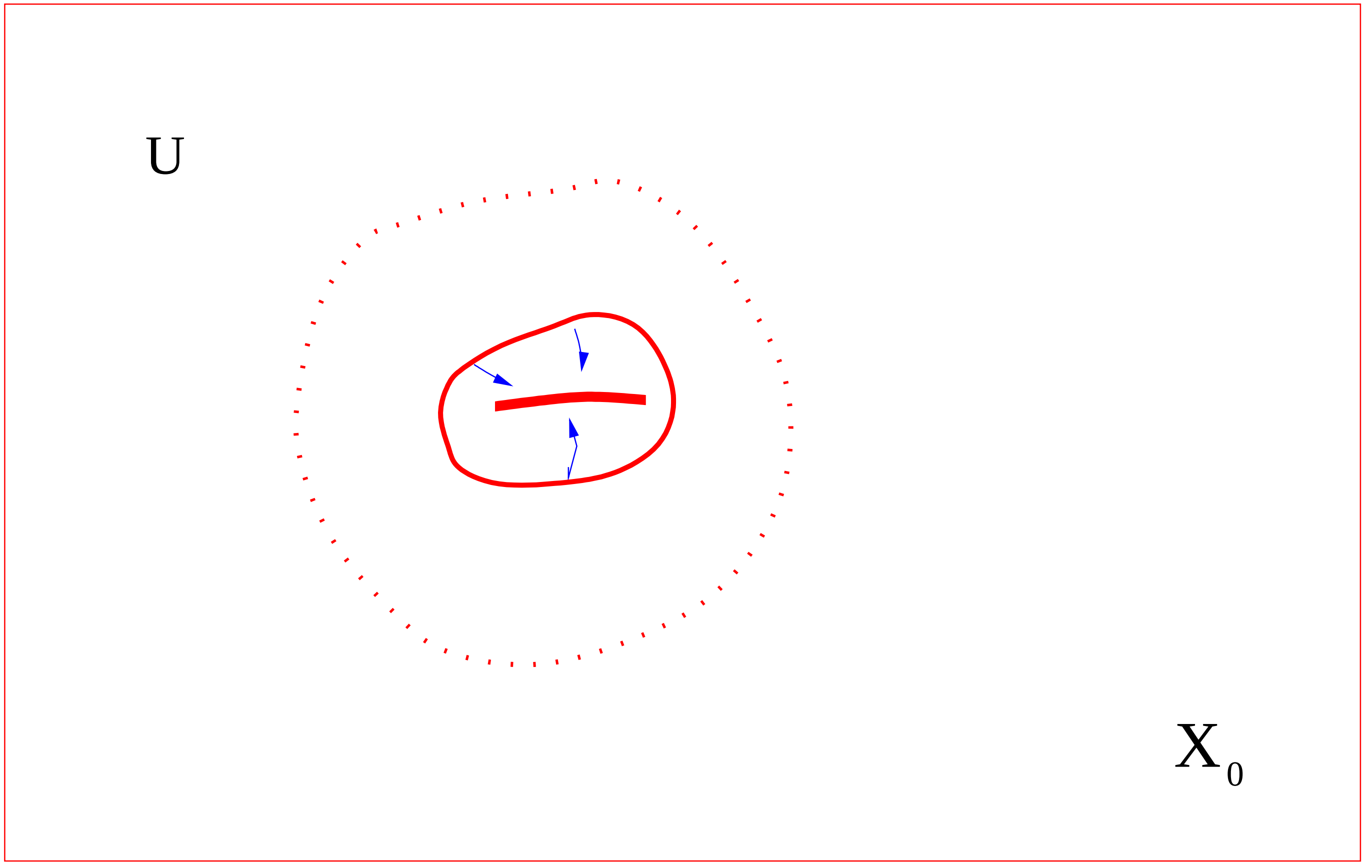
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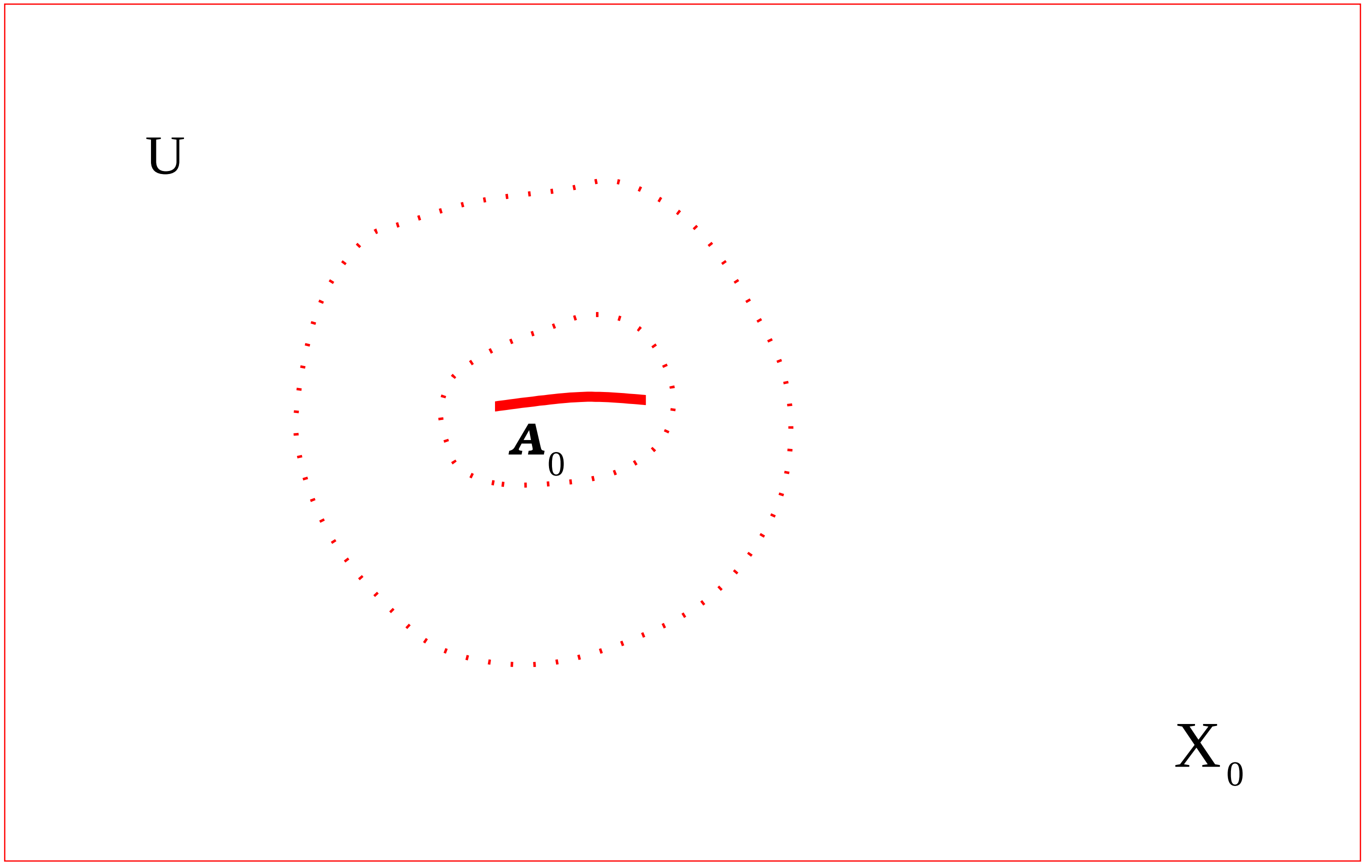
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Attractor: largest compact, invariant set which attracts every bounded set of the phase space.

- It contains all global and bounded orbits: equilibria, periodic orbits, conecting orbits, etc ..

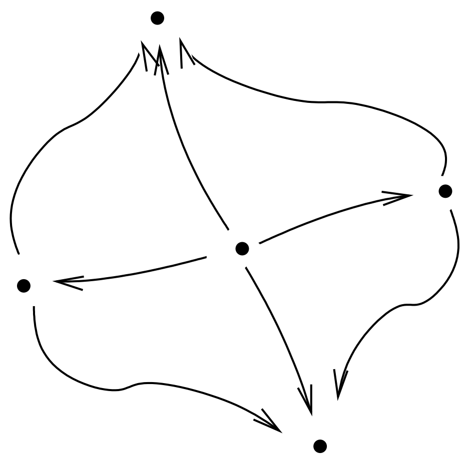
- The dynamics in the attractor contains all the asymptotic dynamics.

- The attractor is a global entity of the dynamical system. Therefore, understanding its structure is far away from being resolved in this generality.

  The attractor may have a very complicated structure and it is not easy to analyze its behavior under perturbations.

Nevertheless, if the dynamical system is gradient, the attractor structure is simpler. It is made of

- Equilibria.
- Conections among equilibria.



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A. Babin, M.I. Vishik “Attractors of evolution equations” North Holland, 1992

R. Temam “Infinite dimensional dynamical systems in mechanics and physics”, Springer 1988

We consider the following problem

$$\begin{cases} x' + A_0 x = F_0(x), & t > 0 \\ x(0) = x_0 \in X_0 \end{cases}$$

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Let us consider a perturbed problem ($0 < \epsilon \leq \epsilon_0$)

$$\begin{cases} x' + A_\epsilon x = F_\epsilon(x), & t > 0 \\ x(0) = x_\epsilon \in X_\epsilon \end{cases}$$

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Questions:

- What is the relation between attractors $\mathcal{A}_0$ and $\mathcal{A}_\epsilon$?.
- Under which conditions we can guarantee that $\mathcal{A}_\epsilon$ is close to $\mathcal{A}_0$?.

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Since we are comparing elements of X_0 with elements of X_ϵ, we need a concept of "closeness" or "convergence" for elements living in different spaces.

If for instance there exists an space Y so that $X_\epsilon \hookrightarrow Y$, $0 \leq \epsilon \leq \epsilon_0$, then we can talk of convergence in Y.

In each case we need to define this concept in a very precise way.

Domain Perturbation

Case 1. General domain perturbation and Neumann boundary conditions.

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

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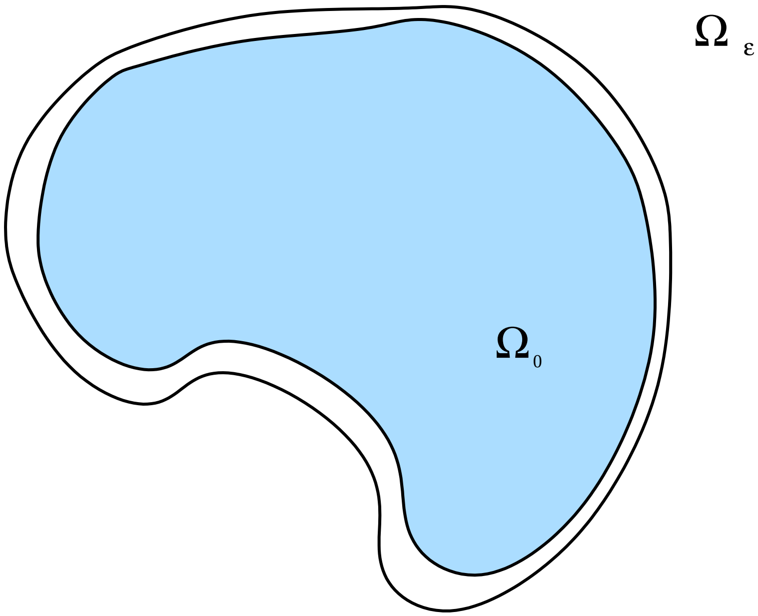

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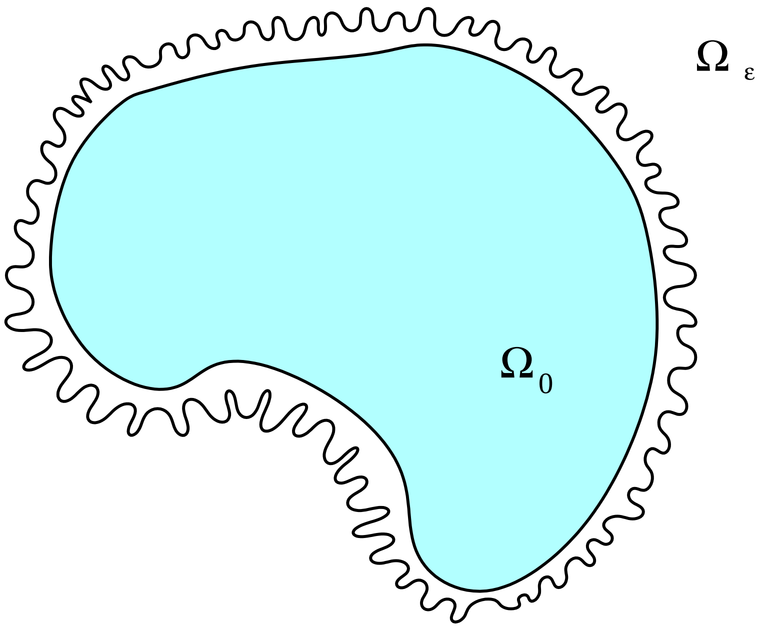
J.A., A.N. Carvalho " Spectral Convergence and Nonlinear Dynamics of Reaction-Diffusion Equations under Perturbations of the Domain " Journal of Differential Equations 199 (2004) pp 143-178

Case 2. Nonlinear boundary conditions and boundary oscillations

$$\begin{cases} u_t - \Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$



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J.A., S.M. Bruschi "Very rapidly varying boundaries in equations with nonlinear boundary conditions", *In preparation*

Dumbell Domain

Case 3. Dumbbell type domain

$$\begin{cases} u_t - \Delta u + u = f(u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

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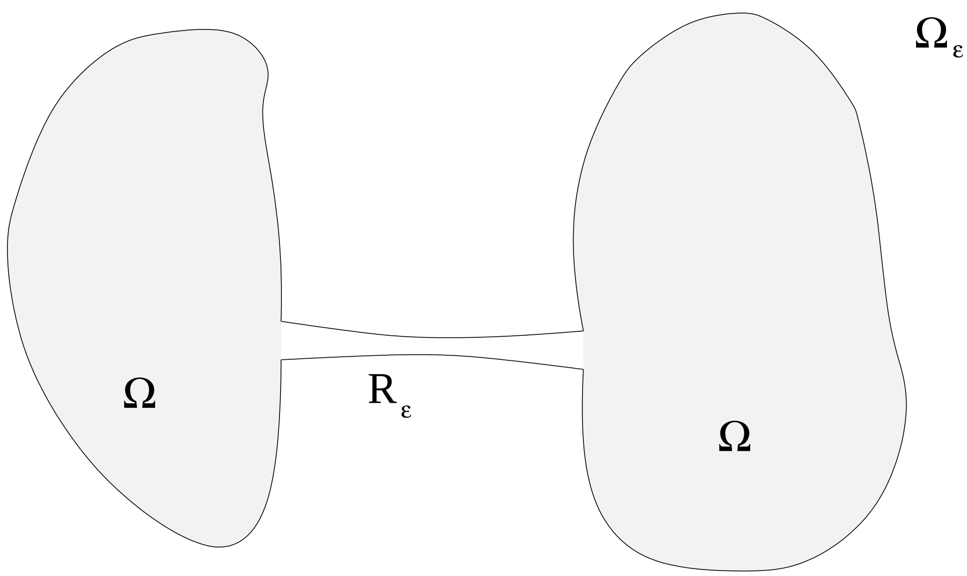
$\Omega_\epsilon = \Omega \cup R_\epsilon \subset \mathbb{R}^N$, $N \geq 2$.



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- J. Hale, J. M Vegas ’83, ’84
- S. Jimbo, Y.Morita ’88-’93
- J. A. ’91-’95
- R. Gadyl’shin ’05

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J.A., A.N. Carvalho, G. Lozada-Cruz "Dynamics in Dumbbell Domains II. Continuity of attractors", In preparation

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, satisfying (to simplify)

$$|f(u)| + |f'(u)| + |f''(u)| \leq M, \quad \forall u \in \mathbb{R}$$

With this hypothesis it is not difficult to see that the equation has an attractor $\mathcal{A}_\epsilon$ for each ϵ which is unifomly bounded in $L^\infty(\Omega_\epsilon)$ by the constant M (using comparison arguments).

In particular, all equilibria (solutions of the nonlinear elliptic problem) are uniformly bounded by M.

The channel R_ϵ is given as:

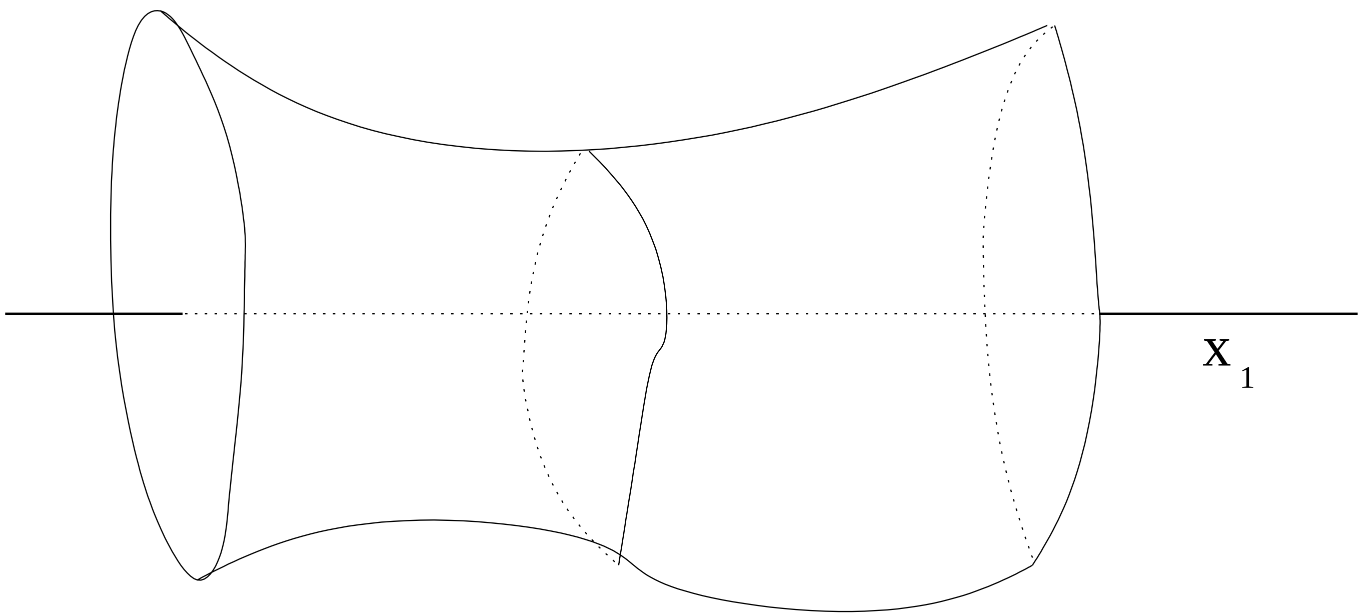
$$R_\epsilon = \{(x_1, \epsilon x') : \ (x_1, x') = x \in R_1\}$$

and

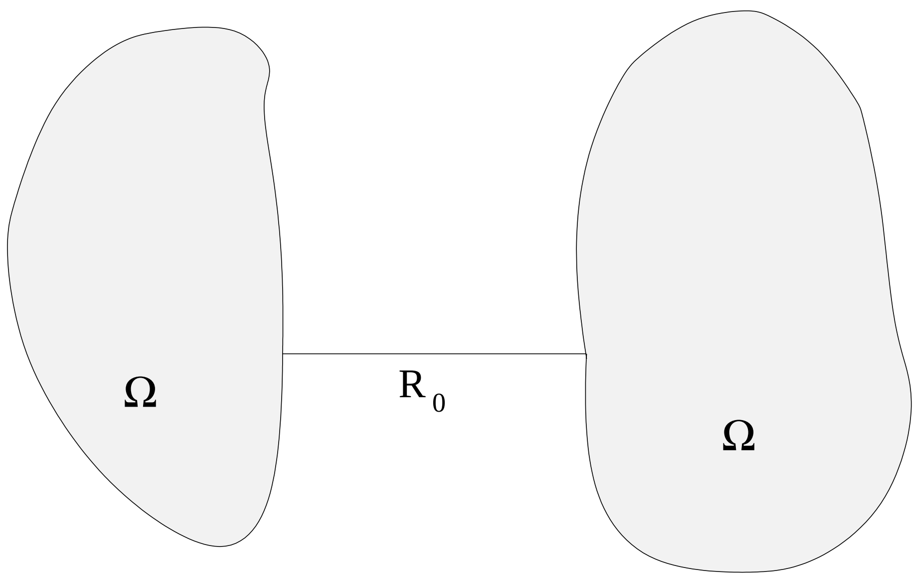
$$R_1 = \{(x_1, x') : 0 \leq x_1 \leq 1, \ x' \in \Gamma_{x_1}\}$$

where Γ_{x_1} is diffeomorphic to the unit ball in $(N-1)$ dimensions and Γ_{x_1} changes smoothly as we move $x_1 \in [0,1]\}$.

In particular, the function $g(x_1) = |\Gamma_{x_1}|$ is smooth.


x_1

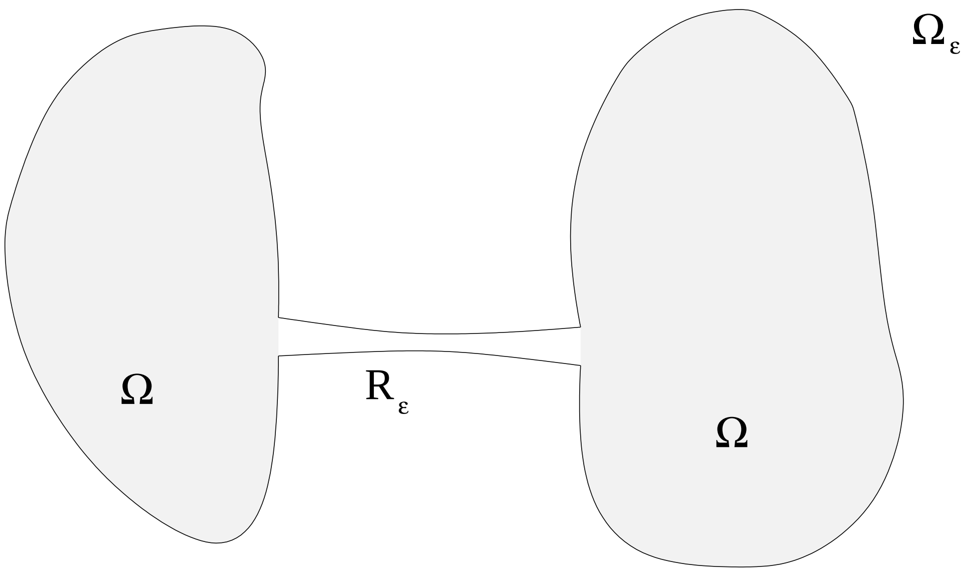
The limit domain is given by



• The behavior of the solutions in the channel R_ϵ is relevant.

For instance, if the equation is $u_t - \Delta u = u - u^3$, we have that $u \equiv 1$ is a constant solution of the equation for all $\epsilon > 0$ and it does not vanish in the channel at least in L^∞ as $\epsilon \to 0$.

This is one difference with respect to the same problem with Dirichlet boundary conditions in which all solutions vanish in the channel as $\epsilon \to 0$.



Other reasons: there exist functions $u_\epsilon \in H^1(\Omega_\epsilon)$ with bounded energy, that is $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C$, which concentrate their mass in the channel R_ϵ.

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Eigenvalue problem: if we denote by $\{(\lambda_n^\epsilon, \varphi_n^\epsilon)\}_{n=1}^\infty$ the set of eigenvalues and orthonormalized eigenfunctions of the operator $-\Delta$ with Neumann boundary conditions in Ω_ϵ,

$$\begin{cases} -\Delta\varphi_n^\epsilon = \lambda_n^\epsilon \varphi_n^\epsilon \quad \text{in } \Omega_\epsilon \\ \frac{\partial \varphi_n^\epsilon}{\partial n} = 0 \quad \text{on } \partial\Omega_\epsilon \end{cases}$$

Then, it is not true that the eigenvalues of λ_n^ϵ converge to the eigenvalues of

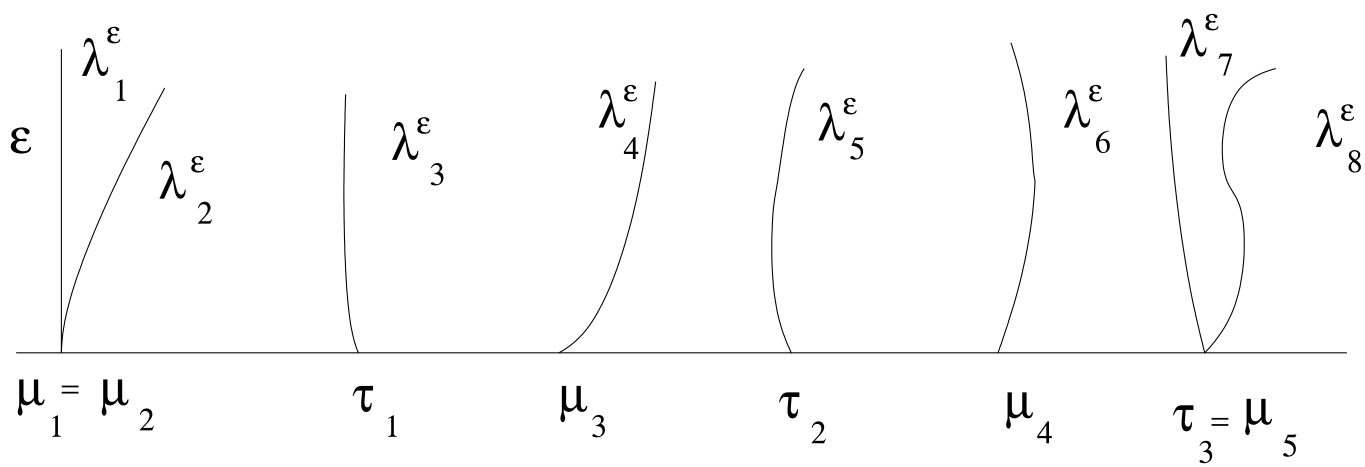
$$\begin{cases} -\Delta\phi_n = \mu_n \phi_n \quad \text{in } \Omega \\ \frac{\partial \varphi_n}{\partial n} = 0 \quad \text{on } \partial\Omega \end{cases}$$

Some of them will do...

But, some eigenfunctions will concentrate on the channel and the eigenvalue will converge to an eigenvalue of the problem

$$\begin{cases} -\frac{1}{g}(g\gamma')' = \tau\gamma \quad \text{in } (0,1) \\ \gamma(0) = \gamma(1) = 0 \end{cases}$$

As a matter of fact, if we denote by $\{\tau_n\}_{n=1}^{\infty}$ the eigenvalues of the problem above and by $\{\mu_n\}_{n=1}^{\infty}$ the eigenvalues in Ω with Neumann boundary conditions we have the following diagram:



• If the behavior in the channel is relevant, then to analyze the convergence of the solutions we should choose a functional space that makes the behavior in the channel relevant.

- bad choices: $H^1(\Omega_\epsilon)$, $L^p(\Omega_\epsilon)$, $1 \leq p < \infty$ with the usual norms

- good but difficult choices: $L^\infty(\Omega_\epsilon)$ or $C(\bar{\Omega}\epsilon)$

- reasonable choices: $U_\epsilon^p = L^p(\Omega_\epsilon)$, $1 \leq p < \infty$ with the following "weighted" norm:

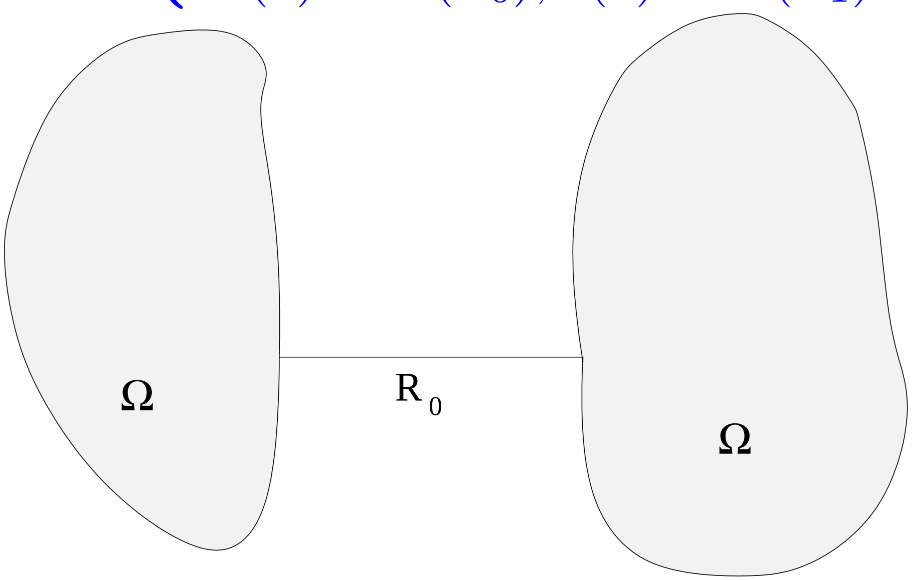
$$\|u\|_{U_\epsilon^p} = \|u\|_{L^p(\Omega)} + \frac{1}{\epsilon^{(N-1)/p}} \|u\|_{L^p(R_\epsilon)}$$

and $U_\epsilon^{1,2} = H^1(\Omega_\epsilon)$ with the norm

$$\|u\|_{U_\epsilon^{1,2}} = \|u\|_{H^1(\Omega)} + \frac{1}{\epsilon^{(N-1)/2}} \|u\|_{H^1(R_\epsilon)}$$

• The limit problem and limit "domain" are

$$\begin{cases} w_t - \Delta w + w = f(w) \quad \text{in } \Omega \\ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \\ \\ v_t - \frac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1) \\ v(0) = w(P_0), v(1) = w(P_1) \end{cases}$$



• The problem is "one sided coupled": w is independent of v but v strongly depends on w through the boundary conditions.

• For a given initial condition (w_0, v_0) we first solve the equation for w,

$$\begin{cases} w_t - \Delta w + w = f(w) \quad \text{in } \Omega \\ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \\ w(0) = w_0 \end{cases}$$

then, we calculate the trace of $w(t, x)$ at $x = P_0 = (0, 0, .., 0)$ and $x = P_1 = (1, 0, .., 0)$, then we solve the v equation

$$\begin{cases} v_t - \frac{1}{g}(gv_x)_x + v = f(v), \quad x \in (0, 1) \\ v(t, 0) = w(t, P_0), v(t, 1) = w(t, P_1) \\ v(0, x) = v_0(x) \end{cases}$$

A natural space to pose the problem is

$$U_0^p = \{u = (w, v) : w \in L^p(\Omega), v \in L_g^p(0,1)\} = L^p(\Omega) \oplus L_g^p(0,1)$$

with the norm:

$$\|u\|_{U_0^p} = \|w\|_{L^p(\Omega)} + \left(\int_0^1 g(x_1)|v(x_1)|^p dx_1 \right)^{\frac{1}{p}}$$

and we require $p > N/2$ to have the trace at P_0 and P_1 of the solution well defined.

We have the following transformations among the spaces: $E_\epsilon : U_0^p \longrightarrow U_\epsilon^p$ given by

$$E_\epsilon(w,v) = \begin{cases} w(x), & x \in \Omega \\ v(x_1), & (x_1, x') \in R_\epsilon \end{cases}$$

which obviously satisfies $\|E_\epsilon(w,v)\|_{U_\epsilon^p} = \|(w,v)\|_{U_0^p}$

We also have $M_\epsilon : U_\epsilon^p \longrightarrow U_0^p$ given by $M_\epsilon u_\epsilon = (w,v) \in U_0^p$ where

$$w = u_{\epsilon|\Omega}$$

$$v(x_1) = \frac{1}{|\Gamma_{x_1}^\epsilon|} \int_{\Gamma_{x_1}^\epsilon} u_\epsilon(x_1, x') dx'$$

Hence

$$\begin{cases} u_t - \Delta u + u = f(u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon, \\ u(0) = \phi_\epsilon \in U_\epsilon^p \end{cases}$$

which can be written as $u_t + A_\epsilon u = F(u)$ in U_ϵ^p where

$$\begin{aligned} A_\epsilon : \mathcal{D}(A_\epsilon) : U_\epsilon^p &\rightarrow U_\epsilon^p \\ u_\epsilon &\rightarrow -\Delta u + u \end{aligned}$$

$$\mathcal{D}(A_\epsilon) = \{u_\epsilon \in W^{2,p}(\Omega_\epsilon) : \frac{\partial u}{\partial n} = 0\}$$

• The equilibria (critical points) are given by the fixed points of

$$u_\epsilon = A_\epsilon^{-1} F(u_\epsilon) \quad \text{in } U_\epsilon^p$$

• Also, the solutions of the nonlinear evolution problem are given by the variation of constants formula:

$$T_\epsilon(t, \phi_\epsilon) = e^{-A_\epsilon t}\phi_\epsilon + \int_0^t e^{-A_\epsilon (t-s)} T(s, \phi_\epsilon) ds$$

and

$$e^{-A_\epsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + A_\epsilon)^{-1} d\lambda$$

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We should study the behavior of $A_\epsilon^{-1} : U_\epsilon^p \longrightarrow U_\epsilon^p$ as $\epsilon \longrightarrow 0$.

Let $f_\epsilon \in U_\epsilon^p$ and let $u_\epsilon = A_\epsilon^{-1} f_\epsilon$, that is

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Let also $(w_\epsilon, v_\epsilon) = A_0^{-1} M_\epsilon f_\epsilon$ be the solution of

$$\begin{cases} -\Delta w_\epsilon + w_\epsilon = f_\epsilon \quad \text{in } \Omega \\ \frac{\partial w_\epsilon}{\partial n} = 0 \quad \text{on } \partial\Omega \\ \\ -\frac{1}{g}(g v_x)_x + v_\epsilon = M_\epsilon f_\epsilon, \quad x \in (0,1) \\ v_\epsilon(0) = w_\epsilon(P_0), v_\epsilon(1) = w_\epsilon(P_1) \end{cases}$$

Proposition. There exists a constant $C > 0$, independent of ϵ and of $f_\epsilon \in U_\epsilon^p$, such that

- If $p > N/2$ then $\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} = \|A_\epsilon^{-1} f_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le C \|f_\epsilon\|_{U_\epsilon^p}$
- If $p > N, \quad 1 \le q < \infty$

$$\|u_\epsilon - w_\epsilon\|_{H^1(\Omega)} + \|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)} \le C\epsilon^{N/2} \|f_\epsilon\|_{U_\epsilon^p}$$

$$\|u_\epsilon - w_\epsilon\|_{L^q(\Omega)} + \|u_\epsilon - v_\epsilon\|_{L^q(R_\epsilon)} \le C\epsilon^{N/q} \|f_\epsilon\|_{U_\epsilon^p}$$

This implies that

$$\|u_\epsilon - w_\epsilon\|_{H^1(\Omega)} + \epsilon^{\frac{-(N-1)}{2}} \|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)} \le C\epsilon^{1/2} \|f_\epsilon\|_{U_\epsilon^p}$$

$$\|u_\epsilon - w_\epsilon\|_{L^q(\Omega)} + \epsilon^{\frac{-(N-1)}{q}} \|u_\epsilon - v_\epsilon\|_{L^q(R_\epsilon)} \le C\epsilon^{1/q} \|f_\epsilon\|_{U_\epsilon^p}$$

Equivalently

$$\|A_\epsilon^{-1} f_\epsilon - E_\epsilon A_0^{-1} M_\epsilon f_\epsilon\|_{U_\epsilon^{1,2}} \leq C\epsilon^{1/2} \|f_\epsilon\|_{U_\epsilon^p}$$
$$\|A_\epsilon^{-1} f_\epsilon - E_\epsilon A_0^{-1} M_\epsilon f_\epsilon\|_{U_\epsilon^q} \leq C\epsilon^{1/q} \|f_\epsilon\|_{U_\epsilon^p}$$

And this represents some convergence of A_ϵ^{-1} to A_0^{-1}.

An appropriate way to define "convergence of operators" which are defined in different spaces is as follows:

Definition 1. $u_\epsilon \xrightarrow{E} u$ *if* $\|u_\epsilon - E_\epsilon u\|_{U_\epsilon} \xrightarrow{\epsilon \to 0} 0$.

Definition 2. $\{u_\epsilon\}_\epsilon$ *is* E*-pre-compact if for each sequence* $\{u_{\epsilon_n}\}$ *there is a subsequence* $\{u_{\epsilon_{n'}}\}$ *and an element* $u \in U_0$ *such that* $u_{\epsilon_{n'}} \xrightarrow{E} u$.

Definition 3. $B_\epsilon \xrightarrow{EE} B_0$ *if* $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$ *whenever* $u_\epsilon \xrightarrow{E} u \in U_0$.

Definition 4. $B_\epsilon \xrightarrow{CC} B_0$ *if*

i) B_ϵ, B_0 *are compact operators.*

ii) $\{B_\epsilon u_\epsilon\}$ *is E-precompact for any family* $\{u_\epsilon\}$ *with* $\|u_\epsilon\|_{U_\epsilon} \leq R$, *for any* $R > 0$.

iii) $B_\epsilon \xrightarrow{EE} B_0$

The Proposition we have proved implies

Theorem 5. *If $p > N$, we have* $\mathcal{L}(U_\epsilon^p) \ni A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1} \in \mathcal{L}(U_0^p)$

This theorem has a lot of consequences:

- $\sigma(A_\epsilon) \longrightarrow \sigma(A_0)$ and the spectral projections behave also continuously.

- If $V_\epsilon \in L^\infty(\Omega_\epsilon)$ and $V_\epsilon \xrightarrow{E} V_0$ and $\exists\,(A_0 + V_0)^{-1}$, then for ϵ small enough $\exists\,(A_\epsilon + V_\epsilon)^{-1}$ and $(A_\epsilon + V_\epsilon)^{-1} \xrightarrow{CC} (A_0 + V_0)^{-1}$

- $A_\epsilon^{-1} \circ f \xrightarrow{CC} A_0^{-1} \circ f$

Proposition 6. *(Uppersemicontinuity of fixed points) If $N_\epsilon : U_\epsilon \to U_\epsilon$ is a sequence of nonlinear operators with $N_\epsilon \xrightarrow{CC} N_0$ and $u_\epsilon^* = N_\epsilon(u_\epsilon^*)$ with $\|u_\epsilon^*\|_{U_\epsilon} \leq R$ then there exists a subsequence $u_{\epsilon'}$ and a fixed point of the limit problem u_0^* such that $u_{\epsilon'} \xrightarrow{E} u_0^*$.*

• Applying this result to $A_\epsilon^{-1} \circ f \xrightarrow{CC} A_0^{-1} \circ f$ and since the equilibria is uniformly bounded in L^∞ (and therefore in U_ϵ^p) we get that the set of equilibria is uppersemicontinuous in U_ϵ^p.

• If $u_\epsilon^* \xrightarrow{E} u_0^*$ then $V_\epsilon(\cdot) = f'(u_\epsilon^*(\cdot)) \xrightarrow{E} f'(u_0^*(\cdot))$. Therefore if $(A_0 - f'(u_0^*) + C)$ is invertible then so is $(A_\epsilon - f'(u_\epsilon^*) + C)$ for ϵ small and

$$(A_\epsilon - f'(u^*(\cdot) + C)^{-1} \xrightarrow{CC} (A_0 - f'(u_0^*(\cdot) + C)^{-1}$$

and this implies that $\sigma(A_\epsilon - f'(u_\epsilon^*)) \to \sigma(A_0 - f'(u_0^*))$

In particular if u_0^* is a hyperbolic equilibria, $\exists\,(A_0 - f'(u_0^*))^{-1}$, and if $u_\epsilon^* \xrightarrow{E} u_0^*$, then u_ϵ^* is also a hyperbolic equilibria. We can actually prove something else:

Proposition 7. *If u_0^* is a hyperbolic equilibrium of the limit problem, there exists $\delta > 0$ small such that the problem in the dumbbell domain has one and only one solution satisfying $\|u_\epsilon^* - Eu_0^*\|_{U_\epsilon^p} < \delta$. Moreover*

- $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon^p} \to 0$.
- $\sigma(A_\epsilon - f'(u_\epsilon^*)) \to \sigma(A_0 - f'(u_0^*))$ *and the spectral projections also converge.*

With respect to the evolution equation

$$(\lambda + A_\epsilon)^{-1} - E_\epsilon(\lambda + A_0)^{-1}M_\epsilon =$$
$$= (I - \lambda(\lambda + A_\epsilon)^{-1}) \circ (A_\epsilon^{-1} - E_\epsilon A_0^{-1}M_\epsilon) \circ (I + E_\epsilon \lambda(\lambda + A_0)^{-1}M_\epsilon)$$

which implies

$$\|(\lambda + A_\epsilon)^{-1} - E_\epsilon(\lambda + A_0)^{-1}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, L^q(\Omega_\epsilon))} \leq C\epsilon^{N/q}(1 + |\lambda|^{1-\alpha})$$

for some $0 < \alpha < 1$

And usign the expresion:

$$e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}((\lambda + A_\epsilon)^{-1} - E_\epsilon(\lambda + A_0)^{-1} M_\epsilon) d\lambda$$

we conclude

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leq C \epsilon^{1/q} e^{\beta t} t^{-2+\alpha}$$

But using that

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L^\infty, L^\infty)} \leq C$$

and interpolation, we get

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leq C e^{\beta t} t^{-\gamma} \theta(\epsilon)$$

Now, with the aid of the variation of constants formula:

$$T_\epsilon(t,\phi_\epsilon) = e^{-A_\epsilon t}\phi_\epsilon + \int_0^t e^{-A_\epsilon(t-s)}T(s,\phi_\epsilon)ds$$

$$T_0(t,\phi_0) = e^{-A_0 t}\phi_0 + \int_0^t e^{-A_0(t-s)}T(s,\phi_0)ds$$

we will be able to compare the nonlinear evolution problems and prove:

Proposition 8. *There exists a* $0 \leqslant \gamma < 1$ *and a function* $c(\epsilon)$ *with* $c(\epsilon) \xrightarrow{\epsilon \to 0} 0$ *such that, for each* $\tau > 0$ *we have*

$$\|T_\epsilon(t, u_\epsilon) - E_\epsilon T_0(t, M_\epsilon u_\epsilon)\|_{U^p_\epsilon} \leq M(\tau) c(\epsilon) t^{-\gamma}, \quad t \in (0, \tau], \quad u_\epsilon \in \mathcal{A}_\epsilon.$$

Moreover, the family of attractors $\{\mathcal{A}_\epsilon : \epsilon \in [0, \epsilon_0]\}$ *is upper semicontinuous at* $\epsilon = 0$ *in* U^p_ϵ*, in the sense that*

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} \left[\inf_{u_0 \in \mathcal{A}_0} \{\|u_\epsilon - E_\epsilon u_0\|_{U^p_\epsilon}\} \right] \to 0, \quad \textit{as } \epsilon \to 0$$

Finally,

● Requiring that all equilibria of the limit problem is hyperbolic

● Analyzing in detail the convergence of the “local unstable manifold” of each equilibria

● Using the gradient structure of the limit equation

we show the lower semicontinuity (and therfore the continuity) of the attractors.