Inequalities in nonlinear Fourier analysis

by **Diogo Oliveira e Silva***

This note is meant as a gentle introduction to nonlinear Fourier analysis. In particular, we highlight the parallel with the classical linear theory, describe some recent developments, and mention some open problems in the field.

1 The Fourier transform

The Fourier transform is an ubiquitous tool in mathematical analysis. Its power stems from the fact that it reveals certain properties about the function which are not readily apparent by inspection. One of the first questions that arises concerns its mapping properties on the scale of Lebesgue spaces L^p. Given a sufficiently nice function $f : \mathbb{R} \to \mathbb{C}$, we shall define its Fourier transform as follows:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\, \mathrm{d}x.$$

One easily checks that the Fourier transform of an integrable function is a uniformly continuous function which decays at infinity, an observation which is historically attributed to early works of Riemann and Lebesgue. With the above normalization, a straightforward application of the triangle inequality shows that the Fourier transform is a contraction from L^1 to L^∞. It also extends to an isometry on the Hilbert space L^2, as a consequence of Plancherel's theorem. In other words, the following hold:

$$\|\hat{f}\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}, \tag{1}$$
$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \tag{2}$$

Estimates (1) and (2) can be interpolated with the classical convexity theorem of Riesz-Thörin. As a consequence, we obtain the Hausdorff-Young inequality, which in turn asserts the following: Given an exponent $1 \leq p \leq 2$, the inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \tag{3}$$

holds for every $f \in L^p$. Here p' denotes the exponent conjugate to p, given by $\frac{1}{p} + \frac{1}{p'} = 1$.

The concept of L^p convergence differs substantially from that of pointwise convergence. A powerful link between the two is provided by *maximal functions*, which are in themselves central objects of study in Fourier analysis. In general terms, one expects L^p-bounds for a maximal function to imply pointwise almost everywhere convergence of the original operator. In our setting, we are led to define the maximally truncated Fourier transform,

$$\mathcal{F}_* f(\xi) = \sup_{y\in\mathbb{R}} \Big| \int_{-\infty}^{y} f(x) e^{-2\pi i x\xi}\, \mathrm{d}x \Big|.$$

The classical Menshov-Paley-Zygmund inequality states that, for every $1 \leq p < 2$, there exists a constant $M_p < \infty$ such that

$$\|\mathcal{F}_* f\|_{L^{p'}(\mathbb{R})} \leq M_p \|f\|_{L^p(\mathbb{R})}, \tag{4}$$

for every $f \in L^p$. The case $p = 2$ of inequality (4) is considerably more subtle, and follows from the celebrated result of

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Carleson [3] on the pointwise convergence of Fourier series of square integrable functions. A powerful *variational* refinement of these results has been recently proved by Oberlin, Seeger, Tao, Thiele, and Wright [13]: Given $1 \le p \le 2$ and $r > p$, there exists a constant $C_{p,r} < \infty$ such that

$$\Big\| \int_{-\infty}^{y} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x \Big\|_{L^{p'}_{\xi}(\mathbb{R}; \mathscr{V}^{r}_{y}(\mathbb{R}))} \le C_{p,r} \|f\|_{L^p(\mathbb{R})}, \tag{5}$$

for every $f \in L^p$. In the case $1 \le p < 2$ and $r = \infty$, inequality (5) reduces to (4). When $p = 2$ and $2 < r < \infty$, inequality (5) strengthens Carleson's theorem by establishing L^2-estimates for the r-variation[1] of the partial sum operator for the Fourier transform.

We conclude this succinct account of linear Fourier analysis by asking the following natural questions: What is the optimal constant in inequality (3)? What are the corresponding *extremizers*? By this we mean functions which saturate the sharp inequality, turning it into an equality. Beckner [2] proved that the inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \le \mathbf{B}_p \|f\|_{L^p(\mathbb{R})} \tag{6}$$

holds with constant $\mathbf{B}_p := p^{\frac{1}{2p}} p'^{-\frac{1}{2p'}}$, which is strictly less than 1 if $1 < p < 2$, and is sharp. Moreover, equality is attained by Gaussians. Lieb [9] later proved that there exist no other extremizers besides the Gaussian functions. More recently, Christ [4] further refined inequality (6), establishing the following stable version: There exists a constant $c_p > 0$ such that

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \le \Big(\mathbf{B}_p - c_p \frac{\operatorname{dist}_p^2(f, \mathfrak{G})}{\|f\|^2_{L^p(\mathbb{R})}} \Big) \|f\|_{L^p(\mathbb{R})}, \tag{7}$$

for every nonzero $f \in L^p$. Here $\operatorname{dist}_p(f, \mathfrak{G})$ denotes the L^p-distance from f to the set of all Gaussians, denoted $\mathfrak{G}$. Sharp inequalities and stable versions thereof, together with a characterization of the corresponding sets of extremizers, have a rich history in mathematical analysis. The brief description given here only scratches the surface of this fascinating topic for the very particular case of the Hausdorff-Young inequality, which will nonetheless be of interest to us further along the discussion.

2 The nonlinear Fourier transform

One of the many useful features of the Fourier transform is that it maps a *linear* partial differential equation into an algebraic equation, which can be explicitly solved, and then pulled back to a solution of the original problem via Fourier inversion. There have been many attempts to find suitable replacements for this mechanism in the world of *nonlinear* partial differential equations.

For the remainder of this note, we shall focus on a simple nonlinear model of the Fourier transform, also known as the Dirac *scattering transform*, or the SU(1, 1)-*scattering transform*. To describe it precisely, let us take a measurable, bounded and compactly supported function $f : \mathbb{R} \to \mathbb{C}$, which will generally be referred to as a *potential*. Given an arbitrary number $\xi \in \mathbb{R}$, consider the initial value problem

$$\begin{gathered} \frac{\partial}{\partial x} \begin{bmatrix} a(x,\xi) \\ b(x,\xi) \end{bmatrix} = \begin{bmatrix} 0 & \overline{f(x)} e^{2\pi i x \xi} \\ f(x) e^{-2\pi i x \xi} & 0 \end{bmatrix} \begin{bmatrix} a(x,\xi) \\ b(x,\xi) \end{bmatrix}, \\ \begin{bmatrix} a(-\infty,\xi) \\ b(-\infty,\xi) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{gathered} \tag{8}$$

This system is well known to have unique absolutely continuous solutions $a(\cdot,\xi)$ and $b(\cdot,\xi)$. Defining functions $a, b : \mathbb{R} \to \mathbb{C}$ via $a(\xi) := a(+\infty,\xi)$ and $b(\xi) := b(+\infty,\xi)$, one is led to study properties of the *forward transform* $f \mapsto (a,b)$. The differential equation (8) forces $|a(\xi)|^2 - |b(\xi)|^2 = 1$, which in particular means that a certain size of the above vector is controlled by the quantity $|a(\xi)|$ alone. It is sometimes convenient to add an extra column and turn this vector into a 2×2 matrix belonging to the classical Lie group

$$\mathrm{SU}(1,1) = \Big\{ \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} : w, z \in \mathbb{C} \text{ and } |w|^2 - |z|^2 = 1 \Big\},$$

which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. It should be emphasized that we are *not* considering the linear Fourier transform on the group SU(1, 1), as the map $f \mapsto (a,b)$ is highly nonlinear. This is at the root of several foundational and technical issues, and among other issues prevents any sort of interpolation scheme from holding in general. On the other hand, the nonlinear Fourier transform enjoys several symmetries which are shared by its linear counterpart, e.g. with respect to L^1-normalized dilations, translations, and modulations.

Sources of motivation for considering this precise instance of the nonlinear Fourier transform include the eigenvalue problem for the Dirac operator, the study of completely integrable systems and scattering theory, and the Riemann-Hilbert problem; see the expository paper [15] for further information. The Dirac scattering transform is the simplest nonlinear model that cannot be solved explicitly of a more general transform, the AKNS-ZS nonlinear Fourier transform; see [1, 17] for details.

We now describe some nonlinear analogues of the classical inequalities for the linear Fourier transform from the first section. First of all, Grönwall's inequality from ODE theory implies the following analogue of the Riemann-Lebesgue estimate (1):

$$\big\| (\log |a(\xi)|^2)^{\frac{1}{2}} \big\|_{L^{\infty}_{\xi}(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}, \tag{9}$$

for every potential f.

[1] See §3.2 below for a discussion of variation norms.

Secondly, the nonlinear PLANCHEREL's theorem is a well-known scattering identity, a variant of which goes back at least to work of VERBLUNSKY from the 1930s. It states that

$$\left\| (\log |a(\xi)|^2)^{\frac{1}{2}} \right\|_{L^2_\xi(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}, \tag{10}$$

for every potential f. Identity (10) can be established via a contour integration argument, see e.g. [11, §6], and it is curious to note that no other proof seems available in the literature. The reader might wonder about the role of the square root of the logarithm on the left-hand sides of inequalities (9) and (10). It helps to notice that both inequalities reduce to their linear analogues in first order approximation. In particular, the linear FOURIER transform coincides with the linearization of the nonlinear Fourier transform at the origin.

Similarly to the linear case, one would like to use interpolation in order to obtain a nonlinear HAUSDORFF-YOUNG inequality but, as previously mentioned, this is not available in the current nonlinear setting. However, the seminal work of CHRIST and KISELEV [5, 6] on the spectral theory of one-dimensional SCHRÖDINGER operators implies the following result: If $1 \leq p < 2$, then there exists a constant $C_p < \infty$ such that

$$\left\| (\log |a(\xi)|^2)^{\frac{1}{2}} \right\|_{L^{p'}_\xi(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}, \tag{11}$$

for every potential f. The proof produces a family of constants C_p which, contrary to the linear case, blows up as $p \to 2^-$. Thus it is natural to ask:

QUESTION 1.— Do the constants C_p from inequality (11) remain uniformly bounded, as p tends to 2?

Question (1) was originally asked by MUSCALU, TAO and THIELE [11], and was solved in a particular toy model by KOVAČ [7], but remains open in its full generality. In §3.1 we shall describe some recent investigations around this circle of problems. On the other hand, a variational refinement generalizing the nonlinear analogue of the MENSHOV-PALEY-ZYGMUND inequality was established by OBERLIN et al. [13]. This has recently been extended to the discrete setting, and we present some details in §3.2 below.

We close this section by mentioning a nonlinear analogue of CARLESON's theorem on the pointwise convergence of FOURIER series. It was originally formulated in [11, Conjecture 1.2], and we record it here.

QUESTION 2.— Does the following inequality hold, for every square integrable function f?

$$\left\| \sup_{x\in\mathbb{R}} (\log |a(x,\xi)|^2)^{\frac{1}{2}} \right\|_{L^2_\xi(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}$$

Question 2 was solved in a particular toy model by MUSCALU, TAO and THIELE [11], but remains open in its full generality. It is known [12] that this fundamental question cannot be settled by estimating the terms in the natural multilinear expansion of the scattering transform.

3 SOME RECENT PROGRESS

3.1 TOWARDS QUESTION 1

By considering truncated Gaussian potentials and linearizing, one may check that the constant C_p from (11) dominates BECKNER's constant from (6), $C_p \geq \mathbf{B}_p$. It may be tempting to conjecture that $C_p = \mathbf{B}_p$. While this is still an open problem, which would immediately provide an affirmative answer to Question 1, the main result from [8] hints at some supporting evidence in this direction. To describe it precisely, fix an exponent $1 < p < 2$, a *height* $H > 0$, and a *width* $W > 0$. We only consider potentials $f : \mathbb{R} \to \mathbb{C}$ of controlled height and width, i.e. such that $|f| \leq H$ and f is supported on an interval of length at most W.

THEOREM 1 ([8]).— There exist $\delta, \varepsilon > 0$, depending on p, H, W, such that

$$\left\| (\log |a(\xi)|^2)^{\frac{1}{2}} \right\|_{L^{p'}_\xi(\mathbb{R})} \leq \left(\mathbf{B}_p - \varepsilon \|f\|^2_{L^1(\mathbb{R})}\right) \|f\|_{L^p(\mathbb{R})}, \tag{12}$$

for every potential f satisfying the above hypotheses and $\|f\|_{L^1} \leq \delta$.

Note that inequality (12) implies (11) with $C_p = \mathbf{B}_p$, but only for the restricted class of potentials considered in the theorem. Since this class is allowed to depend on the exponent p, no uniformity is claimed. The emphasis is rather on the perhaps surprising fact that the nonlinear HAUSDORFF-YOUNG ratio beats the linear one for sufficiently small values of $\|f\|_{L^1}$.

We briefly describe the main idea behind the proof of Theorem 1. The strategy is to split the analysis into two cases, depending on whether or not the potential f is far from the set of Gaussians in the relative L^p-distance. In the former case, one invokes CHRIST's sharpened HAUSDORFF-YOUNG inequality (7) in order to absorb the error terms coming from linearization. In the latter case, one calculates a few terms of the multilinear expansion of $(\log |a|^2)^{1/2}$, and approximates f by a suitable Gaussian. The error terms that appear are controlled by successive applications of the MENSHOV-PALEY-ZYGMUND inequality (4).

3.2 DISCRETE ANALOGUES

There is a close and fruitful connection between the continuous FOURIER transform and discrete FOURIER series. In a similar vein, TAO and THIELE [16] introduced a discrete model for the solution curves of the nonlinear FOURIER transform. To define it, consider a compactly supported, complex-valued sequence F satisfying $|F_n| < 1$, for every n, and transfer matrices $\{T_n\}$ given by

$$T_n(z) = (1 - |F_n|^2)^{-\frac{1}{2}} \begin{pmatrix} 1 & F_n z^n \\ \overline{F_n} z^{-n} & 1 \end{pmatrix},$$

where $z \in \mathbb{T}$ is a unimodular complex number. Note that $T_n(z) \in \mathrm{SU}(1,1)$. The nonlinear Fourier transform of the sequence F is defined as an $\mathrm{SU}(1,1)$-valued function on the unit circle given by the expression

$$(a,b)(z) = \lim_{N\to\infty} \prod_{n=-N}^{N} T_n(z),$$

where the ordered product is seen to converge in an appropriate sense provided $F \in \ell^2$. Discrete analogues of the nonlinear Riemann-Lebesgue, Plancherel and Hausdorff-Young inequalities are available, see [16, §1–3]. To describe a variational refinement of latter, consider the following truncated versions of the linear and the nonlinear Fourier transforms of F, respectively denoted by $\sigma = \sigma(F)$ and $\gamma = \gamma[F]$, and given at level N by

$$\begin{aligned} &\sigma(F)(N;z) = \textstyle\sum_{n=-\infty}^{N} \begin{pmatrix} 0 & F_n z^n \\ \overline{F_n} z^{-n} & 0 \end{pmatrix}, \quad \text{and} \\ &\gamma[F](N;z) = \textstyle\prod_{n=-\infty}^{N} T_n(z). \end{aligned} \tag{13}$$

For fixed $z \in \mathbb{T}$, we shall think of the maps $N \mapsto \gamma[F](N;z)$ and $N \mapsto \sigma(F)(N;z)$ as discrete curves taking values on the Lie group $\mathrm{SU}(1,1)$ and its Lie algebra $\mathfrak{su}(1,1)$, respectively. Endow the Lie algebra with the operator norm $\|\cdot\|_{\mathrm{op}}$, and the Lie group with the distance

$$d(X,Y) = \log(1 + \|X^{-1}Y - I\|_{\mathrm{op}}).$$

Given an exponent $r \geq 1$, we are interested in measuring the r-variation in the variable N of the curves σ and γ. The variation is defined as

$$\mathscr{V}_r(\gamma)(z) = \sup_K \sup_{N_0<\ldots<N_K} \Big(\sum_{j=0}^{K-1} d(\gamma_{N_j}(z), \gamma_{N_{j+1}}(z))^r \Big)^{\frac{1}{r}},$$

and similarly for $\mathscr{V}_r(\sigma)$. Here the supremum is taken over all strictly increasing finite sequences of integers $N_0 < N_1 < \ldots < N_K$ and over all integers K. We are finally in a position to state the following discrete, variational, nonlinear Hausdorff-Young inequality.

Theorem 2 ([14]).— *Let $1 \leq p < 2$ and $r > p$. Then there exists a constant $D_{p,r} < \infty$ such that*

$$\begin{aligned} &\|\mathscr{V}_r(\gamma[F])\|_{L^{p'}(\mathbb{S})} + \|\mathscr{V}_r(\gamma[F])\|^{1/r}_{L^{p'/r}(\mathbb{T}\setminus\mathbb{S})} \leq \\ &\leq D_{p,r} \Big\| \log\Big(\tfrac{1+|F_n|}{1-|F_n|}\Big) \Big\|_{\ell^p(\mathbb{Z})}, \end{aligned} \tag{14}$$

for every $F \in \ell^p$ satisfying $|F_n| < 1$, for every n. Here, $\mathbb{S} := \{z \in \mathbb{T} : \mathscr{V}_s(\gamma[F])(z) \leq 1\}$, where $s = r$ if $p < r < 2$, and $s = (p+2)/2$ if $r \geq 2$.

We briefly describe the proof of Theorem 2, which comprises two parts. The first part is inspired by the adaption of Lyons' theory of rough paths [10] by Oberlin et al. [13] to study variation norms on $\mathrm{SU}(1,1)$. In particular, given a potential F, one shows that the r-variation of the discrete curve $\gamma[F]$ can be controlled by the r-variation of the linearized curve $\sigma(F)$, plus an extra term that accounts for the possible presence of large jumps. The second part amounts to a discrete variational version of the Menshov-Paley-Zygmund inequality (4). This step requires $r > p$, and is accomplished via an adaptation of the original argument of Christ-Kiselev [5] to the variational setting.

We finish by noting that the range of exponents promised by Theorem 2 is almost sharp. Indeed, given $p > 1$, one easily checks that inequality (14) can only hold if $r > p$. On the other hand, extending this inequality to $p = 2$, already in the simplest case $r = \infty$, would provide an affirmative answer to a discrete version of Question 2. This remains a central open problem in the area.

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