# The homology of amalgams of topological groups

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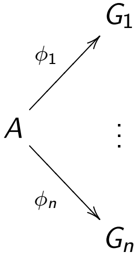
$\phi_i \colon A \to G_i$ group homomorphisms. The *amalgam* $\underset{A}{*} G_i$ is the *colimit* of the diagram:



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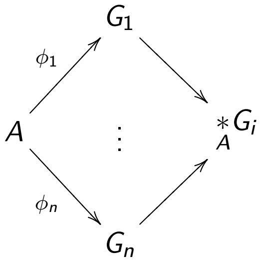

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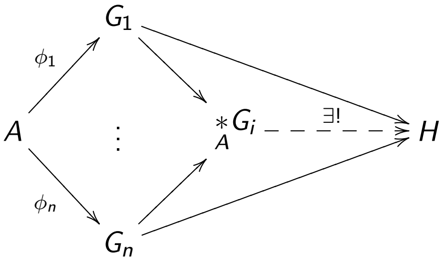

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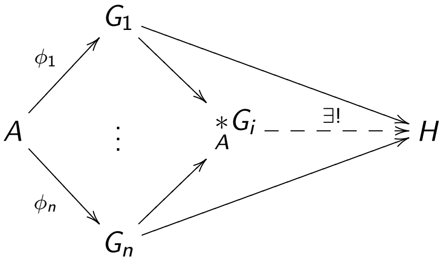


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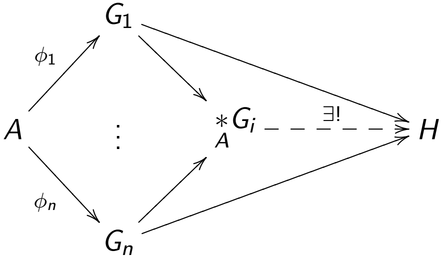


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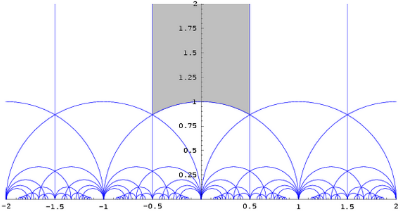
**Aim:** Compute the Pontryagin algebra $H_*\left(\underset{A}{*} G_i\right)$ for $\phi_i$ inclusions.

# Examples of amalgams

1. $SL(2;\mathbb{Z}) = \mathbb{Z}/4 \underset{\mathbb{Z}/2}{*} \mathbb{Z}/6$

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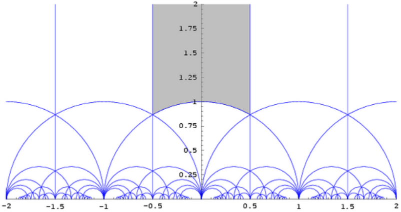


Figure: The tree of $SL(2;\mathbb{Z})$.

2. $\mathrm{Diff}(S^2 \times S^2, \omega) \simeq \mathrm{colim}(S^1 \times SO(3) \leftarrow SO(3) \xrightarrow{\triangle} SO(3) \times SO(3))$ if $\omega(S^2 \times 1) = 1, \omega(1 \times S^2) \in ]1, 2]$.

# More examples of amalgams

3. $\mathrm{Diff}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega) \simeq \mathrm{colim}(U(2) \xleftarrow{(1,0)} S^1 \xrightarrow{(2,1)} U(2))$ if $\omega(\mathbb{C}P^1) = 1, \omega(E) = \lambda \in [1, 2[$.

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4. $K$ simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

$$\mathop{*}_{B} P_i \xrightarrow{\pi} K$$

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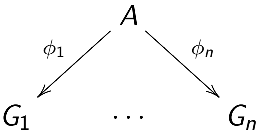
The homomorphism $\pi$ induces a surjection on homology [Kitchloo].

# Main theorem

**Theorem:** Consider the diagram $F\colon I \to \mathrm{TopGps}$ described by

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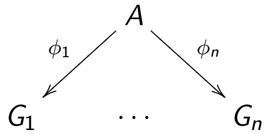


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1. The following canonical map is a weak equivalence

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2. There is a functor $G\colon \Pi_n \to \mathrm{Spaces}$ and a spectral sequence of graded algebras

$$E^2_{k,j} = \operatorname*{colim}_{w\in\Pi_n}{}_j H_k G(w) \Rightarrow H_{j+k}(\mathop{*}_{A} G_i).$$

# Remarks on the main Theorem

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  In this case the $E_2$ term of the spectral sequence is concentrated on the 0-line which is given by

$$E^2_{k,0} = \operatorname*{colim}_{w \in \Pi} H_k G(w) = \left( \mathop{*}_{H_*(A)} H_*(G_i) \right)_k .$$

# Colimits

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$$\begin{array}{ccc} F(i) & & \\ {\scriptstyle F(\alpha)}\downarrow & \searrow^{\phi_i} & \\ F(j) & \xrightarrow{\phi_j} & C \end{array}$$

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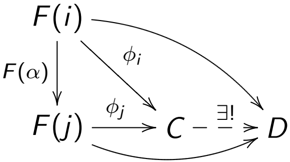

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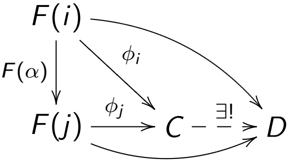


**Examples:** Let $\mathcal{C} = Sets$.

1. $\operatorname{colim}\left(X \xleftarrow{f} A \xrightarrow{g} Y\right) = \left(X \coprod Y\right)/f(a) \sim g(a)$.
2. For $I = G$ a (discrete) group,

$$\operatorname{colim}\left( X \circlearrowleft G \right) = X/G.$$

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$$S^n \qquad\qquad *$$

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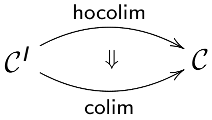
**1st definition of homotopy colimit:** hocolim is the terminal homotopy invariant functor mapping to colim



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This suggests a construction of the homotopy colimit.

# Examples of homotopy colimits of spaces

- $\operatorname{hocolim}\left(X \stackrel{f}{\leftarrow} A \stackrel{g}{\rightarrow} Y\right) =$
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  If the action is free the map $EG \times_G X \to X/G$ is a weak equivalence.
- Inclusions of topological groups are not cofibrations of topological groups!

# Homotopy colimits of topological groups

**Theorem [Kan]:** There is an equivalence of homotopy theories

$$\mathrm{Ho}(\mathrm{TopGps}) \leftrightarrow \mathrm{Ho}(\mathrm{ConnectedSpaces})$$

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$\Pi_n$ is a monoidal category with product given by concatenation. The unit is the empty word.

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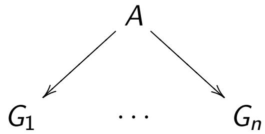
Given a diagram of topological groups




and $w = (a_1, \ldots, a_k) \in \Pi_n$ with $a_i \in \{1, \ldots, n\}$ define

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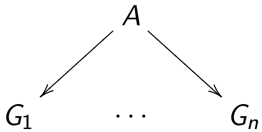


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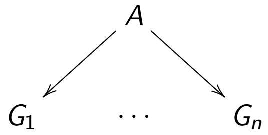


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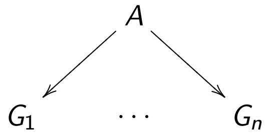


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- $\operatorname*{colim}_{w \in \Pi_n} G(w) = \mathop{*}_A G_i$
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- $\operatorname{colim}_{w \in \Pi_n} G(w) = \underset{A}{*} G_i$
- $G \colon \Pi_n \to (A - \text{Spaces} - A)$ is a monoidal functor, hence
- $\operatorname{hocolim}_{w \in \Pi_n} G(w)$ is a monoid and the canonical map hocolim $\to$ colim is a map of monoids.

# Homotopy colimits of topological groups III

**Prop:** $\operatorname*{hocolim}_{w \in \Pi_n} G(w)$ is homotopy equivalent to

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This implies the first part of the Theorem.

# The spectral sequence

Given $F\colon I \longrightarrow \text{Spaces}$ there is a standard spectral sequence

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Applying this to $G\colon \Pi_n \to$ Spaces gives the spectral sequence in the second part of the Theorem.

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**Example:** For $X \circlearrowleft^{G}$ this is the usual spectral sequence $H_p(G;H_q(X;R)) \Rightarrow H_{p+q}(EG\times_G X;R)$.

Applying this to $G\colon \Pi_n \to \text{Spaces}$ gives the spectral sequence in the second part of the Theorem.

$G$ monoidal $\Rightarrow$ the spectral sequence is multiplicative.

# Rank 1 parabolics

**Recall:** $K$ simply connected unitary form of a Kač-Moody group. There is a surjective homomorphism

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Want to prove the algebra $H_*(\mathop{*}_{T^n} K_i)$ with $K_i$ one of the two groups above is finitely generated.

# A cell decomposition of $SU(2)$

$$SU(2) = \left\{ \begin{bmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{bmatrix} : |z_1|^2 + |z_2|^2 = 1 \right\}.$$

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$e^2$ provides a transverse to both right and left actions of $S^1$ on $SU(2) \setminus S^1$.

$$e^3 = e^1 e^2$$

# The homology DGA of $SU(2)$

The cellular chains form a differential graded algebra

$$C_*(SU(2);\mathbb{Z}) = \mathbb{Z}\langle x_1, z_2\rangle / \langle x_1^2, z_2^2, x_1 z_2 + z_2 x_1\rangle$$

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This gives a simple formula for the differential graded algebra of cellular chains $C_*(\underset{T^n}{*} K_i;\mathbb{Z})$ in this case.

# A simple example

$$C_*(SU(2) \times S^1 \underset{T^2}{*} S^1 \times SU(2); \mathbb{Z}) = \mathbb{Z}[x_1, y_1, z_2, w_2]/J$$

with $J$ the ideal

$$J = \langle x_1^2, y_1^2, z_2^2, w_2^2, x_1z_2+z_2x_1, y_1w_2+w_2y_1, x_1y_1+y_1x_1, z_2y_1-y_1z_2, w_2x_1-x_1w_2$$

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It follows that

$$H_*(SU(2) \times S^1 \underset{T^2}{*} S^1 \times SU(2); \mathbb{Z}) = \mathbb{Z}(A_3, B_3) \otimes \mathbb{Z}[C_4].$$

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More generally one can prove in this way that the homology is finitely generated when the factors are all of type $T^{n-1} \times SU(2)$.