On Farber's Topological Complexity

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Joint work with L. Fernández Suárez,
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24/11/2007

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How many “continuous formulas” are necessary?

$\rightsquigarrow$ **Topological complexity**

Definition. (M. Farber) $TC(X)$ is the least integer n such that $X \times X$ can be covered by $n+1$ open sets on each of which

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Another particular case of this is the Lusternik-Schnirelmann category:

$$\mathrm{cat}(X) = \mathrm{secat}(ev_0 : PX \to X)$$

where $PX = \{\lambda : I \to X, \lambda(1) = *\}$.

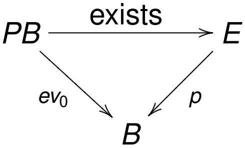
Theorem. (A. Schwarz) For a fibration $p : E \to B$ we have

$$\mathrm{nil}(\ker p^*) \leq \mathrm{secat} p \leq \mathrm{cat} B$$

where

- $p^* : H^*(B) \to H^*(E)$ is the morphism induced by p
- $\mathrm{nil}(\ker p^*) \leq n$ iff every product of length $n+1$ in $\ker p^*$ is 0.

$\mathrm{secat}\, p \leq \mathrm{cat} B$ because:



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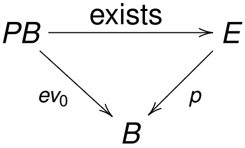

For $p = ev_{0,1} : X^I \to X \times X$ we get

$$\mathrm{TC}(X) \leq cat(X \times X) \leq 2\,\mathrm{cat} X.$$

On the other hand, we have

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X^I \\ & {\scriptstyle \Delta} \searrow \quad \swarrow {\scriptstyle p = ev_{0,1}} & \\ & X \times X & \end{array}$$

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so that, with $H^*() = H^*(-, \Bbbk)$,

$$\mathrm{nil}(\ker p^*) = \mathrm{nil}(\ker \smile)$$

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In $H^*(X) \otimes H^*(X)$, $(a \otimes b)(a' \otimes b') = (-1)^{deg(b)deg(a')} aa' \otimes bb'$.

Theorem. (M. Farber)

$$\left.\begin{array}{c} \mathrm{nil}(\ker \smile) \\ \mathrm{cat}(X) \end{array}\right\} \leq \mathrm{TC}(X) \leq \left\{\begin{array}{ll} 2\mathrm{cat}(X) & \\ \dim(X) & (X \text{ simply-connected}) \end{array}\right.$$

Examples.

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  and $\mathrm{TC}(S^n) = 2$ if n is even.

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  To define the section on U_1 choose a continuous non-zero vector field on S^n. For $(x, y) \in U_1$ go from x to $-x$ through the meridian determined by the vector field at x and then from $-x$ to y by the shortest great circle arc.

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Our objective: Find a computable lower bound of TC which is better than $\mathrm{nil}(\ker \smile)$.

$$\mathrm{nil}(\ker \smile) \;\leq \qquad\quad \uparrow \qquad\quad \leq\; \mathrm{TC}(X)$$

The join of 2 fibrations $p : E \to B$ and $p' : E' \to B$ is the map

$$\begin{array}{rcl} E *_B E' := E \amalg (E \times_B E' \times [0,1]) \amalg E' / \sim & \to & B \\ \langle e, e', t \rangle & \mapsto & p(e) = p(e') \end{array}$$

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This map is a fibration with fibre

$$F * F' = F \amalg F \times F' \times [0,1] \amalg F' / \sim$$

where F and F' are the respective fibres of p and p'.

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Corollary. If X is normal, then

$$\mathrm{TC}(X) \leq n \iff j^n(ev_{0,1}) : *_{X \times X}^n X^I \to X \times X \text{ admits a section.}$$

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  - In particular, $H(A_{PL}(X)) = H^*(X; \mathbb{Q})$.
- Sullivan model of a space X:

$$(\Lambda V, d) \xrightarrow{\sim} A_{PL}(X)$$

If $d(V) \subset \Lambda^{>1}(V)$ the model is said to be *minimal*. In this case $V \cong$ dual of $\pi_*(X) \otimes \mathbb{Q}$.

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If $V = \mathbb{Q}x$, then $\Lambda V = \begin{cases} \mathbb{Q}\cdot 1 \oplus \mathbb{Q}x \oplus \mathbb{Q}x^2 \oplus \cdots & n \text{ even} \\ \mathbb{Q}\cdot 1 \oplus \mathbb{Q}x & n \text{ odd} \end{cases}$

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Hence a Sullivan model of S^n is given by:

$$\begin{cases} \Lambda(x) \text{ with } dx = 0 & n \text{ odd} \\ \\ \Lambda(x, y) \text{ with } dx = 0,\ deg(y) = 2n-1,\ dy = x^2 & n \text{ even} \end{cases}$$

Let $\quad F \to E \xrightarrow{p} B \quad$ a fibration.

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By applying A_{PL} we get:

$$A_{PL}(B) \xrightarrow{A_{PL}(p)} A_{PL}(E) \longrightarrow A_{PL}(F)$$

$$\begin{array}{ccccc} A_{PL}(B) & \longrightarrow & A_{PL}(E) & \longrightarrow & A_{PL}(F) \\ \sim\uparrow & & & & \sim\uparrow \\ (A,d) & & & & (H,0) \end{array}$$

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Such a diagram is a semi-free model of the fibration p.

Model of $*^n F \longrightarrow *^n_B E \stackrel{j^n(p)}{\longrightarrow} B$

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Starting with a semi-free model of p

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Model of $*^n F \to *^n_B E \stackrel{j^n(p)}{\to} B$

Starting with a semi-free model of p

$$(A,d) \rightarrowtail (A \otimes H, d) \longrightarrow (H, 0)$$

and writing $H = \mathbb{Q} \oplus X$ we obtained a semi-free model of $j^n(p)$ of the form

$$(A,d) \rightarrowtail (A \otimes (\mathbb{Q} \oplus s^{-n}(X^{\otimes n+1})), d) \longrightarrow (\mathbb{Q} \oplus s^{-n}(X^{\otimes n+1}), 0)$$

with $(s^{-n}V)^k = V^{k-n}$ and an explicit differential.

Application to TC:

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A semi-free model of $ev_{0,1}$ is:

$$\begin{array}{ccccc}
A_{PL}(X \times X) & \longrightarrow & A_{PL}(X^I) & \longrightarrow & A_{PL}(\Omega X) \\
\uparrow \sim & & \uparrow \sim & & \uparrow \sim \\
(\Lambda(V \oplus V'), d) & \rightarrowtail & (\Lambda(V \oplus V') \otimes \Lambda sV, d) & \longrightarrow & (\Lambda sV, 0)
\end{array}$$

where $(sV)^k = V^{k+1}$ and the differential d is given by

$$dsv = v' - v - \sum_{i=1}^{\infty} \frac{(\zeta d)^i}{i!}(v)$$

where ζ is the derivation given by $\zeta(v) = \zeta(v') = sv$ and $\zeta(sv) = 0$.

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Definition. $\mathrm{MTC}(X)$ is the least n such that the morphism

$$(\Lambda(V \oplus V'), d) \to (J_n, d)$$

admits a retraction of $(\Lambda(V \oplus V'), d)$-modules.

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Proposition. If X is formal, that is $(H^*(X), 0)$ is a CDGA-model of X, then

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Theorem. There exists a (non formal) space $X = S^3 \vee S^3 \cup e^8 \cup e^8$ for which

$$\mathrm{nil}(\ker(\smile)) = 2 \quad \text{and} \quad \mathrm{MTC}(X) = 3.$$

Theorem. (i) For any n, there exists a finite CW-complex X such that

$$\mathrm{MTC}(X) - \mathrm{nil}(\ker \smile) \geq n.$$

(ii) There exists a space X with $\mathrm{MTC}(X) = \infty$ and $\mathrm{nil}(\ker \smile) < \infty$.