Métodos iterativos para a resolução de equações integrais

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Análise Numérica

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Aim

- In this work we consider the numerical solution of a radiative transfer equation for modeling the emission of photons in stellar atmospheres. APPLICATION
- Mathematically, the problem is formulated in terms of a weakly singular Fredholm integral equation defined on a Banach space. MATHEMATICS
- Computational approaches to solve the problem are discussed, using direct and iterative strategies that are implemented in open source packages. COMPUTATIONS

Outline

- Description of the astrophysics problem
- Numerical approach
  - Projection method: Kantorovich
  - Matrix formulation
  - Approximate solution:
    - iterative refinement methods
    - state-of-the-art numerical linear algebra algorithms
    - open source libraries
- Numerical results
- Conclusions

Description of the astrophysics problem

- Solve radiative transfer equation in stellar atmospheres

$$T\varphi = z\varphi + f$$

Fredholm integral equation 2nd kind

- T integral operator defined on $X = L^1(I), I = [0, \tau^*]$

$$(Tx)(\tau) = \int_{\tau^*} g(|\tau - \tau'|)\, x(\tau')\, d\tau'$$

  - τ is the optical depth of a stellar atmosphere
  - τ^* optical thickness of the atmosphere
  - z is on the resolvent set of T
  - $f \in L^1(I)$ is the source term

Description of the astrophysics problem

- g is the kernel defined by $g(\tau) := \frac{\varpi}{2} E_1(\tau),\ 0 < \tau \le \tau^*$
  - $\varpi \in]0,1[$ is the albedo and
  - E_1 is the first exponential-integral function and it belongs to the family

$$E_\nu(\tau) := \int_1^\infty \frac{\exp(-\tau\mu)}{\mu^\nu}\, d\mu\ , \tau > 0\ , \nu \ge 1$$

$$E'_{\nu+1}(\tau) = -E_\nu(\tau);\ E_\nu(0) = \frac{1}{\nu - 1}, \nu > 1$$

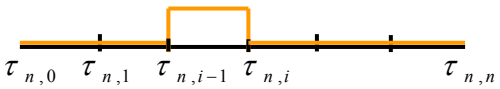
  - g is weakly singular in the sense that

$$\lim_{\tau \to 0^+} g(\tau) = +\infty;\ g \in C^0(]0, \tau^*]) \cap X;\ \sup_{\tau \in [0,\tau^*]} \int_0^{\tau^*} g(|\tau - \tau'|)\, d\tau' < \infty$$

$$g(\tau) > 0 \text{ for all } \tau \in]0, \tau^*];\ g \text{ decreasing function on }]0, \tau^*]$$

Projection method: Kantorovich

- Approximate $T\varphi = z\varphi + f$ by $T_n\varphi_n = z\varphi_n + f$
  - consider a grid $0 = \tau_{n,0} < \tau_{n,1} < \cdots < \tau_{n,n-1} < \tau_{n,n} = \tau^*$
  - define



$$X_n = span\{e_{n,j},\ j = 1,\ldots,n\},\ e_{n,j} \in X$$

- Let π_n be the projection op. $\pi_n\ x = \sum_{j=1}^{n} \langle x, e^*_{n,j} \rangle\, e_{n,j}$

$$T_n x = \pi_n T x = \sum_{j=1}^{n} \langle x, T^* e^*_{n,j} \rangle\, e_{nj}$$

  - where $e^*_{n,j}$ is the adjoint basis of $e_{n,j}$ in X^*

Matrix formulation

JCAM 140 (2002) 13–26

- The solution of the approximate problem

$$T_n\varphi_n = z\varphi_n + f$$

- leads to the solution of a linear system with *n* eq's and *n* unknowns

$$(A_n - zI_n)x_n = b_n$$

  - A_n is the restriction of T_n to X_n: $A_n = \left(\langle e_{n,j}, T^* e^*_{n,i} \rangle\right)_{i,j=1}^{n}$

$$b_n = \left(\langle f, T^* e^*_{n,i} \rangle\right)_{i=1}^{n} \qquad x_n = \left(\langle \varphi_n, T^* e^*_{n,i} \rangle\right)_{i=1}^{n}$$

- we recover φ_n from x_n by $\varphi_n = \frac{1}{z}\left(\sum_{j=1}^{n} x_n(j) e_{n,j} - f\right)$

Error estimates

for n large enough: $\dfrac{\|\varphi - \varphi_n\|_1}{\|\varphi\|_1} \le \dfrac{8 c_0(z)}{q_n} \int_0^{h_n} g(\tau)\, d\tau$

$$\exists\, n_0 : \quad c_0(z) = \sup_{n > n_0} \|R_n(z)\|_1$$

h_n max. amplit. of subintervals,

μ_n min. amplit. of subintervals

$$q_n = \mu_n / h_n$$

Error estimates

for n large enough:

$$\frac{\|\varphi - \varphi_n\|_1}{\|\varphi\|_1} \le c\left(1\text{-}E_2(h_n)\right) = O\left(-h_n \ln h_n\right)$$

$$c \le \frac{10}{q}\left(1 + \frac{1}{q}\right)\left\|(T - I)^{-1}\right\|_1$$

$$q = \inf_{grid} \frac{\min\{h_{n,i} : i = 1,\ldots,n\}}{h_n}$$

$$d_{n,i,j} = \left|\tau_{n,i} - \tau_{n,j}\right|; \;\; h_{n,i} = d_{n,i,i-1}$$

h_n = max. amplit. of subintervals

Matrix coefficients: A_n

grid $(\tau_{n,j})_{j=0}^{n}$ defined on $[0,\tau^*]$, for $i,j\in[1,n]$

$$A_n(i,j)=\frac{\varpi}{2h_{n,i}}\int_{\tau_{n,i-1}}^{\tau_{n,i}}\int_0^{\tau^*}E_1(|\tau-\tau'|)e_{n,j}(\tau')d\tau'd\tau$$

$$=\begin{cases}\dfrac{\varpi}{2h_{n,i}}\left[E_3(d_{n,i-1,j})-E_3(d_{n,i-1,j-1})+E_3(d_{n,i,j-1})-E_3(d_{n,i,j})\right], & i\neq j\\ \varpi\left(1+\dfrac{1}{h_{n,j}}\left[E_3\left(h_{n,j}-\dfrac{1}{2}\right)\right]\right) & ,\ i=j\end{cases}$$

$$d_{n,i,j}=|\tau_{n,i}-\tau_{n,j}|,\ i,j\in[0,n]\qquad h_{n,j}=\tau_{n,j}-\tau_{n,j-1},\ j\in[1,n]$$

RHS coefficients

for $i\in[1,n]$

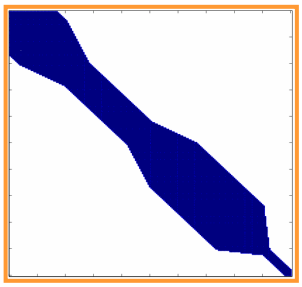
$$b_n(i)=\frac{\varpi}{2h_{n,i}}\int_{\tau_{n,i-1}}^{\tau_{n,i}}\int_0^{\tau^*}E_1(|\tau-\tau'|)f(\tau')d\tau'd\tau\ ,$$

$$f(\tau)=\begin{cases}-1 & if\quad 0\le\tau\le\dfrac{\tau^*}{2}\\ 0 & if\quad \dfrac{\tau^*}{2}<\tau\le\tau^*\end{cases}$$

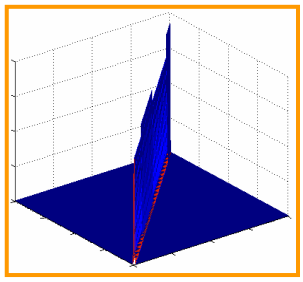
$$=\begin{cases}\dfrac{\varpi}{2h_{n,i}}\left[\begin{array}{l}E_3\left(\dfrac{\tau^*}{2}-\tau_{n,i}\right)-E_3\left(\dfrac{\tau^*}{2}-\tau_{n,i-1}\right)\cdots\\ \cdots+E_3(\tau_{n,i})-E_3(\tau_{n,i-1})-2h_{n,i}\end{array}\right], & \tau_{n,i}\le\dfrac{\tau^*}{2}\\ \dfrac{\varpi}{2h_{n,i}}\left[\begin{array}{l}E_3\left(\tau_{n,i}-\dfrac{\tau^*}{2}\right)-E_3\left(\tau_{n,i-1}-\dfrac{\tau^*}{2}\right)\cdots\\ \cdots-E_3(\tau_{n,i})+E_3(\tau_{n,i-1})\end{array}\right], & \tau_{n,i}>\dfrac{\tau^*}{2}\end{cases}$$

Typical coefficient matrix

$A_n - zI_n, \; z = 1$



band and sparse matrix



strong decay in magnitude from the diagonal

Approximate solution

- How to solve $T_n\varphi_n = z\varphi_n + f$ when the associated coefficient matrix $A_n - zI_n$ has large dimension?
- one can use:
  - direct methods,
  - preconditioned nonstationary iterative methods, or
  - iterative refinement method (defect correction):

$$\begin{cases} given \quad x^{(0)} \\ x^{(k+1)} = x^{(k)} - (T - zI)^{-1}\left(Tx^{(k)} - zx^{(k)} - f\right) \end{cases}$$

  - sequential or parallel computations

Iterative refinement methods

- Jacobian $(T - zI)^{-1}$ can be approximated by
  - scheme A (Atkinson's algorithm): $R_n(z) = (T_n - zI)^{-1}$
  - scheme B (Brakhage's algorithm): $\left(T\,(T_n - zI)^{-1} - I\right)/z$
  - scheme C (Ahues algorithm): $\left((T_n - zI)^{-1}\,T - I\right)/z$

Iterative refinement methods: error estimates

for n large enough: $$\frac{\left\|\varphi - x^{(k)}\right\|_1}{\|\varphi\|_1} \le \left(\frac{c_1(z)}{q_n}\int_0^{h_n} g(\tau)\,d\tau\right)^{k+1}$$

$\exists\, n_0 :\quad c_0(z) = \sup_{n>n_0}\|R_n(z)\|_1,$

$$c_1(z) = 8c_0(z)\max\left\{1, \|T\|_1 / z\right\}$$

h_n max. amplit. of subintervals,

μ_n min. amplit. of subintervals, $q_n = \mu_n / h_n$

Iterative refinement methods: error estimates

for n large enough:

$$\frac{\left\|\varphi - x^{(k)}\right\|_1}{\left\|\varphi\right\|_1} \le \left\|\left[R_n(z)(T - T_n)\right]^{k+1}\right\| \qquad \text{scheme A}$$

$$\cdots \le \left\|\left[\frac{1}{z} R_n(z)(T - T_n)T\right]^{k+1}\right\| \qquad \text{scheme B}$$

$$\cdots \le \left\|\left[\frac{1}{z} TR_n(z)(T - T_n)\right]^{k+1}\right\| \qquad \text{scheme C}$$

Iterative refinement methods

Scheme A:

$$\begin{cases} x^{(0)} = \varphi_n = R_n(z) f \\ x^{(k+1)} = x^{(0)} + R_n(z)\left(T_n x^{(k)} - T x^{(k)}\right), \quad k \ge 0 \end{cases}$$

Scheme B:

$$\begin{cases} x^{(0)} = \frac{1}{z}\left(R_n(z) T - I\right) f \\ x^{(k+1)} = x^{(0)} + \frac{1}{z}\left(R_n(z) T - I\right)\left(T_n x^{(k)} - T x^{(k)}\right), \quad k \ge 0 \end{cases}$$

Scheme C:

$$\begin{cases} x^{(0)} = \frac{1}{z}\left(T\, R_n(z) - I\right) f \\ x^{(k+1)} = x^{(0)} + \frac{1}{z}\left(T\, R_n(z) - I\right)\left(T_n x^{(k)} - T x^{(k)}\right), \quad k \ge 0 \end{cases}$$

Iterative refinement methods

- In practice T is not used. The problem is restricted the to X_m, $m >> n$, considering a finer projection discretization of T, T_m
- T_m restricted to X_m: $A_m = \left(\left\langle e_{m,j}, T^* e^*_{m,i} \right\rangle \right)_{i,j=1}^{m}$ $(m \times m)$
- T_m restricted to X_n: $C = \left(\left\langle e_{m,j}, T^* e^*_{n,i} \right\rangle \right)_{i,j=1}^{n,m}$ $(n \times m)$
- T_n restricted to X_m: $D = \left(\left\langle e_{n,j}, T^* e^*_{m,i} \right\rangle \right)_{i,j=1}^{m,n}$ $(m \times n)$

Matrix coefficients: A_m, C and D

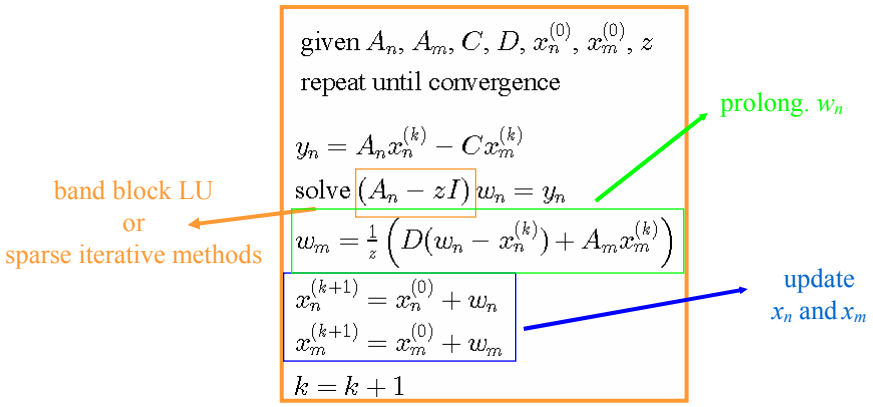
- Matrix A_m computed similarly to A_n
- We can compute matrices C and D:
  - through the integral formulation (similar to A_n and A_m)
  - or by doing some matrix computations using A_m

$$C = RA_m, \quad R(i,k) = \begin{cases} \dfrac{h_{m,k}}{h_{n,i}} & r \times (i-1) + 1 \le k \le r \times i \\ 0 & otherwise \end{cases}$$

$$D = A_m P, \quad P(k,j) = \begin{cases} 1 & r \times (j-1) + 1 \le k \le r \times j \\ 0 & otherwise \end{cases}$$

$$r = \frac{m}{n}$$

Atkinson's scheme



Problem specification

- grid τ^* : nonuniform grid (4 zones)
- parameters: $z = 1$, $\varpi = 0.75$, *tol* : $\varepsilon \le 10^{-7}$ and $\varepsilon \le 10^{-12}$
- data:
  - sparse and band structures
- machine:
  - PIV 2.4 GHz CPU with 256 MB RAM.
- software:
  - F77 & F95, LAPACK, BLAS

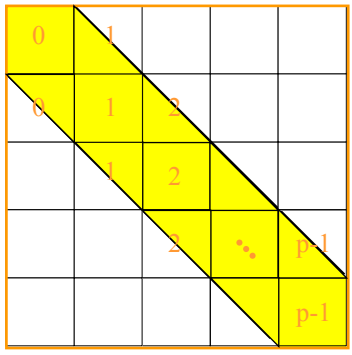
Numerical results: summary for serial computations

AML 18 (2005) 571–576

- Iterative refinement formulæ are competitive with the nonstationary iterative methods (based on Krylov subspaces) applied directly to the large linear system.
- For dense matrix discretizations all three iterative refinement schemes are faster than Krylov methods and than LU factorization even for small values of m.
- In the sparse case, for which Krylov methods are very well suited, iterative refinement execution times are comparable or even better in some cases (lower residual tolerances).
- And what about parallel computations?

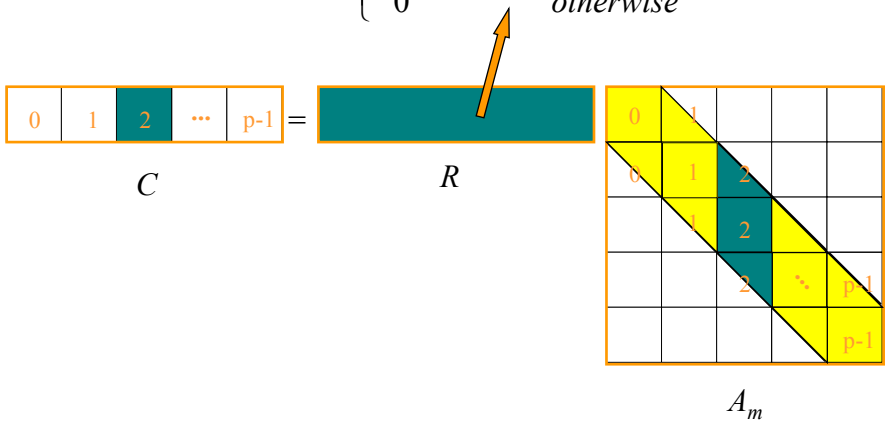
Data distribution

- Distributed memory environement with p processors: p=0,..., p-1



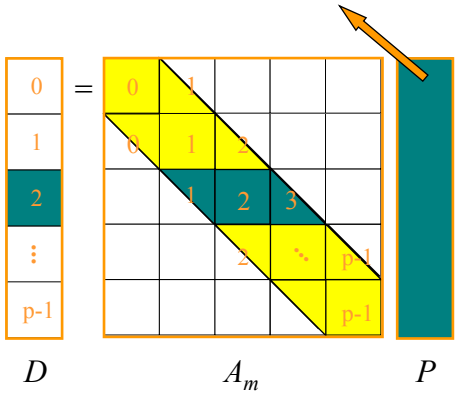
Matrix coefficients: C

$$C = RA_m, \quad R(i,k) = \begin{cases} \dfrac{h_{m,k}}{h_{n,i}} & r \times (i-1) + 1 \le k \le r \times i \\ 0 & otherwise \end{cases}$$



Matrix coefficients: D

$$D = A_m P, \quad P(k,j) = \begin{cases} 1 & r \times (j-1) + 1 \le k \le r \times j \\ 0 & otherwise \end{cases}$$



Matrix coefficients: D

$$D(i,j) = \frac{\varpi}{2h_{m,i}} \int_{\tau_{m,i-1}}^{\tau_{m,i}} \int_{\tau_{n,j-1}}^{\tau_{n,j1}} E_1\left(|\tau - \tau'|\right) d\tau' d\tau$$

$$= \begin{cases} \frac{\varpi}{2h_{m,i}} \left[-E_3(\tau_{n,j} - \tau_{m,i}) + E_3(\tau_{n,j} - \tau_{m,i-1}) + E_3(\tau_{n,j-1} - \tau_{m,i}) - \right. \\ \left. \quad - E_3(\tau_{n,j-1} - \tau_{m,i-1}) \right] , & \tau_{m,i-1} \le \tau_{m,i} \le \tau_{n,j-1} \le \tau_{n,j} \\ \frac{\varpi}{2h_{m,i}} \left[-E_3(\tau_{m,i} - \tau_{n,j}) + E_3(\tau_{m,i-1} - \tau_{n,j}) + E_3(\tau_{m,i} - \tau_{n,j-1}) + \right. \\ \left. \quad - E_3(\tau_{m,i-1} - \tau_{n,j-1}) \right] , & \tau_{n,j-1} \le \tau_{n,j} \le \tau_{m,i-1} \le \tau_{m,i} \\ \varpi \left[1 + \frac{1}{2h_{m,i}} \left(-E_3(\tau_{n,j} - \tau_{m,i}) + E_3(\tau_{n,j} - \tau_{m,i-1}) + E_3(\tau_{m,i} - \tau_{n,j-1}) - \right. \right. \\ \left. \left. \quad - E_3(\tau_{m,i-1} - \tau_{n,j-1}) \right) \right] , & \tau_{n,j-1} \le \tau_{m,i-1} \le \tau_{m,i} \le \tau_{n,j} \end{cases}$$

may lead to scability problems
for large number of processors

Atkinson's parallel scheme

given A_n, $x_n^{(0)}$, $x_m^{(0)}$, z
repeat until convergence

receive $C[i] * x_m[i]$ from all P_i
$y_n = A_n x_n^{(k)} - \sum_{i=1}^{p-1} C[i] * x_m[i]$
solve $(A_n - zI)\, w_n = y_n$
compute $w_n - x_n^{(k)}$
receive $A_m[i] * x_m^{(k)}$
$y_m = y_m - \sum_{i=1}^{p-1} A_m[i] * x_m^{(k)}$
receive $D[i] * \left(w_n - x_n^{(k)}\right)$ from all P_i
in location $w_m[(i-1) * n + 1 : i * n]$
$w_m = \frac{1}{z}(w_m - y_m)$
$x_n^{(k+1)} = x_n^{(0)} + w_n$; $x_m^{(k+1)} = x_m^{(0)} + w_m$
$k = k + 1$

given $A_m[i]$, $C[i]$, $D[i]$, z, $x_n^{(0)}$, $x_m^{(0)}$
repeat until convergence

compute $C[i] * x_m^{(k)}$
compute $A_m[i] * x_m^{(k)}$
receive $\left(w_n - x_n^{(k)}\right)$ from P_0
compute $D[i] * \left(w_n - x_n^{(k)}\right)$
receive $x_n^{(k+1)}$ and $x_m^{(k+1)}$ from P_0
$k = k + 1$

send
broadcast

Problem specification and computer environment

- grid τ^* : nonuniform grid (4 zones)
- parameters: $z = 1$, $\varpi = 0.75$, tol : $\varepsilon \leq 10^{-12}$
- Machine located at FEUP (out of service):
  - Beowulf cluster: 20 PIII 450 MHz processors (128 Mb RAM) connected via fast Ethernet switch (100 Mpbs)
- software:
  - MPI, F77 & F95, LAPACK, SPARSEKIT2

Nb. of iterations and elapsed times for the 3 iterative refinement methods

- $m = 10000$, $n = 1000$

	scheme A	scheme B	scheme C
p	43 iterations	21 iterations	21 iterations
1	3472.7	3472.4	3472.2
2	1749.4	1749.0	1748.9
4	885.0	884.4	884.4
5	714.1	713.6	713.6
10	375.3	375.1	375.1

- $m = 100000$, $n = 5000$

20	14859.7	14855.0	14854.2

Speedup and efficiency for the 3 iterative refinement methods

- $m = 10000, n = 1000$

$$S_p = \frac{t_1}{t_p}; \quad E_p = \frac{S_p}{p}; \quad t_p = \text{elapsed time using } p \text{ processors}$$

	schemes A, B and C	
p	S_p	E_p
1	1	1.00
2	1.99	0.99
4	3.93	0.98
5	4.86	0.97
10	9.26	0.93

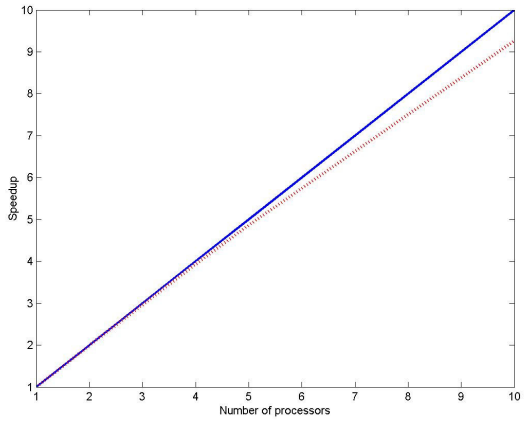
Speedup up to 10 processors for the 3 iterative refinement methods

$t_p = \text{elapsed time using } p \text{ processors}$

$$S_p = \frac{t_1}{t_p}$$

$m = 10000$

$\frac{m}{n} = 10$



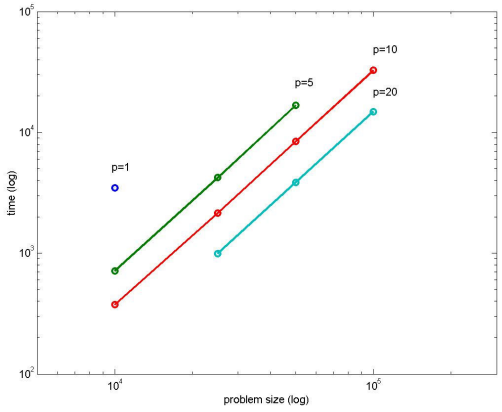
LNCS 3732 (2006) 864–871

Scalability

- Isoefficiency or isogranularity (ScaLAPACK sense)
  - a constant memory use per node allows efficiency to be maintained

time for several size problems using up to 20 processors

$$\frac{m}{n} = 20$$



Solving the problem in the m-D space

- We can solve $T_m \varphi_m = z\varphi_m + f$ for the finer grid approximated matricial problem $A_m - zI_m = b_m$
- Our goal is to experiment with robust and portable algorithm implementations from the:
  - ACTS Collection - Advanced CompuTational Software
  - LBNL and DOE
- Direct methods: SuperLU
- Preconditioned nonstationary iterative methods: PETSc

Problem specification

- grid τ^* : nonuniform grid (4 zones)
- parameters: $z = 1$, $\varpi = 0.75$ and $\varpi = 0.9$; tol : $\varepsilon \le 10^{-12}$
- machines located at LBNL/NERSC
  - SGI Altix 350: 32 64-bit 1.4 GHz Intel Itanium-2 processors, with 192 GBytes of shared memory
  - AMD Opteron Cluster: 356 dual-processor nodes, 2.2 GHz/node, 6 GB/node, interconnected with a high-speed InfiniBand network
  - IBM SP: 380 compute nodes with 16 Power 3+ processors/node, 16 GB memory/node
- machine located at IRIC.UP
  - AMD Opteron Cluster: 24 64-bit dual-processor nodes, 2.4 Ghz/node, 4 GB/node, interconnected with a Gigabit Ethernet
- software:
  - MPI, F77 & F95, LAPACK, SPARSEKIT2, SuperLU, PETSc

Normalized times for the generation phase and system solution for various matrix sizes (m), on the SGI Altix and using SuperLU

$\varpi = 0.75$	generation	solution	
m		factor	solve
1000	3.26E+03	6.95E+01	1.00E+00
2000	2.12E+04	1.65E+02	3.00E+00
4000	9.71E+04	3.59E+01	6.00E+00
8000	4.26E+05	7.51E+02	1.80E+01
16000	1.80E+06	1.54E+03	3.00E+01
32000	7.36E+06	3.12E+03	5.35E+01

1.00E+00=1.953E-03

Normalized times and nb. it. for various matrix sizes (m) on up to 32 processors (p), on the Opteron cluster and using PETSc

a constant memory use per node allows efficiency to be maintained

$\varpi = 0.75$		generation	GMRES	BiCGStab
m	*p*		22 iterations	14 iterations
10000	1	5.40E+03	7.54E+00	7.95E+00
	2	2.67E+03	4.02E+00	4.58E+00
	4	1.39E+03	2.32E+00	2.56E+00
	8	6.90E+02	1.80E+00	1.97E+00
	16	3.51E+02	1.15E+00	1.25E+00
	32	1.79E+02	1.15E+00	1.36E+00
25000	4	8.41E+03	5.42E+00	5.61E+00
	8	4.28E+03	3.02E+00	3.15E+00
	16	2.16E+03	2.05E+00	1.83E+00
	32	1.07E+03	1.00E+00	1.15E+00
50000	16	8.57E+03	3.14E+00	3.20E+00
	32	4.24E+03	1.53E+00	1.86E+00

1.00E+00=5.9E-02

Normalized times and nb. it. for various matrix sizes (m) on up to 32 processors (p) on the Opteron cluster

$\varpi = 0.90$	$\varpi = 0.75$	generation	22	GMRES	14	BiCGStab
m	*p*			24 iterations		37 iterations
10000	1	2.83E+03		6.14E+00		6.99E+00
	2	1.36E+03		3.62E+00		4.12E+00
	4	7.20E+02		2.20E+00		2.31E+00
	8	3.59E+02		1.67E+00		1.76E+00
	16	1.80E+02		1.11E+00		1.30E+00
	32	9.19E+01		1.0		1.00E+00
25000	4	2.83E+03		4.[illegible]E+00		5.37E+00
	8	2.22E+03		2.96E+00		3.39E+00
	16	1.11E+03		1.78E+00		2.11E+00
	32	5.55E+02		1.33E+00		1.46E+00
50000	16	4.36E+03		2.82E+00		3.14E+00
	32	2.21E+03		2.15E+00		2.05E+00

1.5x-2.5x smaller

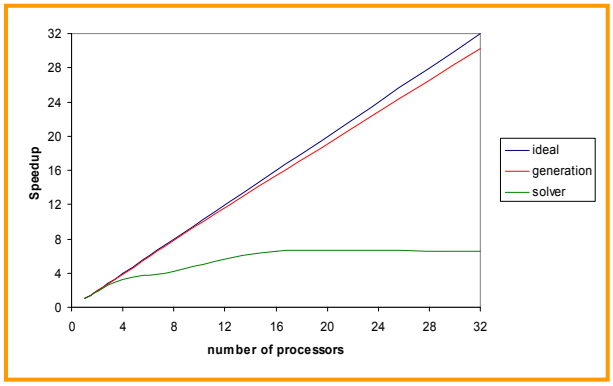
1.00E+00=1.14E-01

Speedup up to 32 processors on the Opteron cluster

t_p = elapsed time using p processors

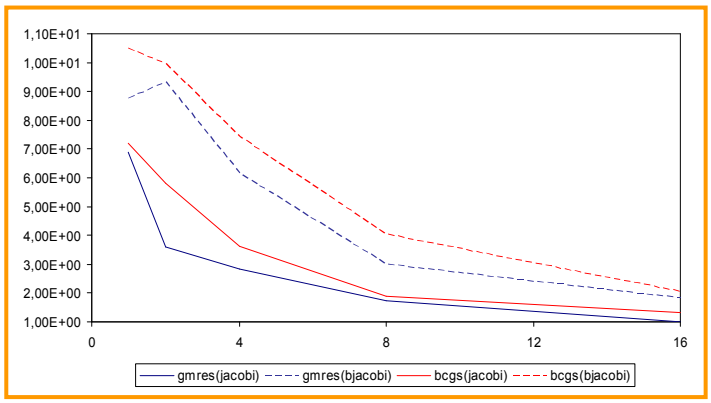
$$S_p = \frac{t_1}{t_p}$$

$m = 10000$



Normalized times for Jacobi and block Jacobi preconditioners on the Opteron cluster

$m = 50000$



Parallel IRM vs. Parallel preconditioned GMRES

- Compared to Iterative Refinement Techniques, the approach using parallel nonstationary iterative methods (GMRES, BiCGstab from PETSc), and for m=50000 and p=5
  - leads to 40% savings in time in the generation phase
  - reduces the number of communications required for mapping the coarse problem into the fine one
    - up to 5x for Atkinson and
    - up to 4x for Brakhage and Ahues' schemes
- To improve the performance of IRM:
  - parallelize the moderate size dimensional system (n-D linear systems)
  - explore the performance of 32 bit floating point arithmetic to obtain 64 bit accuracy => mixed-precision iterative refinement

Conclusions

- We discussed the numerical solution of a radiative transfer equation for modelling the emission of photons in stellar atmospheres.
- A good knowledge of the:
  - mathematics of the problem,
  - numerical methods, as well as,
  - computational aspects,

  were crucial for the implementation of an efficient and reliable solution.
- The parallelization of the generation phase greatly reduces the computation time and enables the solution of large dimensional problems.
- The selection of appropriate linear solvers is important for delivering performance and portability. The use of state-of-the-art linear algebra building blocks can ensure this task.
- Further work: using iterative refinement techniques and developing new algorithms for spectral computations

Main references

- M. Ahues, F. D. d'Almeida, A. Largillier, O. Titaud and P. Vasconcelos
  An L^1 refined projection approximate solution of the radiation transfer equation in stellar atmospheres
  J. Comput. Appl. Math., 140 13-26, 2002
- P.B. Vasconcelos and F. D. d'Almeida
  A Parallel Code for Integral Equations on a Cluster of Computers
  Integral methods in Science and Engineering: Analytic and Numerical, Techniques, Birkhaüser, 261-266, 2004
- B. Rutily
  Multiple scattering theoretical and integral equations
  Integral methods in Science and Engineering: Analytic and Numerical, Techniques, Birkhaüser, 211-231, 2004
- L.A. Drummond and O. Marques
  An Overview of the Advanced CompuTational Software (ACTS) Collection
  ACM TOMS, 31:282–301, 2005
- P.B. Vasconcelos and F. D. d'Almeida
  Performance evaluation of a parallel algorithm for a radiative transfer problem
  LNCS, 3732: 864–871, 2006
- M. Ahues, F. D. d'Almeida, A. Largillier, O. Titaud and P. Vasconcelos
  Defect correction for spectral computations for a singular integral operator
  Comm. on Pure and Appl. Anal., 5:2, 241–250, 2006