Spectra in mathematics and in physics: from the dispersion of light to nonlinear eigenvalues

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1 Introduction

This lecture deals with the concept of spectrum in various epochs, with various meanings and for various disciplines. Its content can be motivated by a quotation of L.A. Steen [53]:

> Not least because such different objects as atoms, operators and algebras all possess spectra, the evolution of spectral theory is one of the most informative chapters in the history of contemporary mathematics. […] In 1928 and 1930 Norbert Wiener developed a theory of spectral analysis for functions in an attempt to analyze mathematically the spectrum of the white light, while twenty years later Arne Beurling inaugurated the complementary study of spectral synthesis.

and a quotation of N. Wiener [60]:

> The author sees no compelling reason to avoid a physical terminology in pure mathematics when a mathematical concept corresponds closely to a concept already familiar in physics. [...] The "spectrum" of this book merely amounts to rendering precise the notion familiar to the physicist, and may as well be known by the same name.

We shall return soon to the origin of the use of the word "spectrum" in physics and in mathematics. The mathematical spectrum is partly made of "eigenvalues", a strange word which has not been immediately adopted, as observed by S.H. Gould in [17]:

> The concept of an eigenvalue is of great importance in both pure and applied mathematics. […] The German word *eigen* means *characteristic* and the hybrid work eigenvalue is used for characteristic numbers in order to avoid confusion with the many other uses in English of the word *characteristic.* […] There can be no doubt that *eigenvalue* will soon find its way into the standard dictionaries. […] The English language has many such hybrids: for example *liverwurst.*

Previous work has already been devoted to the development of spectral theory in mathematics, and the reader can find further information and remarks in [28, 31, 36, 59, 61].

2 Light and colours: the chemical unity of the Universe

The first occurence of the word "spectrum" in science seems to be found in a letter of Isaac Newton to the *Royal Society,* in 1672, where he uses the word to denote the oblong colored image, with the colors of a rainbow, produced on a white paper by a beam of Sun light dispersed by a glass prism. The expression is repeated in his book *Opticks* (London, 1704), and, in no case, Newton makes any comment on the choice of the word. "Spectrum" means "vision" in Latin, and comes from "spectare", to look at.

Little progress is made in Newton's experiment in the eighteenth century, except Melvill's observation in 1752 that a flame of salted alcohol only gives a yellow spectrum.

The beginning of the nineteenth century sees the discovery of the infrared and ultra-violet extensions of the spectrum, respectively by William Herschel and Ritter, the crucial discovery by Wollaston of seven dark lines in the solar spectrum, and the association of the color to the frequency in Young's ondulatory theory of light. In 1814, the Bavarian optician Fraunhofer constructs the first spectroscope. This allows him to establish the first map of the Solar spectrum, and to identify the position of one of its dark lines with the bright Natrium D-line.

After some pioneering work of John Herschell, Talbot, Foucault, Kelvin, Stokes and Angström, the mathematical physicist Kirchhoff, associated to the chemist Bunsen, discovers in 1859 the fundamental *laws of spectral analysis*: each line of a spectrum is due to the presence of a given element and conversely, the appearence of a line spectrum can be used as an analytical test for the presence of an element. Furthermore, a substance traversed by a source of light with continuous spectrum gives rise to dark lines having the same position. Consequently, the dark lines in the Solar spectrum reveal the composition of its atmosphere: astrophysics is born and stellar spectroscopy, with the pioneering work of men like HUGGINS, MILLER and SECCHI, reveals a fact of fundamental philosophical importance: the chemical unity of the Universe, some two hundred years after Newton's gravitation had shown its physical unity. Let us quote, in this respect, POINCARÉ [46]:

> Auguste Comte has said, I do not remember where, that it would be vain to try to find the composition of the Sun, because this knowledge would not be useful to Sociology. How could he be so short-sighted ? [...] First, one has recognized the nature of the Sun, that the founder of positivism wanted to forbid us, and one has found there substances which exist on the Earth and had remained unnoticed; for example Helium [...]. This was already for Comte a first flat contradiction. But we owe to spectroscopy a much more precious lesson [...]. We know now [...] that the laws of our chemistry are general laws of Nature, and do not follow from the chance which has made us born on the Earth.

Through the red-shift and the Döppler-Fizeau effect, galaxy spectra have also revealed to expansion of our Universe.

But the importance of spectroscopy is not less in the infinitely small, as spectra appear like signatures of atoms and molecules. After ANGSTRÖM classifies in 1853 the lines of the emission spectrum of Hydrogen in *series,* and after some pioneering work of MASCART, the Swiss teacher BALMER finds heuristically, in 1885, a formula giving the wave numbers ($\nu = 1/\lambda = c\nu'$, ν' the frequency) of one of those series:

$$\nu = R\left(\frac{1}{2^2} - \frac{1}{m^2}\right), \quad (m = 3, 4, \dots),$$

where $R = 109.677,7 \ cm^{-1}$ is the Rydberg's constant. The lines accumulate near the limit wave number $\nu_l = R/4$, corresponding to the limit wave length $\lambda_l = 3645,6$ Å.

In 1908, RITZ states his *combination principle*: for each type of atom, it is possible to find a sequence of numbers, the *spectral terms,* such that the frequency of any spectral line of this atom is equal to the difference of two of those spectral terms. For example, the Hydrogen atom is characterized by the spectral terms R/n^2, $(n = 1, 2, \dots)$. This principle implies the *generalized Balmer formula*

$$\nu = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right), \quad (m = n+1, n+2, \dots, \ n = 1, 2, \dots),$$

suggesting the existence of Hydrogen lines with new wave numbers, later observed by PASCHEN, BRACKETT, and PFUND in the infra-red, and by LYMAN in the ultraviolet. The reader can consult [55] for the historical development of spectroscopy and its influence on chemistry and astrophysics.

In 1913, BOHR proposes his quantified planetary model for the Hydrogen atom, from which he deduces mathematically the generalized Balmer formula with $R = 2\pi^2\mu e^4/ch^3$. Here μ is the mass of the electron, e its charge, c the speed of light, h is Planck's constant, and the computation gives a value very close to Rydberg's constant. However Bohr's model is based on some contradictory assumptions, and we may leave again to POINCARÉ [47] some prophetic conclusion:

> Following the work of Balmer, Runge, Kaiser, Rydberg, those lines are distributed in series, and, in each series, follow simple laws. The first idea is to relate those laws to those of harmonics. In the same way as a vibrating string has infinitely many degrees of freedom, allowing it to produce an infinity of sounds whose frequences are multiple of the fundamental frequency, [...] could the atom produce, for identical reasons, infinitely many different lights ? You know that this so simple idea has failed, because, according to the laws of spectroscopy, it is the frequency and not its square which has a simple expression; because the frequency does not become infinite for the harmonics of infinitely high rank. The idea must be modified or must be abandoned.

It is time to have a look at those vibrating strings to which Poincaré refers.

3 MUSIC AND HARMONICS: THE PARADIGM OF THE WAVE EQUATION

The relation between vibrating strings and mathematics can be traced at least to the Pythagorian tradition, but the development of musical theory at the Renaissance has led to physical discussions of the frequency of a vibrating string. GALILEO and MERSENNE, around 1640, study the dependence of the *fundamental frequency* of vibration with respect to the length, the tension and the mass of the string. At the end of the seventeenth century, WALLIS, ROBARTES and SAUVEUR describe the connection between the number of nodes and the *overtones* of a vibrating string. See [11, 50] for details and references.

In 1714, TAYLOR assumes the isochronism of the oscillations for all the points of the string, and their simultaneous passage through the horizontal equilibrium position. He shows analytically that the fundamental frequency is $\nu = (1/2l)(\sqrt{T/\sigma})$, ($T$ is the tension, l the length, and σ the linear density) and the shape of the string is $y = A\sin(\pi x/l)$. In 1732, Jean BERNOULLI determines the fundamental frequency of a discrete string made of six masses. None of them mentions the higher modes, which are considered in 1738 by Daniel BERNOULLI for an oscillating suspended string, both in the discrete and in the continuous case (anticipating Bessel functions). Modeling the propagation of sound in the air, EULER obtains in 1750, the characteristic frequencies and the general solution (as a sum of simple harmonic modes)

$$y_k = \sum_{r=1}^{n} A_r \sin\frac{rk\pi}{n+1}\cos\left(2\frac{\sqrt{K}t}{\sqrt{M}}\frac{\sin(\pi r/2)}{n+1}\right),$$

of the discrete model of a horizontal string

$$M\ddot{y}_k = K\left(y_{k+1} - 2y_k + y_{k-1}\right), \quad (k = 1, 2, \dots, n),$$

already written by Jean BERNOULLI in 1727.

Through a limit process, D'ALEMBERT deduces from it, in 1746, the one-dimensional *wave equation*

$$\frac{\partial^2 y(t,x)}{\partial t^2} = a^2\frac{\partial^2 y(t,x)}{\partial x^2},$$

where $a^2 = T/\sigma$. He determines the solutions, satisfying the boundary conditions

$$y(t,0) = 0, \quad y(t,l) = 0, \quad (t \in \mathbb{R}),$$

through the change of independent variables still used to-day. In 1752, he introduces the method of *separation of variables.*

In 1749, EULER mentions that all possible motions of the vibrating string are periodic with the period of the fundamental mode, and that individual modes whose period is half, third,... of the fundamental one can occur. He writes those particular solutions in the form

$$y(t,x) = \sum a_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$

for the initial configuration

$$y(0,x) = \sum a_n \sin\frac{n\pi x}{l},$$

without precising if the sum is finite or not.

After reading the papers of d'Alembert and Euler on wave equation, DANIEL BERNOULLI claims in 1755 that there are enough free constants a_n to represent all the possible initial shapes as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l},$$

and that all the subsequent motions are given by

$$y(t,x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}.$$

Those conclusions are refuted by EULER and D'ALEMBERT for different reasons, and the warm quarrell between those three giants lasts for some ten year, without conclusion, despite some deep comments of LAGRANGE in 1759 (see [50]). For references to the original sources and historical development, see [4, 8, 29, 56, 57].

4 HEAT AND POTENTIAL: THE IRRESISTIBLE ASCENT OF FOURIER SERIES

FOURIER modelizes the conduction of heat in a memoir submitted to the *Académie des Sciences de Paris* in 1807, rejected by the referees LAGRANGE, LAPLACE and LEGENDRE, revised in 1811, awarded the *Grand Prix de Mathématiques de l'Académie* in 1812, and only published in 1824-26 in its *Mémoires*, after Fourier has became its permanent secretary. In the meantime, Fourier has published a variant as a book, the famous *Théorie mathématique de la chaleur* (Paris, 1822).

Fourier establishes that the temperature $T(x,y,z,t)$ in a point (x,y,z) of a homogeneous and isotropic body satisfies the *heat equation*

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = k^2\frac{\partial T}{\partial t},$$

where the constant k^2 depends upon the material. He studies several special cases by separation of variables, raising again the question of representing an arbitrary function by a trigonometric series, and obtaining, in a complicated way, the formula relating the coefficients of the series to the function. For references on Fourier's work, see [9, 18, 32].

Fourier's results motivate STURM and LIOUVILLE [34, 35, 54] to study in 1836-37 the general problem of eigenvalues and eigenfunctions for an arbitrary second order ordinary linear differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \lambda\rho(x)y = 0, \quad (p(x) > 0,\ \rho(x) > 0),$$

with the boundary conditions

$$y'(a) - h_1 y(a) = 0,\ y'(b) + h_2 y(b) = 0,$$

where $h_1 \geq 0$, $h_2 \geq 0$ and $a < b$. They prove that the problem has a nontrivial solution only when λ takes one of the values of an increasing sequence of positive numbers λ_n tending to infinity (*eigenvalues*), that the solutions v_n corresponding to the eigenvalue λ_n (*eigenfunctions*) are orthogonal in the sense

$$\int_a^b v_m(x)v_n(x)\rho(x)\,dx = 0, \quad (m \neq n),$$

and that each C^2 function satisfying the boundary conditions can be developed into a uniformly convergent series $f(x) = \sum_{n=1}^{\infty} c_n v_n(x)$, where the generalized Fourier coefficients $c_n = \int_a^b f(x)v_n(x)\rho(x)\,dx$ satisfy the *Parseval equality*

$$\int_a^b f^2(x)\rho(x)\,dx = \sum_{n=1}^{\infty} c_n^2$$

For the first time, general results are obtained which do not depend upon some explicit form of the solution of the differential equation. For studies of the work of Sturm and Liouville, see [2, 39, 40].

The Sturm-Liouville theory motivates of course the obtention of similar conclusions for the simplest partial differential equation case, namely the *eigenvalue problem for the Laplacian* on a general planar or spatial domain Ω (excluding the use of separation of variables)

$$\Delta u + \lambda u = 0 \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega.$$

Schwarz [52] proves in 1885 the existence of the first eigenvalue and eigenfunction, and shows that a smaller Ω gives a larger λ_1. Picard [43] obtains in 1893 the existence of the second eigenvalue, and Poincaré [45] proves in 1894 the existence and the essential properties of all the eigenvalues and eigenfunctions, by showing that the solution of

$$\Delta u + \lambda u = f \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega,$$

can be expressed as a meromorphic function of λ, whose poles are real and are the eigenvalues. Physically, f can be considered as a force applied to the vibrating membrane or body, and its free oscillations are those for which the forced oscillations become infinite. See [10, 41] for more details.

Motivated by Poincaré's work, Fredholm [15] publishes in 1903 a systematic study of the *integral equations of second type*

$$u(x) + \int_0^1 K(x,\xi)u(\xi)\,d\xi = f(x).$$

Following an idea of Volterra, he approximates the integral equation by the finite algebraic linear system

$$u_n\left(\frac{i}{n}\right) + \sum_{j=1}^{n} K\left(\frac{i}{n},\frac{j}{n}\right)u_n\left(\frac{j}{n}\right)\frac{j}{n} = f\left(\frac{i}{n}\right), \ (i = 1, 2, \dots, n).$$

Using finite-dimensional linear algebra and a limit process, he obtains the necessary and sufficient conditions of solvability, but does not emphasize the corresponding eigenvalue problem

$$u(x) + \lambda \int_0^1 K(x,\xi)u(\xi)\,d\xi = 0.$$

This will be done by Hilbert, as we shall see later. This finite-dimensional linear algebra has been developed in the nineteenth century, to answer some questions raised by analytical and celestial mechanics in the eighteenth century.

5 Stability of the Solar system: the origin of the secular equation

Motivated by elasticity problems, Euler anounces in 1743 the resolution of the linear ordinary differential equations with constant coefficients

$$Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + D\frac{d^3y}{dx^3} + \dots + L\frac{d^ny}{dx^n} = 0.$$

Functions of the form $y(x) = e^{rx}$ are solutions if and only if r is a solution of the *characteristic* or *indicial* or *auxiliary* equation

$$A + Br + Cr^2 + \dots + Lr^n = 0,$$

and the general solution is obtained as a linear combination of the n special solutions associated to its roots.

d'Alembert, in his *Traité de dynamique* (Paris, 1743), studies second order systems of the form

$$y_i'' + \sum_{k=1}^{n} A_{ik}y_k = 0, \quad (i = 1, 2, \dots, n),$$

with $n = 2, 3$ and, for $n = 3$, special values of the coefficients A_{ik}, and, in 1750, first order systems of the type

$$\begin{aligned} x' + ax + by + cz &= 0,\\ y' + ex + fy + gz &= 0,\\ z' + hx + my + nz &= 0, \end{aligned}$$

but he obtains only partial results.

In 1766 Lagrange considers the general second order system above, and, through the substitution $y_i = x_i e^{\rho t}$, $(i = 1, 2, \dots, n)$, shows that

$$x = (x_1, x_2, \dots, x_n)$$

must verify the linear system (in modern matrix notations, with $A = (A_{ik})$, $A^T = (A_{ki})$)

$$\rho^2 x + A^T x = 0.$$

The elimination of the x_i in this system implies that ρ^2 must verify an algebraic equation of degree n. Lagrange is interested in modeling situations where the equilibrium $x = 0$ is stable, and concludes from this *a priori* physical stability and from the form of the solutions that the roots ρ^2 must be real, negative, and simple! In 1778, Lagrange obtains, in linearizing some equations of celestial mechanics, the first order system

$$h_i' + \sum_k A_{ik} l_k = 0, \quad l_i' - \sum_k A_{ik} h_k = 0, \quad (i = 1, 2, \dots, n),$$

where $h_i^2 + l_i^2 = e_i^2$, the square of the excentricity of the orbit, and uses the same approach, ending with the same physical "proof" for the properties of the roots of the corresponding algebraic equation, he calls *secular equation* [30]:

> One must notice that although we have supposed the roots [...] of the [secular] equation [...] real and distinct, it can happen that imaginary [=complex] ones exist; [...] we only observe that the quantities will increase with t; consequently, the above solution will stop to be exact after some time; but happily those case do not seem to occur in the system of the world.

Laplace is convinced that a mathematical proof of the properties of the secular roots should be preferred. In 1787, he deduces the *a priori* boundedness of the solutions from a first integral, first obtained in an approximate way, but rigorously proved two years later. Laplace's arguments are used by Lagrange, in his *Mécanique analytique* (Paris, 1788), for the study of small motions around an equilibrium, using this time the well known energy integral.

In his *Leçons sur les applications du calcul infinitésimal à la géométrie* (Cours de l'Ecole polytechnique, Paris, 1826), Cauchy associates the reduction of a quadric to its axes to an eigenvalue problem and its characteristic equation, invariant for any orthogonal change of coordinates, and proves rigorously that all eigenvalues are real. Sturm uses in 1829 his theorem of the number of real zeros of a real polynomial to prove the reality of the roots of the secular equations introduced by Lagrange and Laplace. The same year, Cauchy [5] gives another proof, shows the analogy of the problem of characteristic values in problems of analytic geometry, differential equations, solid and celestial mechanics, introduces the term *characteristic polynomial* or *equation*, which will finally overcome earlier or even later terminologies like *S-equation, determining, secular* or *latent equation.*

In the second half of the XIX^{th} century, the study of this equation becomes a topics of pure algebra, considered in the language of matrices or forms. For example, Hermite [25] gives in 1854 the standard proof of the reality of the characteristic roots of a *Hermitian form.* All this is carefully described in [19, 20, 21, 22, 29], with references to the original papers.

6 Algebra and geometry in infinite dimensional space: the birth of functional analysis

Motivated by Fredholm's theory, Hilbert publishes, between 1904 and 1910, a series of six articles, later reproduced in book form, under the title *Grunzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Leipzig, 1912). He first follows essentially Fredholm's approach but considers an integral equation containing explicitely the complex parameter λ

$$u(x) - \lambda \int_0^1 K(x,y) u(y)\, dy = 0.$$

Hilbert supposes then $K(s,t)$ *symmetrical* ($K(s,t) = K(t,s)$), and uses the theory of finite quadratic forms to prove the reality of the eigenvalues and the orthogonality of the eigenfunctions. He generalizes to this setting the theorem of principal axes of analytical geometry, the variational characterization of eigenvalues due to Liouville-Weber-Poincaré that we shall consider later, and proves the *Hilbert-Schmidt expansion theorem.* In his own words:

> The method [...] consists in starting from an algebraic problem, namely the problem of the orthogonal transformation of quadratic forms in n variables in a sum of squares, and, through a rigorous limit process for $n = \infty$, to succeed in solving the considered transcendental problem.

Hilbert then forgets the initial motivation by integral equations and considers directly the infinite bilinear form in the sequences $x = (x_j)$, $y = (y_k)$

$$B(x,y) = \sum_{p,q=1}^{\infty} k_{pq} x_p y_q,$$

when $\sum_{j=1}^n |x_j|^2$ et $\sum_{j=1}^n |y_j|^2$ converge, and B is bounded on the corresponding unit ball. He again generalizes the theorem of principal axes. For this, he must introduce, in addition to the discrete spectrum make of the eigenvalues, a *continuous* or *band spectrum,* a word first used in a mathematical setting by WIRTINGER [62] in 1897, by analogy with spectra of molecules, in his discovery of band spectra for Hill's equation. Let us quote Dieudonné [10]:

> We now return to the most original part of Hilbert's 1906 paper, in which he discovered the entirely new phenomenon of the "continuous spectrum". [...] In 1897 Wirtinger developed similar ideas for Hill's equation
>
> $$y'' + \lambda q(x) y = 0.$$
>
> [...] The similarity with the optical spectra of molecules leads him to speak of the "Bandesspectrum" of Hill's equation. [...] Although Hilbert does not mention Wirtinger's paper, it is probable that he had read it (it is quoted by several of his pupils), and it may be that the name "Spectrum" which he used came from it.

To eliminate the continuous spectrum, Hilbert defines the concept of *completely continuous* quadratic forms. He applies his results to integral equations, introducing explicitely the notion of *complete orthogonal system* of functions.

Hilbert's work is simplified and geometrized by Ehrard SCHMIDT in 1907, who introduces the concept of *orthogonal projector;* the same year F. RIESZ extends Hilbert theory to $L^2(0,1)$ and shows its isomorphism with l^2.

In 1908, H. WEYL considers *singular integral equations*

$$u(x) + \lambda \int_I K(x,y) u(y)\, dy = 0,$$

where integration is made on an unbounded interval I, and shows the existence of band spectra. His work is generalized by CARLEMAN in 1923. The famous two volumes monograph *Methoden der mathematischen Physik* (Berlin, 1924) of COURANT-HILBERT describes the state of the art of the mathematical tools of classical physics, before becoming the bible for the new physics. As noticed by C. REID [48]:

> The Courant-Hilbert book on mathematical methods of physics, which had appeared at the end of 1924, before both Heisenberg's and Schrödinger's work, instead of being outdated by the new discoveries, seemed to have been written expressly for the physicists who now had to deal with them.

Excellent surveys of the development of Hilbert's ideas can be found in [1, 23, 24, 42, 49].

7 QUANTUM MECHANICS: UNIFYING THE PHYSICAL AND MATHEMATICAL SPECTRA

In 1923, L. DE BROGLIE recovers Bohr's formula for hydrogen atom by associating to each particle a wave of some frequency and identifying the stationary states of the electron to the stationary character of the wave.

As the observable lines of an atomic spectrum can be represented by the infinite matrix ($\nu_{nm} = T_n - T_m$) of the differences between the spectral terms, HEISENBERG proposes in 1925 to replace the position q of an electron by an infinite matrix $q_{nm} e^{2\pi i \nu_{nm} t}$, and similarly for its momentum p. The diagonal elements of the corresponding Hermitian infinite matrices correspond to a stationary state and the other ones to corresponding transitions. The matrices q and p satisfy the *Born-Jordan non-commutativity condition* $pq - qp = \frac{h}{2\pi i} I$ and Hamilton-type canonical equations of motion. In 1926, PAULI deduces the Bohr formula for Hydrogen atom from matrix mechanics.

Independently and the same year, SCHRÖDINGER proposes to express the Bohr's quantification conditions as an eigenvalue problem [51]:

> In this communication I wish first to show in the simplest case of the Hydrogen atome (nonrelativistic and undistorted) that the usual rules for quantization can be replaced by another requirement, in which mention of 'whole numbers' no longer occur. Instead the integers occur in the same natural way as the integers specifying the number of nodes in a vibrating string. The new conception can be generalized, and I believe it touches the deepest meaning of the quantum rules. [...] The equation contains a "proper value parameter" E, which corresponds to the mechanical energy in macroscopic problems [...]. In general the wave or vibration equation possesses *no* solutions, which together with their derivatives are one-valued, finite and continuous througout the configuration space, *except* for certain special values of E, the *proper values.* These values form the "proper value spectrum" which frequently includes continuous parts (the "band spectrum", not expressly considered in most formulae [...]) as well as discrete points (the "line spectrum"). The proper values either turn out to be identical with the "energy levels" [...] of the quantum theory as hitherto developed, or differ from them in a manner which is confirmed by experience.

Starting from Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = E,$$

Schrödinger sets $S = K \log \psi$ (K is an action) and obtains

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E.$$

For the electron of the Hydrogen atom this equation is

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0,$$

where $r^2 = x^2 + y^2 + z^2$. Schrödinger introduces the problem of finding an extremum of

$$\int_{\mathbb{R}^3} \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2\right] dx\, dy\, dz,$$

among all sufficiently smooth functions ψ tending zero at infinity. The correspond Euler-Lagrange equation

$$\Delta\psi + \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0,$$

is *Schrödinger's equation.* Using spherical coordinates and separation of variables ($\psi(r,\theta,\varphi) = u(r)v(\theta)w(\varphi)$), Schrödinger reduces the problem to finding nontrivial solutions tending to 0 when $r \to \infty$ for the ordinary differential equations

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \left(\frac{2\mu E}{K^2} + \frac{2\mu e^2}{K^2 r} - \frac{n(n+1)}{r^2}\right)u = 0,$$

where $n = 0, 1, 2, \ldots$.

With the help of WEYL, Schrödinger shows that this equation has solutions with the required asymptotic properties if and only if $E > 0$ or

$$E < 0 \text{ and } \frac{\mu e^2}{K\sqrt{-2\mu E}} = j, \quad (j = n+1, n+2, \ldots).$$

For $n = 0$, those conditions become

$$E_j = -\frac{\mu e^4}{2K^2 j^2}, \quad (j = 1, 2, \ldots),$$

and reduce to Bohr's ones by taking $K = h/2\pi$. Schrödinger has reduced the problem of finding the energy spectrum of the Hydrogen atom to an eigenvalue problem on $\mathbb{R}^3$

$$L\psi + \frac{2\mu}{K^2}E\psi = 0,$$

for some differential operator L. Its mathematical spectrum exactly corresponds to the physical spectrum. Poincaré's program is realized.

Schrödinger gives later the now classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to DE BROGLIE's ideas. He also develops a *perturbation method,* inspired by the work of RAYLEIGH in acoustics, gives the *time-dependent Schrödinger's equation*

$$\frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = \Delta\psi - \frac{8\pi^2}{h^2}V(t,x,y,z)\psi.$$

and proves the equivalence between his wave mechanics and Heisenberg's matrix mechanics. Mathematically, this fact is linked to the isomorphism between l^2 and L^2. Indeed, as observed by CONDON [6], physicists could have saved some time and energy if they had taken Hilbert more seriously:

> I remember that David Hilbert was lecturing on quantum theory that fall [1926], although he was in very poor health at the time. [...] Hilbert was having a great laugh on Born and Heisenberg and the Göttingen theoretical physicists because when they first discovered matrix mechanics they [...] had gone to Hilbert for help and Hilbert said the only time he had ever had anything to do with matrices was when they came up as a sort of by-product of the eigenvalues of the boundary-value problem for a differential equation. So if you look for the differential equation which has these matrices you can probably do more with that. They had thought it was a goofy idea and that Hilbert didn't know what he was talking about. So he was having a lot of fun pointing out to them that they could have discovered Schrödinger's wave mechanics six months earlier if they had paid a little more attention to him.

See [26, 27] for the development of quantum mechanics.

Quantum theory gives in return a huge impetus to the mathematical development of spectral theory for unbounded linear operators. According to L.A. STEEN [53]:

> The mathematical machinery of quantum mechanics became that of spectral analysis and the renewed activity precipitated the publication by Aurel Wintner of the first book devoted to spectral theory in 1929.

In 1927, VON NEUMANN defines axiomatically the concept of *abstract Hilbert space* and developes, between 1927 and 1929, a spectral theory for *unbounded self-adjoint operators* in a Hilbert space. He synthetizes his approach in the epoch-making book *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932), and, the same year, STONE publishes his *Linear Transformations in Hilbert Spaces* (Providence, 1932), the first systematic treatise on the spectral theory of unbounded linear operators.

8 VARIATIONAL CHARACTERIZATION OF EIGENVALUES: THE WAY TO A NONLINEAR SPECTRAL THEORY

Using Lagrange multipliers, LAGRANGE and CAUCHY (1829-30) are already well aware that the smallest and the largest eigenvalue of a symmetric quadratic form

$$Q(u) = \sum_{j,k=1}^{n} a_{jk}u_j u_k, \quad (a_{jk} = a_{kj}),$$

can be obtained my minimizing and maximizing it on the unit sphere $\sum_{j=1}^{n} u_j^2 = 1$. If the corresponding extremum is reached at u^*, then u^* is an associated eigenvector, an approach later developed by RAYLEIGH.

In the setting of integral or partial differential equations, LIOUVILLE, in unpublished papers written around 1850, H. WEBER [58] in 1869, and POINCARÉ [44] in 1890, independently propose a *recursive variational method* to determine all eigenvalues $\lambda_1 \leq \lambda_2 \leq$

$\ldots \leq \lambda_n$ and corresponding eigenvectors $u^1, u^2, \ldots, u^n$ of Q:

$$\lambda_1 = \min_{\|u\|=1} Q(u) \quad (= Q(u^1)),$$

$$\lambda_j = \min_{\|u\|=1, \langle u,u^1\rangle=0,\ldots,\langle u,u^{j-1}\rangle=0} Q(u) \quad (= Q(u^j)), \quad (j = 2, \ldots, n).$$

Further considerations of Poincaré lead to a non-recursive *minimum-maximum principle* explicitely given by Fischer [13] in 1905:

$$\lambda_j = \min_{\{X^j \subset \mathbb{R}^n : dim X^j = j\}} \max_{\{u \in X^j : \|u\|=1\}} Q(u).$$

Weyl introduces in 1912 a *maximum-minimum principle*:

$$\lambda_j = \max_{\{p_1,\ldots,p_{j-1} \in \mathbb{R}^n\}} \min_{\{\|u\|=1, \langle u,p_i\rangle=0, 1\leq i\leq j-1\}} Q(u),$$

and Courant widely uses those principles in various existence and approximation questions of mathematical physics (see the survey [7]). The principles are easily extended to the abstract setting of symmetric bilinear forms in Hilbert spaces.

In 1930, Lusternik and Schnirel'mann [37, 38] extend this theory by replacing Q by an differentiable function f and the unit sphere by a finite dimensional compact differentiable manifold M. To replace the dimension of vector spaces, they introduce the concept of *category* $cat_X(A)$ of a closed set A in a topological space X, namely the least integer k such that A can be written as $\cup_{j=1}^k A_j$, with closed subsets A_j contractible in X. Lusternik and Schnirel'mann prove that the number of critical points of f on M is at least $cat_M(M)$, and that the corresponding values of f at the critical points (*critical values*) are given by

$$c_k = \inf_{A \in A_k} \sup_{u \in A} f(u),$$

where $A_k = \{A \subset M : A \text{ closed}, \mathrm{cat}_M(A) \geq k\}$ for $k = 1, 2, \ldots$.

Of course one has to check that A_k is non empty, which requires topological considerations. In particular they prove that if $F : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and even, then the system

$$F'(u) = \lambda u$$

has at least n pairs of solutions $[(\lambda, u), (\lambda, -u)]$ with $\|u\| = 1$. A version of this result is given in 1939 by Lusternik when $\mathbb{R}^n$ is replaced by a real infinite-dimensional separable Hilbert space, F' is compact and satisfies some other conditions.

In the fifties and the sixties, further extensions of Lusternik-Schnirel'mann theory to some infinite-dimensional problems are given by Krasnosel'skii, J.T. Schwartz and Palais (see references in [63]). F. Browder [3] refines and extends them to study nonlinear spectral problems in a Hilbert or a suitable reflexive Banach space X, which are of the form

$$F'(u) = \lambda G'(u),$$

where $F, G : X \to \mathbb{R}$ are suitable sufficiently smooth even nonlinear functionals. He finds conditions upon F and G which insure the existence of infinitely many critical levels.

The special case of $X = W_0^{1,p}(\Omega)$, $p > 1$, Ω a bounded domain of $\mathbb{R}^N$, $F(u) = \int_\Omega |\nabla u(x)|^p\, dx$ and $G(u) = \int_\Omega |u(x)|^p\, dx$ leads to the eigenvalue problem for the so-called *p-Laplacian operator* Δ_p, defined by

$$\Delta_p u(x) := \mathrm{div}\left(|\nabla u(x)|^{p-2} \nabla u(x)\right),$$

with the Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega.$$

An *eigenvalue for* $-\Delta_p$ with the Dirichlet boundary conditions is a λ such that the problem

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a nontrivial solution.

The Lusternik-Schnirel'mann technique implies the existence of a sequence of eigenvalues given by a minimax characterization.

When $N = 1$, it follows from direct computations that this sequence constitutes the whole spectrum, but the problem remains open for $N \geq 2$. For the corresponding *ordinary vector p-Laplacian* $u \mapsto \left(\|u'\|^{p-2} u'\right)'$ where $u : [0, 1] \to \mathbb{R}^m$, $m \geq 2$, the spectrum is completely known in the case of Dirichlet conditions $u(0) = u(1) = 0$, but not in the case of periodic boundary condition $u(0) - u(1) = u'(0) - u'(1) = 0$.

The corresponding forced problem is always solvable (although not uniquely) when λ is not an eigenvalue, but solvability conditions at an eigenvalue remain almost *terra incognita.*

9 Spectra for asymmetric nonlinear operators: a possible tool for suspension bridges

The above extensions preserve the $\mathbb{Z}_2$-symmetry of the linear situation. Motivated by some asymmetric asymptotically linear boundary value problems, FUČIK and DANCER have independently introduced in 1976 the study of problems of the form

$$-\Delta u = \mu u^+ - \nu u^- \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $u^+ = \max(u, 0), u^- = \max(-u, 0)$. An *eigenvalue* of this problem is now a couple (μ, ν) of reals such that the above problem has a nontrivial solution, and the set of eigenvalues is usually called the *Fučik* or the *Dancer-Fučik spectrum* of the corresponding Dirichlet problem. Abstract extensions in suitable ordered Banach spaces exist as well. LAZER and MCKENNA have emphasized the possible interest of those problems in modeling suspension bridges and explaining their possible instability (see [33]).

In the ordinary differential case ($N = 1$ and $\Omega =]0, 1[$), the Fučik spectrum is completely determined and made of the family of hyperbolic type curves

$$\frac{m}{\sqrt{\mu}} + \frac{n}{\sqrt{\nu}} = \frac{1}{\pi}, \quad (m, n = 0, 1, 2, \ldots),$$

whose intersection with the diagonal reproduces of course the standard spectrum

$$\{k^2\pi^2 : k = 1, 2, \ldots\}.$$

Very little is know in contrast when $N \geq 2$, except some properties for the first non trivial curve, some information on the shape near the diagonal points (λ_k, λ_k) associated to the classical eigenvalues, and some generic results about the structure in curves of the spectrum.

Here again the solvability of the forced problem is rather well understood when (μ, ν) is not in the Fučik spectrum, but much remains to be done in finding solvability conditions when (μ, ν) belongs to the spectrum.

Needless to say that the situation is still less developed in the study of the *Fučik spectrum of the p-Laplacian*

$$-\Delta_p u = \mu |u|^{p-2} u^+ - \nu |u|^{p-2} u^- = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

References can be found in the monographs [14, 12].

10 CONCLUSION

Many other modern aspects of spectral theory could have been discussed here, like bifurcation theory, Gelfand C^*-algebras, the spectrum of Riemannian manifolds, inverse spectral problems, perturbation theory or the relation between the spectrum of Schrödinger equations and the solution of some nonlinear partial differential equations. This would have taken the lecture beyong its time schedule, and the author beyond his abilities.

I hope that the story above has revealed the immensively creative power of unplanified research, as well as its unavoidable tortuous development. According to the Chinese tradition, only devils follow straight lines.

The conclusion will be left, like the Introduction, to some quotations, one from the middle, and one from the end of this century. They may convey some changes in mentalities in the fifty years period. The first one is due to R. GODEMENT [16]:

> We believe that the human mind is a "meteor" in the same way as the rainbow – a natural phenomenon; and that Hilbert realizing the "spectral decomposition" of linear operators, Perrin analyzing the blue color of the sky, Monet, Debussy and Proust recreating, for our wonder, the scintillation of the light on the see, all worked for the same aim, which will also be that of the future: the knowledge of the whole Universe.

The second one is due to M. ZWORSKI [64]:

> Eigenvalues of self-adjoint operators describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life [...]. In most situation however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate. [...] Eigenvalues are yet another expression of humanity's narcissic desire for immortality.

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Coming Events

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Pedro Nunes Lectures — 2011

Open questions leading to a global perspective dynamics by Jacob Palis

Pedro Nunes Lectures is an initiative of Centro Internacional de Matemática (CIM) in cooperation with Sociedade Portuguesa de Matemática (SPM), with the support of the Fundação Calouste Gulbenkian, to promote visits of notable mathematicians to Portugal. Each visitor is invited to give two or three lectures in Portuguese Universities on the recent developments in mathematics, their applications and cultural impact. Pedro Nunes Lectures are aimed to a vast audience, with wide mathematical interests, especially PhD students and youth researchers.

Jacob Palis (IMPA)
February 23, 2011 (15:00) Universidade do Porto.
March 02, 2011 (16:30) Universidade de Lisboa.

Open questions leading to a global perspective in dynamics

Abstract.—We will address one of the most challenging and central problems in dynamical systems, meaning flows, diffeomophisms or, more generally, transformations, defined on a closed manifold (compact, without boundary or an interval on the real line): can we describe the behavior in the long run of typical trajectories for typical systems? Poincaré was probably the first to point in this direction and s tress its importance. We shall consider finite-dimensional parameterized families of dynamics and typical will be taken in terms of Lebesgue probability both in parameter and phase spaces. We will discuss a conjecture stating that for a typical dynamical system, almost all trajectories have only finitely many choices, of (transitive) attractors, where to accumulate upon in the future. Interrelated conjectures will also be discussed.