Soergel bimodules and 2-representation theory

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1 Introduction

For almost four decades, the canonical bases of certain quantum algebras have been at the core of representation theory. Historically, the first ones were the Kazhdan-Lusztig bases of Hecke algebras associated to Coxeter groups [8]. For lack of space in this review, we will mostly concentrate on these, although the canonical bases of quantum groups form another interesting class of examples.

The KL bases, and the associated KL polynomials, have remarkable *positive integrality properties*, which were conjectured by Kazhdan and Lusztig in [8]. For example, the multiplication constants of the Hecke algebra w.r.t. the KL basis belong to $\mathbb{N}[v, v^{-1}]$, where v is a formal parameter. (We assume that the Hecke algebras and their modules are defined over $\mathbb{C}(v)$.)

In a subsequent paper [9], Kazhdan and Lusztig proved their conjectures for finite and affine Weyl groups, by interpreting the KL bases in terms of the local intersection cohomology of Schubert varieties. In that approach, the aforementioned multiplication constants become dimensions of cohomology groups and are therefore positive integral. Eventually, a geometric proof for Weyl groups of symmetrizable Kac-Moody algebras was found [1, 2, 7]. However, the geometric arguments do not work for other Coxeter groups.

Therefore, Soergel [17, 18] introduced an algebraic/combinatorial approach, using certain bimodules, designed to prove that the KL positive integrality properties hold for any Coxeter group. This huge project, after important partial results by himself and others [17, 18, 5, 4], was eventually completed by Elias and Williamson [3].

Following standard terminology in this field, we say that Soergel's monoidal categories, resp. the indecomposable bimodules, *categorify* the Hecke algebras, resp. the KL basis elements. Alternatively, we can say that the latter *decategorify* the former.

Since the interest in Hecke algebras stems from their representation theory, it is natural to study the 2-representation theory of Soergel's monoidal categories, in which modules are replaced by their categorical analogue, called 2-modules.

In their systematic approach to 2-representation theory, Mazorchuk and Miemietz [16] proved a categorical version of the Jordan-Hölder theorem. This led them to define the notion of a *simple transitive 2-module*, which is the correct categorical analogue of a simple module, although its decategorification is often not simple. Thus arises naturally the problem of classifying all simple transitive 2-modules of Soergel's monoidal category for any finite Coxeter type.

In this review, we will recall what is known about this classification.

2 Coxeter groups and Hecke algebras

In this section we will briefly recall some well-known facts about Coxeter groups, Hecke algebras and Kazhdan-Lusztig bases. More material and proofs can be found in [6, 8, 12].

2.1 Coxeter groups

Let S be a finite set. A Coxeter *matrix* $(m_{st})_{s,t\in S}$ is a symmetric matrix such that $m_{ss} = 1$ for all $s \in S$, and $m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t \in S$. Furthermore, let W be a group.

Definition 1.— We say that (W, S) is a Coxeter system if there exists a Coxeter matrix $(m_{st})_{s,t\in S}$ such that $W \cong F(S)/N$, where $F(S)$ is the free group generated by S and $N \lhd F(S)$ the normal subgroup generated by the elements

$$(st)^{m_{st}} \tag{1}$$

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for all $s, t \in S$ with $m_{st} < \infty$.

We call W the COXETER group and S the set of simple reflections of the COXETER system (W, S).

By definition, the *rank* of (W, S) is the order of S, which is finite by assumption. However, this does not necessarily imply that W is of finite order.

The only COXETER group of rank 0 is the trivial group, and the only one of rank 1 is $\mathbb{Z}/2\mathbb{Z}$. But there is an infinite family of rank 2 Coxeter groups, indexed by $m_{st} = m_{ts} = n \in \{2, 3, 4 \ldots, \} \cup \{\infty\}$ with $S = \{s, t\}$. These are isomorphic to the dihedral groups of order $2n$ (which can be infinite), with st corresponding to a rotation of degree $2\pi/n$ when n is finite.

The finite COXETER groups are classified by the finite type COXETER diagrams [6, Sections 2.4 and 6.4], which are a generalization of the DYNKIN diagrams of finite-dimensional complex semisimple LIE algebras.

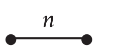
For example, for any $n \in \mathbb{N}$, the symmetric group on $n+1$ letters can be seen as a COXETER group of type A_n, with $S = \{s_1, \ldots, s_n\}$ the set of simple transpositions. Its COXETER diagram is

Numbering the vertices of the diagram from left to right by $1, 2, \ldots, n$, and writing s_i for the simple reflection associated to the vertex i, we have

$$m_{ij} = \begin{cases} 3 & \text{if } |i-j| = 1, \\ 2 & \text{if } |i-j| > 1, \\ 1 & \text{if } |i-j| = 0. \end{cases}$$

This is the general rule for obtaining the COXETER matrix from a COXETER diagram and vice-versa, with one exception: if $m_{st} > 3$ for two neighboring vertices in the diagram, then that number is written above the corresponding edge.

For example, for any $n > 3$, the dihedral group of order $2n$ can be seen as a COXETER group of type $I_2(n)$ with COXETER diagram



Note that $I_2(3) = A_2$, since they have the same COXETER diagram.

For any $w \in W$, a *reduced expression* for w is by definition a shortest string $s_1, \ldots, s_\ell \in S$ such that $w = s_1 \cdots s_\ell$. We call ℓ the *length* of the string. In general, there can be more than one reduced expression for w, but two of them can always be related by applying (3) a finite number of times, as shown by MATSUMOTO and TITS' theorem (see [12, Thm. 1.9]).

This allows us to define the length function $\ell : W \to \mathbb{Z}_{\geq 0}$, which associates to each $w \in W$ the length of a reduced expression for w (see [6, Sect. 1.6]).

Furthermore, it allows us to define the BRUHAT *order* $\preceq$ on W, which is a partial order defined by: $u \preceq w$ iff u can be obtained as a (not necessarily reduced) subexpression of a reduced expression for w (see [6, Sect. 5.9 and 5.10]).

If W is finite, then it has a unique longest element, denoted w_0, which is also maximal w.r.t. the BRUHAT order.

2.2 HECKE ALGEBRAS

Let (W, S) be any COXETER system. In the group algebra $\mathbb{C}[W]$, the relations $s^2 = e$ and $(st)^{m_{st}} = e$ can be rewritten as

$$(s+e)(s-e) = 0, \tag{2}$$
$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}}. \tag{3}$$

The next definition is obtained by v-deforming the relation in (2).

DEFINITION 2.— The HECKE algebra $\mathscr{H}$ associated to (W, S) is the unital associative $\mathbb{C}(v)$-algebra generated by T_s, for $s \in S$, subject to the relations

$$\begin{aligned} (T_s + 1)(T_s - v^{-2}) &= 0, \\ \underbrace{T_s T_t T_s \cdots}_{m_{st}} &= \underbrace{T_t T_s T_t \cdots}_{m_{st}}, \end{aligned} \tag{4}$$

for all $s, t \in S$. By convention, we write $T_e = 1$.

Note that $T_s^2 = (v^{-2} - 1)T_s + v^{-2}$ and $T_s^{-1} = v^2 T_s + v^2 - 1$.

For any $w \in W$, choose a reduced expression $w = s_1 \cdots s_{\ell(w)}$, with $s_i \in S$, and define

$$T_w := T_{s_1} \cdots T_{s_{\ell(w)}}.$$

By MATSUMOTO and TITS' theorem, the element T_w does not depend on the choice of reduced expression. Moreover, $\{T_w \mid w \in W\}$ is a linear basis of $\mathscr{H}$, called the *standard basis* (see [12, Prop. 3.3]). In particular, this implies that $\mathscr{H}$ is a flat deformation of the group algebra of W.

The KL *basis* $\{b_w \mid w \in W\}$ is harder to define. Let $^-$ be the *bar involution* on $\mathscr{H}$, which is the $\mathbb{C}$-linear involution given by

$$\overline{v} := v^{-1} \quad \text{and} \quad \overline{T_w} := T_{w^{-1}}^{-1}.$$

The KL *basis elements* $b_w \in \mathscr{H}$ are uniquely determined by the two properties [8, Thm 1.1]:

$$\overline{b_w} = b_w, \tag{5}$$
$$b_w = v^{\ell(w)} \sum_{y \preceq w} P_{y,w} T_y, \tag{6}$$

where $P_{y,w} \in \mathbb{Z}[v^{-2}]$ has negative v-degree strictly less than $\ell(w) - \ell(y)$ for $y \prec w$ and $P_{w,w} = 1$.

Note that the matrix $(P_{y,w})_{y,w \in W}$ is unitriangular, so the fact that the b_w form a basis follows immediately from the fact that the T_w form a basis.

In general, there is no simple formula expressing b_w in terms of the T_y. Only the KL generators are easy to compute:

$$b_e = 1 \quad \text{and} \quad b_s = v(T_s + 1) \quad \text{for all } s \in S. \tag{7}$$

However, in type $I_2(n)$ we can write down all KL basis elements explicitly [12, Ch. 7]:

$$b_w = v^{\ell(w)} \sum_{y \preceq w} T_y \quad \text{for all } w \in W.$$

A short calculation shows that, in any Coxeter type, we have

$$b_s^2 = (v + v^{-1}) b_s \quad \text{for all } s \in S. \tag{8}$$

It is also easy to see that $b_s b_t = b_{st}$ for all $s \neq t \in S$. But, in general, the product of a finite number of KL basis elements is not a KL basis element, e.g. in type $I_2(3) = A_2$ we have

$$b_s b_t b_s = b_{sts} + b_s \quad \text{and} \quad b_t b_s b_t = b_{tst} + b_t,$$

where $b_{sts} = b_{tst}$ because $sts = tst$ in W. Nevertheless, if we choose a reduced expression $w = s_1 \cdots s_{\ell(w)}$ for each $w \in W$, and define $b_{\underline{w}} := b_{s_1} \cdots b_{s_{\ell(w)}}$, then $\{b_{\underline{w}} \mid w \in W\}$ is yet another basis of $H_v(W)$, called the Bott-Samelson *basis*. This follows from the fact that $b_{\underline{w}} = v^{\ell(w)} T_w + \text{l.o.t.}$, where l.o.t. is a linear combination of T_y with $y \preceq w$. Note, however, that $b_{\underline{w}}$ depends on the choice of reduced expression for w.

3 Soergel bimodules

For any Coxeter group W, take $\mathfrak{h}$ to be the complexification of Soergel's finite-dimensional real W-module in [18, Prop. 2.1], which generalizes the usual representation of an affine Weyl group on the Cartan subalgebra of an affine Kac-Moody algebra.

Let R be the complex algebra of regular functions on $\mathfrak{h}$, equipped with a $\mathbb{Z}$-grading such that $\deg(\mathfrak{h}^\star) = 2$. The action of W on $\mathfrak{h}$ extends naturally to an action on R by degree-preserving automorphisms.

Let R–fmod–R be the monoidal category of all finitely generated graded R–R bimodules, where the monoidal product is given by the tensor product over R. By definition, the morphisms are the degree-preserving bimodule maps. Note that R–fmod–R is additive, because we can also take the direct sum of two bimodules. Furthermore, the homogeneous direct summands of the hom-spaces are all finite-dimensional complex vector spaces and composition is bilinear, so R–fmod–R is $\mathbb{C}$-linear.

For any $s \in S$, let R^s be the graded subalgebra of s-invariant polynomials and define

$$B_s := R \otimes_{R^s} R\{1\},$$

where $\{1\}$ indicates a downward grading shift of 1. This is a graded $R - R$ bimodule with left and right actions given by $a \cdot (x \otimes y) \cdot b := (ax) \otimes (yb)$, for any $a, b, x, y \in R$.

We have $R \cong R^s \oplus R^s\{-2\}$ as R^s-bimodules, so

$$B_s \otimes B_s \cong B_s\{+1\} \oplus B_s\{-1\} \quad \text{for all } s \in S.$$

This isomorphism categorifies the equality in (8).

More generally, for any finite number of simple reflections $s_1, \ldots, s_m \in S$, the corresponding Bott-Samelson *bimodule* is defined as

$$B_{s_1} \otimes_R B_{s_2} \otimes_r \cdots \otimes_R B_{s_m}.$$

Let $w \in W$ and suppose $w = s_1 \cdots s_{\ell(w)}$ is a reduced expression. Then we denote the corresponding Bott-Samelson bimodule by $B_{\underline{w}}$.

Definition 3.— The monoidal category of Soergel bimodules $\mathcal{S}$ is the full subcategory of R–fmod–R containing all direct sums of direct summands of Bott-Samelson bimodules with grading shifts.

The additive category $\mathcal{S}$ is idempotent complete and Krull-Schmidt [18, Rem. 1.3].

Before we state the categorification theorem for Soergel bimodules, recall that the *split* Grothendieck *algebra* of $\mathcal{S}$, denoted $[\mathcal{S}]$, is by definition the $\mathbb{C}(v)$-vector space spanned by the isoclasses of the Soergel bimodules, subject to the relations:

$$[U \oplus V] = [U] + [V] \quad \text{and} \quad [U\{t\}] = v^t[U]$$

for all Soergel bimodules U, V and $t \in \mathbb{Z}$. It becomes an algebra after putting

$$[U \otimes_R V] := [U][V]$$

for all Soergel bimodules U, V. By the above, it follows that $\{[B_w] \mid w \in W\}$ is a basis of $[\mathcal{S}]$.

The first three points in the following theorem are due to Soergel [18, Thm. 1.10 and Satz 6.16]. The fourth point is due to Elias and Williamson [3, Thm. 1.1].

Theorem 4.— Let (W, S) be an arbitrary Coxeter system. Then

1. there is a well-defined isomorphism of $\mathbb{C}(v)$-algebras $\rho_S : \mathcal{H} \to [\mathcal{S}]$ uniquely determined by
$$b_s \mapsto [B_s] \quad \text{for all } s \in S;$$
2. for every $w \in W$, there exists an indecomposable B_w in $\mathcal{S}$, unique up to degree-preserving isomorphism, that is a direct summand of the Bott-Samelson bimodule $B_{\underline{w}}$, for any reduced expression for w, and is not a direct summand of $B_{\underline{u}}\{t\}$ for any $u \prec w$ and $t \in \mathbb{Z}$; in particular, the isoclass of B_w does not depend on the choice of reduced expression for w;

3. every indecomposable Soergel bimodule is isomorphic to $B_w\{t\}$ for some $w \in W$ and $t \in \mathbb{Z}$;

4. for every $w \in W$, we have $\rho_S(b_w) = [B_w]$.

Note that this theorem immediately implies that the KL basis of $\mathscr{H}$ is positive integral: for any $u, v \in W$, we have

$$b_u b_v := \sum_{w \in W} \gamma^w_{u,v} b_w$$

such that $\gamma^w_{u,v} \in \mathbb{N}[v, v^{-1}]$, because these multiplication constants are equal to the graded decomposition numbers of $[B_u \otimes B_v]$ in terms of the $[B_w]$

4 2-Representation theory

From now on, let (W, S) be an finite type Coxeter system, i.e. we assume that W is a finite group.

Recall that a category is *graded finitary* if it is additive, $\mathbb{C}$-linear, idempotent complete and Krull-Schmidt, such that the homogeneous direct summands of its hom-spaces are finite-dimensional and it has finitely many isoclasses of indecomposable objects up to grading shifts, e.g. the category of finitely generated graded projective modules over a non-negatively graded algebra which is finite-dimensional in each degree.

Let $\mathscr{S}$ be the monoidal category of Soergel bimodules for (W, S). A 2-*module* of $\mathscr{S}$ is by definition a graded finitary category $\mathscr{M}$ on which the Soergel bimodules act as linear endofunctors and the bimodule maps as natural transformations, such that all structures (including the grading) are preserved. In general, the 2-action is allowed to be weak in a restricted sense, but this is not the right place to explain such technical details.

A 1-*intertwiner* between two 2-modules of $\mathscr{S}$ is by definition a degree-preserving $\mathbb{C}$-linear functor between the underlying categories which commutes with the 2-action. Again, we suppress all technical conditions which control the level of weakness that is allowed. Two 2-modules are called equivalent if there is a fully faithful and essentially surjective 1-intertwiner between them.

Finally, there is a next layer of structure, formed by natural transformations between 1-intertwiners which satisfy additional conditions. We call these 2-intertwiners.

Together, the 2-modules of $\mathscr{S}$ and the 1 and 2-intertwiners between them form a 2-category, which we denote by $\mathscr{S}$-2fmod.

Note that we only consider additive 2-modules. In this review, we do not discuss abelian or triangulated 2-modules.

4.1 Cell modules

The decategorified story of cell modules of Hecke algebras is due to Kazhdan and Lusztig [8].

Definition 5.— We define the *left pre-order* $\geq_L$ on W by putting $w \geq_L v$ if $\gamma^w_{u,v} \neq 0$ for some $u \in W$.

We set $w \sim_L u$ provided that $u \geq_L w$ and $w \geq_L u$. The equivalence classes of this equivalence relation are called the *left cells* of W.

The right and two-sided pre-orders $\geq_R$ and $\geq_J$, and the right and two-sided cells for the corresponding equivalence relations $\sim_R$ and $\sim_J$ are defined similarly, using multiplication from the right and from both sides respectively.

Note that each left (resp. right) cell is contained in a two-sided cell, that each two-sided cell is the disjoint union of the left (resp. right) cells it contains, and that W is the disjoint union of all two-sided cells.

In general, it is not so easy to compute cells explicitly. In type A_n, Kazhdan and Lusztig [8] proved that $u \sim_L w$ iff $Q(u) = Q(w)$, where Q is the recording tableau in the Robinson-Schensted correspondence. Similarly, $u \sim_R v$ iff $P(u) = P(w)$, where P is the insertion tableau.

In type $I_2(n)$, the computation of the cells is straightforward and gives:

$$\mathscr{J}_e = \mathscr{L}_e = \mathscr{R}_e = \{e\}$$

$$\mathscr{J}_s = \mathscr{J}_t = \begin{array}{c|c|c|c} & \mathscr{R}_s & \mathscr{R}_t & \\ \hline & s, sts, \ldots & ts, tsts, \ldots & \mathscr{L}_s \\ \hline & st, stst, \ldots & t, tst, \ldots & \mathscr{L}_t \\ \hline \end{array}$$

$$\mathscr{J}_{w_0} = \mathscr{L}_{w_0} = \mathscr{R}_{w_0} = \{w_0\}.$$

If $\mathscr{L}$ is a left cell of W, we write $w \geq_L \mathscr{L}$ if $w \geq_L u$ for all $u \in \mathscr{L}$, and we write $w >_L \mathscr{L}$ if $w \geq_L \mathscr{L}$ and $w \notin \mathscr{L}$. Let $M_{\geq_L \mathscr{L}}$ and $M_{>_L \mathscr{L}}$ be the subvector-spaces of $\mathscr{H}$ spanned by all b_w satisfying $w \geq_L \mathscr{L}$, and $w >_L \mathscr{L}$ respectively. Both are left ideals of $\mathscr{H}$ and $M_{>_L \mathscr{L}} \subset M_{\geq_L \mathscr{L}}$.

Definition 6.— The *left cell module* $C_{\mathscr{L}}$ is defined as

$$C_{\mathscr{L}} := M_{\geq_L \mathscr{L}} \big/ M_{>_L \mathscr{L}}$$

with the natural left $\mathscr{H}$ action.

Note that $C_{\mathscr{L}}$ inherits a KL-basis, consisting of all b_w with $w \in \mathscr{L}$. Clearly, this provides the cell module with a positive integral basis, i.e. on the KL-bases of $\mathscr{H}$ and $C_{\mathscr{L}}$, the *action constants* all belong to $\mathbb{N}[v, v^{-1}]$.

As we already remarked, the left cells in type A_n are parametrized by standard tableaux. As a matter of fact, every left cell-module of $\mathscr{H}$ is simple and its isomorphism class is determined by the partition underlying the corresponding standard tableau. This establishes a bijection between the isoclasses of left cell-modules and the isoclasses of simple modules, which is atypical: in other COXETER types most simple modules do not have a positive integral basis and most cell modules are not simple.

For example, consider type $I_2(n)$. Any one-dimensional module is completely determined by its character $\chi\colon \mathscr{H} \to \mathbb{C}(q)$. By the quadratic relations in $\mathscr{H}$, we must have $\chi(T_s) = \epsilon_1$ and $\chi(T_t) = \epsilon_2$ with $\epsilon_1, \epsilon_2 \in \{v^{-2}, -1\}$. If n is even, there is no extra condition, so there are four different characters. If n is odd, then $\epsilon_1 = \epsilon_2$ is required to hold, so there are only two different characters. We denote the corresponding one-dimensional modules by $V_{\epsilon_1,\epsilon_2}$.

We have $C_{\mathscr{L}_e} \cong V_{-1,-1}$, because $b_s = v(T_s + 1)$ and $b_t = v(T_t + 1)$ act as zero. Similarly, we have $C_{\mathscr{L}_{w_0}} \cong V_{v^{-2},v^{-2}}$, because b_s and b_t both act as multiplication by $v+v^{-1}$. When n is even and at least 4, the modules $V_{v^{-2},-1}$ and $V_{-1,v^{-2}}$ are not equivalent to cell modules, because there are no more one-element left cells.

All other simple modules are known to be of dimension two. Since $C_{\mathscr{L}_s}$ and $C_{\mathscr{L}_t}$ have dimension $n-1$, they cannot be simple for $n \geq 4$.

Furthermore, in type $I_2(n)$ there are other interesting modules of $\mathscr{H}$ with a positive integral basis, as we will explain below.

4.2 Cell 2-modules

There is a natural categorification of the (left) cell-modules, due to MAZORCHUK and STROPPEL [13] in the case of finite WEYL groups, and MAZORCHUK and MIEMIETZ [15] in general (see also [16, Sec. 3.3]). Let $\mathscr{L}$ be a left cell and take $\mathscr{M}_{\geq_L \mathscr{L}}$ to be the full subcategory of $\mathcal{S}$ generated by the B_w for $w \geq_L \mathscr{L}$. This subcategory contains a unique ideal $\mathcal{I}_{\mathscr{L}}$ which is maximal in the set of all $\mathcal{S}$-stable ideals.

DEFINITION 7.— The left cell 2-module associated to $\mathscr{L}$ is defined as

$$\mathscr{C}_{\mathscr{L}} := \mathscr{M}_{\geq_L \mathscr{L}} \big/ \mathcal{I}_{\mathscr{L}}$$

with the natural 2-action of $\mathcal{S}$.

By construction, we have $C_{\mathscr{L}} \cong [C_{\mathscr{L}}]$ as $\mathscr{H}$-modules.

4.3 Simple transitive 2-modules

MAZORCHUK and MIEMIETZ [16] found that the correct categorification of the notion of simple module, is that of *simple transitive 2-module*. A 2-module $\mathscr{M}$ of $\mathcal{S}$ is *transitive* if for any two indecomposable objects X, Y in $\mathscr{M}$, there exists a SOERGEL bimodule B in $\mathcal{S}$ such that X is a direct summand of BY. A transitive 2-module $\mathscr{M}$ is SIMPLE TRANSITIVE if it has no non-zero proper $\mathcal{S}$-stable ideals. Any transitive 2-module has a simple transitive quotient [16, Lem. 4]. By construction, any cell 2-module is simple transitive.

In type A_n the converse is also true: any simple transitive 2-module is equivalent to a cell 2-module [15, Sec. 7.1].

However, in type $I_2(n)$ there are simple transitive 2-modules that are not equivalent to cell 2-modules. There is an ADE-classification for the simple transitive 2-modules in type $I_2(n)$, and only the ones of type A are equivalent to cell 2-modules.

To explain this, we first note that any simple transitive 2-module $\mathscr{M}$ has an underlying quiver, which can be graded, so that $\mathscr{M}$ becomes equivalent to the category of graded finitely generated projective modules of the quiver algebra after modding out by a virtually nilpotent ideal. As it turns out, for type $I_2(n)$ SOERGEL bimodules, the quiver underlying a simple transitive 2-module can always be obtained from a simply laced DYNKING diagram of finite type. The main part of the following theorem can be found in [10, Thm. 1 and Sec. 6], with only a construction of the simple transitive 2-modules of DYNKIN type E missing, which can be found in [14].

THEOREM 8.— Let $\mathcal{S}$ be the monoidal category of SOERGEL bimodules of type $I_2(n)$. For any $n > 2$, $\mathcal{S}$ has two inequivalent cell 2-modules of rank one, namely $\mathscr{C}_{\mathscr{L}_e}$ and $\mathscr{C}_{\mathscr{L}_{w_0}}$.

Furthermore, there are two cell 2-modules of rank $n-1$, namely $\mathscr{C}_{\mathscr{L}_s}$ and $\mathscr{C}_{\mathscr{L}_t}$, whose underlying graph is of Dynkin type A_{n-1}. They are equivalent iff n is odd.

1. If $n = 2k+1 > 2$ or $n = 4$, then all simple transitive 2-modules are equivalent to the above cell 2-modules.
2. If $n = 2k > 4$, there are two additional inequivalent simple transtive 2-modules, whose underlying graph is of DYNKIN type D_{k+1}.
3. If $n = 12$, 18 or 30, there are also two inequivalent exceptional simple transitive 2-modules, whose underlying graph is of DYNKIN type E_6, E_7 and E_8 respectively.

The above gives a total classification of the simple transitive 2-modules of $\mathcal{S}$.

It is interesting to note that the two inequivalent simple transitive 2-modules of DYNKING type E_6 decategorify to isomorphic $\mathscr{H}$-modules. The same happens for DYNKIN type E_8, but the two inequivalent simple transitive 2-modules of DYNKIN type E_7 have non-isomorphic decategorifications.

We also note that the decategorified story was already known to LUSZTIG [11, Prop. 3.8].

The classification of the simple transitive 2-modules of $\mathcal{S}$ in other finite COXETER types is very incomplete. In [10] the following (very) partial result was proved. For every finite type COXETER system (W, S), there is a unique lowest order two-sided cell $\mathcal{J}_S$ which does not contain e. One simple description of $\mathcal{J}_S$ is that it consists of all $w \neq e \in W$ with a unique reduced expression. Now assume that (W, S) has rank > 2. Then any simple transitive 2-module of $\mathcal{S}$ that is annihilated by all B_w with $w >_J \mathcal{J}_S$, is equivalent to a cell 2-module [10, Thm. 1].

The rest of the classification is unknown and forms an interesting but difficult open problem, except for COXETER type A_n where the cell 2-modules exhaust the simple transitive 2-modules.

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