REPORT

On the Madeira Klein Conference

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The programme of the conference "Didactics of Mathematics as a Mathematical Discipline", that was held in Funchal last October 1-4, 2009, can be found in http://glocos.org/index.php/dm-md/. *This conference was the first one associated with the IMU/ICMI Klein project,* http://www.mathunion.org/index.php?id=805, *The Klein Project, inspired by the Felix Klein's famous book* Elementarmathematik vom höheren Standpunkte aus, *published one century ago, is intended as a stimulus for mathematical teachers, so to help them to make connections between the mathematics they teach, or can be asked to teach, and the field of mathematics, while taking into account the evolution of this field over the last century. The project will have three outputs: a book simultaneously published in several languages, a resource DVD to assist teachers wishing to bring some of the ideas to realisation in their classes, and a wiki-based web-site seen as a vehicle for the many people who will wish to contribute to the project in an on-going way. This report to the Design Team of the Klein project will focus on themes and contributions that arose in discussion at the Madeira meeting, that were strongly debated, or that received some consensus.*

After an introductory session from the Rector of the University of Madeira, José Manuel Castanheira, the Presidents of ICMI and CIM, respectively Michèle Artigue and José Francisco Rodrigues, who also presented a brief overview on Felix Klein, and Bill Barton, as convenor of the Klein Project, the discussion raised two important points that need further consideration. The first was an organisational issue, the idea of local or regional "Writing Workshops". It was suggested, that when the project was progressed a little more, it would be possible to have writing workshops involving a group of people coming together to draft material for the Klein Project (either for the book or for webpages or resources). It would be understood that the product of these workshops would not necessarily be included, but would be submitted to the Design Team for consideration (and maybe further development). However, the Workshops would provide a wide opportunity for involvement, and, if they included both mathematicians and mathematics educators, would become part of the process of the project. I had not thought before about the process of the Klein Project having some developmental aims separate from the project itself. The second point raised was that of "problems sets" as either on organising idea for the book and/or a technique for writing. It was not clear exactly what constitutes a "problem set" (and more than one idea appeared to be present in the discussion), but there was general agreement about the usefulness of this idea.

Figure 1: *The opening session chaired by the Rector of the University of Madeira.*

Thomas Banchoff's talk on Midpoint Polygons using a geometric environment raised the general issue of the way technology has changed geometry itself as well as its pedagogy. As an aside, he reminded us of the power of counterexampleswith a nice example of a conjecture that appears to be true and is then (moving one corner of a pentagon left the area of the midpoint polygon constant, but moving another changed it). Gert

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Schubring's talk on Klein and his vision included the key ideas of historical shifting (that is the gradual elementarisation of mathematical topics over time) and the consequential hysteresis, a gap of more or less 30 years between the origin of a mathematical idea in its original complexity and the integration of the concept as an organic part of mathematics. Nevertheless, it was noted, new unprocessed (but suitably presented) mathematical ideas can motivate and inspire teachers. In discussion, it was noted that we must solve contemporary problems, which are different from those facing Klein although similarities exist, (but what are they exactly?). In particular, the Wiki-site frees the book from the tyranny of having to choose. Nevertheless, we should not shy away from the fact that the Klein project will be a filter of the essence of mathematics. The choice of examples/topics/problems which can convey this essence is not unique. So focus on the vision we wish to present for teachers. One suggested vision was: mathematics as a human construction in order to resolve classes of problems in a certain domain, where mathematicians pursue both the techniques for solution but also the underlying structure that makes those techniques work. Another point raised was that we know that doing mathematics supports its understanding. This applies for teachers as well as students. How can we incorporate this idea into the Klein Project?

Sebastian Xambó introduced us to Clifford's conception of geometric algebra, the link with physics and the writing of David Hestenes http://modelingnts.la.asu.edu/. Hestenes paper on the occasion of the Oersted Medal and Xambó's slide 18 of his presentation in particular. The session prompted discussion about the idea of "Chapters" in the book, the linking of ideas, and raised the idea of Case Studies (e.g. of elliptic curves, codes, complex numbers, algebraic topology, FLT, etc) rather than (as well as?) Chapters. It also highlighted the way mathematics is part of the frontier to human knowledge geometric algebra could be an example of a living research area exemplifying the culture of modern mathematics. Will we have a summary, somewhere, of the achievements of mathematics in the 20th century? Mário Dias Carneiro showed us more interconnections, topological ideas in differential equations (tent maps), and illustrated the importance of normal forms. He spoke more about the way research has changed with new technology. He introduced the idea of "The Better Book", that is a book that continues to evolve with new contributions as they mature being contained in new editions.

Ulrich Kortenkamp demonstrated the power of computing in many ways, both as a mathematical tool, and as a presenter of mathematical ideas, e.g. a lovely illustration of the many notions of angle, and another of the midpoint theorem showing how theorems arise from definitions but are not true in a universal or mystical sense. We need to utilise technology. See ¡madepedia.de¿. He spoke of criteria for the Wiki-site: citable, authors visible, interactive, and an editorial board. What else? Manuel Silva spoke of algorithmic thinking, giving examples of algorithmic proofs, including an induction example. He argued for Erdos' Probabilistic Method to be included. Discussion questioned how profoundly we can (or should) study algorithms, and then asked meta-questions about algorithms: how do we choose them, how do we critique them, how do we choose between them, how do we know if they are correct or not, etc? It was argued that programming is part of mathematics, or, rather, that formulating mathematics in a programming language is mathematics. Is a programming language a new language of mathematics? The SAGE Project (William Stein at Washington State) was mentioned.

Figure 2: *An aspect of the audience with B. Barton, B. Hogdson and T. Banchoff in the first row.*

Jaime Carvalho e Silva reminded us that Klein's was not the only vision of his era, alerting us to wonder what people will say about the Klein Project book 100 years from now. João Caramalho Domingues spoke about a proof of Cunha, which raised the issue of results no longer used showing us mathematical development. Discussion included the following formulation of the Klein project as saying to teachers "You are teaching elementary mathematics, but this is why what you are teaching is important". That is, it is neither exposition, curriculum, text, nor popularisation. National schools of thought were suggested as needing inclusion. Similarly for different approaches: the genetic (historic) approach, the intuitive approach, the experimental approach, the axiomatic (logico-deductive) approach, and the pedagogical approach. (Any others?). It was noted that "the work of logical analysis is to distinguish the acts of intuition and help successive abstractions and so proceed in the development of mathematical intuition into higher spaces".

Abraham Arcavi refocused our attention on school mathematics, and the students whom our target au-

dience will be teachingnot all are going to be mathematicians. Joao Pedro Ponte pursued this theme by focussing on the need for students to experience mathematical discovery and investigationthereby raising the issue of how those themes will be represented in the Klein Project book. James King brought us back to mathematical development with a discussion of affine geometry. In discussion the inverse phrase "advanced mathematics from an elementary viewpoint" was raised again is this what the Klein Project is about? (Hans Rademacher, Higher Mathematics from an Elementary Point of View, Birkhäuser, 1983). How far can you go mathematically without getting into formal mathematics or doing a transposition or didactically engineer an advanced topic? Another question asked was how the book can capture both the present state and what is still to be done. Part of the answer is to ensure mathematics is presented openly, open problems (in the sense of showing that solved problems lead to other problems, and in the sense of unsolved problems, or that problems can be expanded).

Figure 3: *The workshop was held at the University of Madeira sixteenth century building.*

Margarida Oliveira presented dynamical modelling and simulation using a geometry environment, and Elsa Fernandes presented the use of robots in the classroom. Together the presentations inspired reflection on the way that teachers are able to take new ideas and transform them in the classroomfreeing the Klein Project to the task of presenting interesting ideas, not directing classroom practice. Frank Quinn ranged over some historical developments in the methodology of mathematics, and thereby raised the questions of who defines "significant" change, when does change in mathematics imply change in classrooms, and the deep nature of the discontinuity between schools and research mathematics. It also raised again the issue of the diversity of mathematicians philosophies, ways of working, and approaches to mathematics.Yuriko Baldin emphasised the importance of the concept of manifold and transformation groups, and linear algebra as a basic tool. She suggested that topology is directed towards global results in geometry, analysis towards local properties. She referred us to a television documentary on the Poincaré conjecture. Emanuel Martinho, Maria Margarida Pinto and Virgínia Amaral reminded us of the difficulties of writing a text, thereby pointing out some problems we can avoid. Arsélio Martins spoke of some negative influences of technology, giving an example where dynamic geometry can lead to important mathematical thinking being avoided. He noted the importance of examples that "look right, but are not". He urged the Klein Project to present problems to teachers so that they are not problems to be solved, but are rather situations by which to develop further mathematics. Discussion on counterexamples mentioned Falsehoods in Mathematics by Maxwell; Ed Barbeau, Mathematical Fallacies, Flaws and Flimflam, MAA, 2000; and Counterexamples by Dudley. Another reference is Proofs from the Book by Erdös, where we will find proofs that are not key results as much as paradigmatic of proofs. There needs to be something about proof and how they help us understand. The idea of showing proofs where the "obvious" way was not the right one was put forward. Another related idea mentioned later was that it is also important to prove that some things cannot be true. Another phrase that caught attention was "we can take students and teachers to the beach to see the openness of the seabut only the brave can sail to the edges of the horizon".

Luís Esteves used trigonometry to model a fun park, and Adelaide Carreira, Leila Ângelo, and Ana Valdez discussed topics in analysis and calculus. This led us to consider the way that software generates problemsdoes this happen in research? What are examples? Another possibly guiding idea to arise was that of a digest of books: think of the set of available books, and ask what is missing or what genre is missing. The Klein Project might also provide a guide to these books. The teachers who had presented were then asked what they would like. After some comments referring to curriculum/text issues, the following emerged: "I hope that the book might close the gap between secondary school and the intentions of university"; "I want it to broaden my horizons in an accessible way". I also need some simple examples to be able to answer my students when they ask me "what is this for? Both applications, careers and mathematics"; "Recent mathematicians results can be applied in schoolsbut we don't know how to do it. We know GPS has mathematics insidebut how?"; "I want to share the beauty of advances in mathematics. I teach linear algebra very well, but what is it good for? Where is this going?"; "I would like to see the main topics that can be foundations and then the links with the development of mathematics." The website/book For All Practical Purposes Comat, was mentioned as a resource that is updated, but note it is for students who are ending their mathematical study. The book needs

to explain WHY: why do we need to study algebraic fractions, factoring polynomials, rational functions up to the graphis it because we have been trained to do so, or is it fundamental?

José Carlos Santos argued that Group theory needs to be in the Klein Project introduced through group actions, in particular on geometrical objects. João Fernandes, speaking about mathematics in astronomy, gave criteria for examples to be used in the Project. Pedro Patrício reiterated the complexity (and importance) of crypto-coding. Discussion mentioned extending groups to crystals and noted that this is a nice example because it was not done by mathematician. Another astonishing application of group theory is the work of a Polish mathematician who almost broke ENIGMA code, but it got changed. He sent his discoveries to UK when he knew Poland was being invaded and that helped the English break the code later in the war. It was noted that a contemporary feature of mathematics is that it is digital, so key theorems include compressing information, and capacity limits for transmit information. Further discussion on cryptography mentioned the headaches presented to mathematicians by security; and asked how it was possible to make elliptic curves "elementary". It was suggested that the Klein Project will affect the curriculum whether it is intended or not. But that nudging the curriculum (in no particular direction, just asking questions of it) is a good outcome.

In the final sessions, Luis Sanchez and Michèle Artigue both discussed analysis, its foundations and how it might be presented. Dinis Pestana discussed statistics and the Central Limit Theorem, and John Mason presented five possibilities for the Klein Project:

- presenting contemporary mathematics to teachers (and others)
- presenting mathematics as the solving of problems
- as the explaining of phenomena
- ways to bridge school-university divide
- a unification of mathematics and its didactics.

Bernard Hodgson emphasised: the importance of integrating and using explicitly an historical vision; the role of visual proofs; and a chapter on logic, presenting some topics with a strong mathematical logic connection. Discussion noted that each of Mason's possibilities implies a different genre of the book (or resource). Also that we must represent a 20th century vision of how mathematics CAN be presented. Is there an opportunity for a fresh voice. We need to ask for whom examples are illustrative or inspiring, and what we expect people to do with them.

Figure 4: *Bill Barton in the summarizing session chaired by M. Artigue.*

In my summing up, and thanking the organisers, I emphasised the value of the conference, especially as the first conference and as the model it represented of productive discussions between mathematicians and mathematics educators. The Project is indebted to Centro Internacional de Matemática (CIM) and its director for organising this first Klein conference. Appreciation also to the Centres of Mathematics at the University of Coimbra, and at the Universidade do Minho, Centro de Matemática e Aplicações Fundamentais at the University of Lisbon, and the University of Madeira for their support, particularly the latter who provided a magnificent venue. We give thanks to the Programme Committee of José Francisco Rodrigues (Pres. CIM), Elfrida Ralha (Univ. Minho), Jaime Carvalho e Silva (Univ. Coimbra), Suzana Nápoles (Univ. Lisboa), Pedro Patrício (Univ. Minho) and the Local Organising Committee of José Manuel Castanheira (Univ. Madeira), Elsa Fernandes (Univ. Madeira), Sandra Mendonça (Univ. Madeira). There is no doubt that this was an extremely successful conference thanks to their efforts.