Feature Article

Hydrodynamic limit of particle systems

Patrícia Gonçalves

Centro de Matemática da Universidade do Minho, Campus de Gualtar
4710-057 Braga, Portugal
patg@math.uminho.pt

Abstract

In these notes I present a classical particle system, namely exclusion type processes evolving on the one-dimensional lattice. I consider both settings, the nearest neighbor jumps of symmetric and asymmetric rates and long-jumps. For nearest neighbor jumps, for the symmetric and asymmetric case, one obtains the particle density evolving according to the heat equation and the inviscid Burgers equation, respectively, while for the long-jumps one gets to the fractional heat equation. From this global behavior I present a simple argument to obtain the limit distribution for the position of a tagged particle from the joint distribution of the empirical density of particles plus the current through the origin.

1 Introduction

The study of Interacting Particle Systems goes back to the late seventies and were introduce by Spitzer. The goal was to understand the macroscopic temporal evolution of physical systems through the underlying microscopic dynamics, ie the dynamics between the molecules constituting the physical system. The scenario is the following, first one supposes to have two scales for space and time and to have, for example, a fluid or a gas evolving in a certain volume. The idea is to split this volume into a certain number of cells and in each one of these cells one can have a random number of molecules that move according to a fixed rule, a probability transition rate. For details on the formal definition of Interacting Particle Systems, I refer the reader to [13].

The microscopic behavior of a physical system is very hard to obtain in a reasonable way, since the number of molecules is huge, typically of the Avogadro's number and in order to have some meaningfully description some simplifications have to be assumed. In this theory it is supposed to have a stochastic motion of molecules instead of a deterministic one, and with this assumption a probabilistic analysis of the system can be performed. The underlying motion relies on having each molecule waiting a random time and each one of them performing a random walk subjected to local restrictions. So, **Interacting Particle Systems** consist in a random motion of a collection of particles, each one waiting an exponential random time after which it moves from one cell to another, according to a probability transition rate. Probabilistic speaking, since the random times are variables with exponential law, these processes belong to the class of Markov processes and since the microscopic space is discrete these processes have compact space state.

In the **Hydrodynamic Limit** theory one is interested in deducing the macroscopic hydrodynamic equation that governs the temporal evolution of some physical quantity of interest, see [12] and [17]. So, for processes in which the microscopic dynamics conserves a macroscopic thermodynamic quantity, as for example, the density or energy, I will deduce the partial differential equation that governs the temporal evolution of this quantity of interest, through the random motion between particles. This partial differential equation is known as the **hydrodynamic equation** of the particle system.

In these review I will concentrate on particle systems which are of **exclusion type** and **ergodic**. The exclusion type means that, at each cell one has at most one particle per site, but nevertheless one could also consider more general Interacting Particle Systems in which one can have any number of particles per site, called Zero-Range processes, see [11], [12] and references therein, but in order to keep the presentation simple and capture the main ideas I will consider only exclusion type models of ergodic kind. The ergodicity property means that one can split the state space of the process into a disjoint set of invariant pieces, namely the hyperplanes with a fixed number of particles, but each one of them being a unique ergodic piece in the sense that fixing any two configurations on the same hyperplane it is possible to get from one to the other by allowed jumps of the dynamics. Anyway, it is also possible to consider more general particle systems in which the invariant pieces split into the ergodic component plus an isolated set of blocked configurations - these systems belong to the class of kinetically constrained lattice gases, which are of very well interest in the physics community since they model, for example, the liquid/glass transition. In these models particles can only move from on cell to the other if there is a certain number of particles in the cells at their vicinity, otherwise they are blocked [6]. So, for these systems there is a phase transition, since for higher densities of particles each hyperplane is a unique erdogic piece, but below a critical density each hyperplane splits into the irreducible component plus a number of blocked configurations. In [6] it was first proved the hydrodynamic limit for a non ergodic particle system of **gradient type**. The result for non-gradient systems is still open as well as for the case of non-erdogic gradient systems for which below the critical density each hyperplane splits into the irreducible component, plus isolated blocked configurations, plus a set in which there is a mixture of the behavior in each one of this sets, lets say to have a path of possible moving configurations that get blocked after a certain number of jumps. For details on the universe of this kind of constrained models, we refer the reader to [6], [17] and references therein.

After having the hydrodynamic result, one has the knowledge about the global macroscopic behavior of the system. Now, one can focus the attention on a single **tagged particle** and analyze its motion. Since each particle performs a random walk, it is known that if instead having a system with an arbitrary number of particles, we consider it to have just one, then the limit distribution of the position of this tagged particle is given by a **Brownian motion** or a **Levy process**, depending on the properties of the probability transition rate. But what about the limit distribution for one fixed or tagged particle when the system is in the presence of more than one particle? It is known since the work of Robert Brown that for a transition rate with finite second moment, the movement of a single particle in a random medium is given by a Brownian motion. In this setting the motion of a single particle is influenced by the position of the other particles in the system, but can one get to the Brownian motion in the limit as well, or do other processes come along as the Levy processes? Here I am going to present, an argument which allows to deduce the limit distribution for the position of a tagged particle, through the global analysis of the system in the equilibrium setting. Recently there have been obtained partial results about the limit distribution for the position of a tagged particle when the system is out of equilibrium, for details see for example [8], [9], [10] and references therein. When we are restricted to one-dimensional systems with jumps to neighboring sites, the initial order of particles is preserved and the main idea is to relate the position of the tagged particle with the current of particles through a fixed bond together with the empirical density of particles in a certain box. From this relation the limit distribution for the position of a tagged particle is an easy consequence of the joint Central Limit Theorem (C.L.T.) for the current and the empirical measure. In these notes I will define different dynamics for particle systems, that lead us to very different hydrodynamic equations and for which the limit distribution for the position of a tagged particle, one will get to the Brownian motion, to the Fractional Brownian motion and to a Levy process.

Here follows an outline of this review. On the second section I introduce the particle systems by means of their generators and describe its invariant measures. On the third section the empirical measure is introduced and I present an heuristic argument to get to the hydrodynamic equation from the explicit definition of the microscopic dynamics. On the fourth section is stated the hydrodynamic limit result for the processes considered here. On the fifth section I present the C.L.T. for the empirical measure and in the following section the current through the origin is defined and its C.L.T. is stated. On the last section I present the C.L.T. for the position of a tagged particle.

2 The Particle Systems

Here I consider the most classical example of an Interacting Particle System: the **Exclusion process**. In order to capture the fundamental ideas behind the hydrodynamic limit theory I will restrict this exposition to one-dimensional particle systems evolving on $\mathbb{Z}$. For more general processes and larger state space we refer the reader to [13].

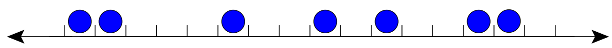

Figure 1: one possible configuration of the Exclusion process

At first one fixes a probability $p(\cdot)$ on $\mathbb{Z}$ and each particle, independently from the others, waits a mean one exponential time, at the end of which being at the site x it jumps to $x + y$ at rate $p(y)$.

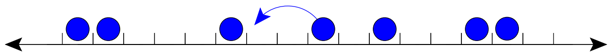

Figure 2: a jump possible jump occurring at rate $p(-2)$

For this process there can be at most one particle per site, and if a clock rings and a particle attempts to jump to an occupied site, the jump is suppressed in order to respect the exclusion rule and after that the clock restarts.

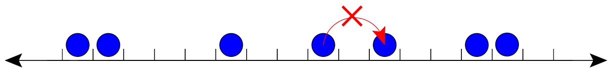

Figure 3: a forbidden jump

The space state for this Markov process is $\Omega = \{0,1\}^{\mathbb{Z}}$ in such a way that a 1 means to have the site occupied while the 0 denotes that it is empty. Now, we give the precise definition of the process by means of its generator.

The Exclusion process is denoted as the Markov process $\eta_t \in \Omega$ with generator given on local functions $f : \Omega \to \mathbb{R}$ by

$$\mathscr{L} f(\eta) = \sum_{x,y\in\mathbb{Z}} \eta(x)(1-\eta(x+y))p(y)[f(\eta^{x,x+y}) - f(\eta)],$$

where

$$\eta^{x,x+y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, x+y \\ \eta(x+y), & \text{if } z = x \\ \eta(x), & \text{if } z = x+y \end{cases}.$$

I recall here that a core for the operator $\mathscr{L}$ is the set of local functions, ie functions defined on the state space of the process that depend on the configuration η only through a finite number of coordinates $\eta(x)$, see [13] for a proof of this result. So, in this process configurations are denoted by η, so that $\eta(x) = 0$ if the site x is vacant and $\eta(x) = 1$ otherwise, as mentioned above. In these notes I consider three different kind of dynamics:

- Symmetric Simple Exclusion Process (ssep), for which the probability transition rate is given by $p(1) = p(-1) = 1/2$, ie jumps occur to neighboring sites at the same rate.
- Asymmetric Simple Exclusion Process (asep), for which the probability transition rate is given by $p(1) = 1 - p(-1) = p > \frac{1}{2}$, ie jumps also occur to neighboring sites but with a drift to the right.
- Long-jump Exclusion Process, for which the probability transition rate satisfies

$$p(x,y) = |y-x|^{-(1+\alpha)}, \alpha \in (0,2),$$

ie jumps occur from any site x to y, but the further the distance the smaller the probability of jumping.

Following the Boltzmann ideas from Statistical mechanics the first step to do when one analyzes the temporal evolution of a macroscopic thermodynamical quantity of a physical system, is to obtain the knowledge of its **invariant states**. For particle systems, the invariant states are translated as invariant measures of the system. So in this setting, μ is an invariant measure of the system, if starting the process from μ, ie if the distribution of η_0 is μ, then for any time t, the distribution of the system at time t, ie the distribution of η_t is again given by μ - this means that the trajectory of the measure distributions is constant in time and equal to μ.

Now we describe a set of **invariant measures** for the processes considered above. Fix $0 \leq \rho \leq 1$ and denote by ν_ρ the Bernoulli product measure on Ω with density ρ, ie its marginal at the site x is given by:

$$\nu_\rho(\eta : \eta(x) = 1) = \rho.$$

So, for any site x, $\eta(x)$ has Bernoulli distribution of parameter ρ and since ν_ρ is a product measure $(\eta(x))_{x\in\mathbb{Z}}$ are independent random variables. It is known that $(\nu_\rho)_\rho$ with $\rho \in [0,1]$ is a family of invariant measures for the exclusion process. I note here that this family is homogeneous (since the marginal at the site x does not depend on x) and translation invariant (since it is invariant by the shift application).

3 Hydrodynamic equation

In this section I deduce the hydrodynamic equation for two of the processes described above by means of the random microscopic dynamics. For simplicity I present here the computations for the Simple Exclusion Process: symmetric and asymmetric rates. For details on the hydrodynamic limit for the Long-jump process we refer the reader to [7], but the hydrodynamic equation for this process is the fractional heat equation $\partial_t \rho(t,u) = -(-\Delta)^{\alpha/2}$, where $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian.

Now, I introduce the **empirical measure** associated to the Markov process $\eta_\cdot$. For each configuration η, denote by $\pi^N(\eta, du)$ the measure given by

$$\pi^N(\eta, du) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta(x)\delta_{\frac{x}{N}}(du)$$

and define the process of empirical measures by $\pi^N_t(\eta, du) = \pi^N(\eta_t, du)$. Here δ_u is the Dirac measure at u.

In the sequence I present an heuristic argument to obtain the conservation law that describes the temporal evolution of the density of particles. Any of the dynamics introduced above, does not create or destroy particles, it simply move particles according to some pre-determined rule and as a consequence the number of particles is a conserved quantity. So, the **density of particles** is the **thermodynamical quantity** of interest for these processes.

From the classical theory of Markov processes it is known that for a test function H

$$M_t^{N,H} =< \pi_t^N, H > - < \pi_0^N, H > - \int_0^t \mathscr{L} < \pi_s^N, H > ds$$

is a martingale with respect to the natural filtration. Note that $< \pi_t^N, H >$ denotes the integral of H with respect to the measure π_t^N. For a particle system with generator $\mathscr{L}$ and whose dynamics conserves the number of particles $\mathscr{L}(\eta(x)) = W_{x-1,x}(\eta) - W_{x,x+1}(\eta)$, where for a site x and a configuration η, $W_{x,x+1}(\eta)$ denotes the instantaneous current between the sites x and $x+1$, namely it is the difference between the jump rate from x to $x+1$ and the jump rate from $x+1$ to x. This **gradient property** allows us to perform a summation by parts and write down the martingale as

$$\begin{aligned} M_t^{N,H} &= < \pi_t^N, H > - < \pi_0^N, H > \\ &- \int_0^t \frac{1}{N^2} \sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big) W_{x,x+1}(\eta_s) ds, \end{aligned} \tag{3.1}$$

where $\nabla^N H$ denotes the discrete derivative of H.

From this point on, we split the argument depending on the behavior of the expectation of the current. For the asep, the instantaneous current between the sites x and $x+1$ is given by $W_{x,x+1}(\eta) = p\eta(x)(1-\eta(x+1)) - q\eta(x+1)(1-\eta(x))$ and its expectation with respect to the invariant measure ν_ρ equals to $(p-q)\rho(1-\rho)$ which is non-zero for $\rho \in (0,1)$. For the ssep, the instantaneous current between the sites x and $x+1$ is given by $W_{x,x+1}(\eta) = \frac{1}{2}\Big(\eta(x) - \eta(x+1)\Big)$ and its expectation with respect to the invariant measure ν_ρ vanishes. This property of the expectation of the instantaneous current is crucial to the following conclusions.

We proceed by closing the integral part of the martingale as a function of the empirical measure. Since, for the asep, the expectation of the current does not vanish, by re-scaling time by tN and performing a change of variables, one gets to

$$\begin{aligned} M_{tN}^{N,H} &= \frac{1}{N}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\Big(\eta_{tN}(x) - \eta_0(x)\Big) - \\ &\int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big) W_{x,x+1}(\eta_{sN}) ds. \end{aligned}$$

Now we introduce the notion of **conservation of local equilibrium**. Physical reasoning suggests that due to the huge number of particles, physical systems may not present a global equilibrium picture, but microscopically by the interaction among particles its reasonable to assume that for a macroscopic time t the system is **locally in equilibrium**. This means, loosely speaking, that the expectation of η_{tN}, is close to the expectation of $\eta(0)$ with respect to the equilibrium measure of the system, but with parameter predicted by the hydrodynamic equation:

$$E(\eta_{tN}(x)) \sim E_{\nu_{\rho(t,x/N)}}[\eta(0)] = \rho(t, x/N).$$

Applying expectation with respect to the distribution of the system at the microscopic time tN to the martingale above and since this martingale vanishes at time 0, it holds that

$$\begin{aligned} &\frac{1}{N}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\Big(\rho(t,x/N) - \rho(0,x/N)\Big) = \\ &\int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big)\tilde{W}(\rho(s,x/N))ds \end{aligned}$$

Taking the limit as $N \to +\infty$, $\rho(t,u)$ is identified as a **weak solution** of the hyperbolic conservation law:

$$\begin{cases} \partial_t \rho(t,u) + \nabla \tilde{W}(\rho(t,u)) = 0 \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} \tag{3.2}$$

where $\tilde{W}(\rho) = E_{\nu_\rho}[W_{0,1}] = (p-q)\rho(1-\rho)$. This partial differential equation is known as the **inviscid Burgers equation.**

Now we analyze the case for the ssep. Since for this process $W_{x,x+1}(\eta) = \frac{1}{2}\Big(\eta(x) - \eta(x+1)\Big)$, this allows us to perform a double summation by parts in the integral part of the martingale and write (3.1) as:

$$\begin{aligned} M_t^{N,H} &= < \pi_t^N, H > - < \pi_0^N, H > - \\ &\int_0^t \frac{1}{N^2}\sum_{x\in\mathbb{Z}} \Delta^N H\Big(\frac{x}{N}\Big)\frac{1}{2}\eta_s(x) ds. \end{aligned}$$

We note here that particle systems for which the instantaneous current can be written as a function minus its translation are called **gradient systems**. The ssep is an example of a gradient system since $W_{x,x+1}(\eta) = \frac{1}{2}\eta(x) - \frac{1}{2}\eta(x+1)$. Now, following the same argument as above, by re-scaling time by tN^2, performing a change of variables and by the local equilibrium assumption, when taking the limit as $N \to +\infty$, $\rho(t,u)$ is a **weak solution** of the well known **heat equation**:

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2}\Delta \rho(t,u) \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases} \tag{3.3}$$

4 Law of Large Numbers for the empirical measure

In order to introduce the notion of hydrodynamic limit we have to fix some notation. Let $\rho_0 : \mathbb{R} \to [0,1]$ be an initial profile and denote by $(\mu_N)_{N\geq 1}$ a sequence of probability measures defined on Ω. Assume that a time 0, the system starts from a initial measure μ_N that is associated to the initial profile ρ_0:

Definition 1. *A sequence $(\mu^N)_{N\geq 1}$ **is associated to** ρ_0, if for every continuous function $H:\mathbb{R}\to\mathbb{R}$ and for every $\delta>0$*

$$\lim_{N\to+\infty}\mu^N\Big[\eta:\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}H\Big(\frac{x}{N}\Big)\eta(x)-\int_{\mathbb{R}}H(u)\rho_0(u)du\Big|>\delta\Big]=0.$$

Note that the term on the left hand side of the expression above, corresponds to the integral of H with respect to π_0^N. Thus the above definition corresponds to asking that empirical measure at time 0 satisfy a law of large numbers, namely that the sequence $\pi^N(\eta,du)$ converges in μ_N-probability to $\rho_0(u)du$.

The goal in hydrodynamic limit consists in showing that, if at time $t=0$ the empirical measures are associated to some initial profile ρ_0, then at the macroscopic time t they are associated to a profile ρ_t which is the solution (in some topology) of the corresponding hydrodynamic equation. In other words, the aim is to prove that the random measures $\pi_\cdot^N$ converge in probability to the deterministic measure $\rho(t,u)du$, which is absolutely continuous with respect to the Lebesgue measure and whose density evolves according to the hydrodynamic equation.

Since the work of Rezakhanlou in [16], it is known that for the asep starting from a sequence of measures $(\mu_N)_N$ associated to a profile $\rho_0(\cdot)$ and some additional hypotheses (see [16] for details) under the **hyperbolic time scale** tN

$$\pi_{tN}^N\xrightarrow[N\to+\infty]{}\rho(t,u)du,$$

in $\mu^N S_N(t)$-probability, where $\rho(t,u)$ is the entropy solution of (3.2) and $S_N(t)$ is the semigroup associated to the generator of the asep.

For the ssep, the hydrodynamic limit can be derived by the entropy method since the local equilibrium convergence holds for this process, see [12] for details. Under the **parabolic time scale** tN^2, it holds that

$$\pi_{tN^2}^N\xrightarrow[N\to+\infty]{}\rho(t,u)du,$$

in $\mu^N S_N(t)$-probability, where $\rho(t,u)$ is the weak solution of (3.3) and $S_N(t)$ is the semigroup associated to the generator of the ssep.

Probabilistic speaking, hydrodynamic limit is a **Law of Large Numbers** for the empirical measure associated to a Markov process starting from a general set of initial measures. A natural question that follows has to do with the fluctuations of this measure around the equilibrium state. "Does a Central Limit Theorem holds?" and "How is the behavior of the limit process?" In order to analyze the C.L.T. for the empirical measure, we consider the simplest case in which the process is equilibrium, ie the initial measure is ν_ρ. This will be developed in the next section. The scenario out of equilibrium is much harder to obtain and few cases are know, here we leave this issue out of discussion.

5 Central Limit Theorem for the empirical measure

In this section we state the C.L.T. for the empirical measure for the simple exclusion process: ssep and asep. Let $\mathscr{S}(\mathbb{R})$ denote the Schwartz space of test functions. Fix ρ and an integer k. Denote by $\mathscr{Y}_\cdot^N$ the density fluctuation field, ie a linear functional acting on functions $H\in\mathscr{S}(\mathbb{R})$ as

$$\begin{aligned}\mathscr{Y}_t^N(H) &= \sqrt{N}\Big[<\pi_t^N,H>-\mathbb{E}_{\nu_\rho}<\pi_t^N,H>\Big]\\ &= \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}}H\Big(\frac{x}{N}\Big)(\eta_t(x)-\rho).\end{aligned}$$

For an integer $k\geq 0$, let $\mathscr{H}_k$ be the Hilbert space induced by $\mathscr{S}(\mathbb{R})$ and $<f,g>_k=<f,K_0^k g>$, where $<\cdot,\cdot>$ denotes the inner product of $L^2(\mathbb{R})$, $K_0=x^2-\Delta$ and denote by $\mathscr{H}_{-k}$ its dual. Let $D(\mathbb{R}^+,\mathscr{H}_{-k})$ (resp. $C(\mathbb{R}^+,\mathscr{H}_{-k})$) be the space of H_{-k}-valued functions, right continuous with left limits (resp. continuous), with the uniform weak topology, by Q_N the probability measure on $D(\mathbb{R}^+,\mathscr{H}_{-k})$ induced by $\mathscr{Y}_\cdot^N$ and ν_ρ.

Theorem 5.1 (G. [4])**.** *Fix an integer $k>2$. Let $\eta_\cdot$ be the asep evolving on the time scale tN, starting from the invariant measure ν_ρ and Q_N be the probability measure on $D(\mathbb{R}^+,\mathscr{H}_{-k})$ induced by $\mathscr{Y}_\cdot^N$ and ν_ρ. Denote by Q the probability measure on $C(\mathbb{R}^+,\mathscr{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$E_Q[\mathscr{Y}_t(H)\mathscr{Y}_s(G)]=$$
$$\rho(1-\rho)\int_{\mathbb{R}}H(u+(p-q)(1-2\rho)(t-s))G(u)du$$

for every $0\leq s\leq t$ and H, G in $\mathscr{H}_k$. Then, $(Q_N)_N$ converges weakly to Q.

For the asep the limit density field satisfies

$$\mathscr{Y}_t(H)=\mathscr{Y}_0(H)-\int_0^t\mathscr{Y}_s((p-q)(1-2\rho)\nabla H)ds,$$

ie $\mathscr{Y}_t$ satisfies:

$$d\mathscr{Y}_t=(p-q)(1-2\rho)\nabla\mathscr{Y}_t dt.$$

In this case we obtain a simple expression for $\mathscr{Y}_t$ given by $\mathscr{Y}_t(H)=\mathscr{Y}_0(T_tH)$ with $T_tH(u)=H(u+(p-q)(1-2\rho)t)$, which is the semigroup associated to $(p-q)(1-2\rho)\nabla$. Restricted to $\mathscr{F}_0$ (the σ-algebra on $D([0,T],\mathscr{H}_{-k})$ generated by $\mathscr{Y}_0(H)$ and H in $S(\mathbb{R})$) Q is a Gaussian field with covariance given by $E_Q(\mathscr{Y}_0(G)\mathscr{Y}_0(H))=\rho(1-\rho)<G,H>$. In this case the limit density field at time t is just a translation (by the characteristics velocity) of the initial limit density field, so the systems does only depends on the initial configuration of the system.

For the ssep a more sophisticated process appears in the limit, it is known as an Onrstein-Uhlenbeck process and the temporal evolution depends highly on the initial configuration of the system but also on the randomness of the dynamics.

Theorem 5.2 (Ravishankar [15]). *Fix an integer $k > 3$. Let $\eta_\cdot$ be the ssep evolving on the time scale tN^2, starting from the invariant measure ν_ρ and Q_N be the probability measure on $D(\mathbb{R}^+, \mathscr{H}_{-k})$ induced by $\mathscr{Y}^N_\cdot$ and ν_ρ. Denote by Q be the probability measure on $C(\mathbb{R}^+, \mathscr{H}_{-k})$ corresponding to a stationary mean zero generalized Ornstein-Uhlenbeck process with characteristics $\mathfrak{A} = 1/2\Delta$ and $\mathfrak{B} = \sqrt{\chi(\rho)}$. Then $(Q_N)_N$ converges weakly to Q.*

For the ssep the limit density field satisfies

$$d\mathscr{Y}_t = \frac{1}{2}\Delta\mathscr{Y}_t dt + \sqrt{\chi(\rho)}\nabla dB_t$$

where B_t is a Brownian motion.

For the C.L.T. for the long-jump process I refer the reader to [7].

6 Current fluctuations

In this section I introduce the notion of flux or current of particles through the origin. Let $J_{-1,0}(t)$ be the number of particles that jump from the site -1 to 0 minus the number of particles that jump from the site 0 to -1 during the time interval $[0,t]$. Since

$$J_{-1,0}(t) = \sum_{x\geq 0}\Big(\eta_t(x) - \eta_0(x)\Big),$$

the current can be written in terms of the density fluctuation field as

$$\frac{1}{\sqrt{N}}\Big\{J_{-1,0}(t) - \mathbb{E}_{\nu_\rho}[J_{-1,0}(t)]\Big\} = \mathscr{Y}^N_t(H_0) - \mathscr{Y}^N_0(H_0),$$

where H_0 is the Heaviside function, $H_0 = 1_{[0,+\infty)}$. Using this relation and the C.L.T. for the empirical measure, a C.L.T. for the current through the origin can be obtained.

Theorem 6.1 (G. [4]). *Let $\eta_\cdot$ be the asep evolving on the time scale tN and starting from ν_ρ. Then*

$$\frac{1}{\sqrt{N}}\Big(J_{-1,0}(tN) - \mathbb{E}_{\nu_\rho}(J_{-1,0}(tN))\Big) \xrightarrow[N\to+\infty]{} \sigma_a(J)\mathscr{B}_t$$

where $(\sigma_a(J))^2 = \rho(1-\rho)|(p-q)(1-2\rho)|$ and $\mathscr{B}_t$ is the standard Brownian motion.

Theorem 6.2 (Arratia [1], De Masi-Ferrari [2], Peligrad-Sethuraman [14]). *Let $\eta_\cdot$ be the ssep evolving on the time scale tN^2 and starting from ν_ρ. Then*

$$\frac{1}{\sqrt{N}}J_{-1,0}(tN^2) \xrightarrow[N\to+\infty]{} \sigma_s(J)\mathscr{W}^H_t$$

where $(\sigma_s(J))^2 = \sqrt{\frac{2}{\pi}}\rho(1-\rho)$ and $\mathscr{W}^H_t$ is the Fractional Brownian motion of Hurst parameter $H = 1/4$.

7 Tagged Particle

Now we want to prove the C.L.T. for a single tagged particle that we suppose to be initially at the origin:

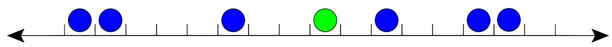

Figure 4: tagged particle at the origin

For that, let η be a configuration of Ω such that $\eta(0) = 1$. For the other sites x consider $\eta(x)$ distributed according to the invariant measure of the system, denoted by ν_ρ. This means that now we are starting the process from the measure ν_ρ conditioned on configurations with a particle at the origin:

$$\nu^*_\rho(\cdot) = \nu_\rho(\cdot|\eta(0) = 1).$$

This measure is no longer an invariant measure of the system, since the clock at the origin can ring and the particle initially at the origin can move to an empty site according to the transition probability rate $p(\cdot)$.

In order to keep track of the position of this particle, let X_t denote the position at time t of the tagged particle initially at the origin $(X(0) = 0)$.

Since in the one-dimensional setting and for nearest neighbor jumps the order of particles is preserved, this allows us to obtain a simple relation between the position of the tagged particle, the current through the origin and the empirical density of particles as:

$$\Big\{X(t) \geq n\Big\} = \Big\{J_{-1,0}(t) \geq \sum_{x=0}^{n-1}\eta_t(x)\Big\}$$

Theorem 7.1 (Ferrari-Fontes [3], G. [4]). *Let $\eta_\cdot$ be the asep evolving on the time scale tN, starting from ν^*_ρ and let $X(tN)$ denote the position at time tN of the particle initially at the origin. Then*

$$\frac{1}{\sqrt{N}}\Big(X(tN) - \mathbb{E}_{\nu^*_\rho}(X(tN))\Big) \xrightarrow[N\to+\infty]{} \sigma_a(X)\mathscr{B}_t$$

where $(\sigma_a(X))^2 = |p-q|(1-\rho)$ and $\mathscr{B}_t$ is the standard Brownian motion.

Theorem 7.2 (Arratia [1], De Masi-Ferrari [2], Peligrad-Sethuraman [14], G.-Jara [5]). *Let $\eta.$ be the ssep evolving on the time scale tN^2 and starting from ν_ρ^* and let $X(tN^2)$ denote the position at time tN^2 of the particle initially at the origin. Then*

$$\frac{1}{\sqrt{N}}X(tN^2) \xrightarrow[N\to+\infty]{} \sigma_s(X)\mathscr{W}_t^H,$$

where $(\sigma_s(X))^2 = \sqrt{\frac{2}{\pi}}\frac{1-\rho}{\rho}$ and $\mathscr{W}_t^H$ is the Fractional Brownian motion of Hurst parameter $H = 1/4$.

When one considers the long-jump process the relation above between the position of the tagged particle, the current and the density of particles does not hold since now particles can move to sites arbitrarily away from each other - so the order of particles is no longer preserved. Anyway for this process the C.L.T. for the tagged particle is achieved by considering the process seen from an observer sitting on the position of the tagged particle, namely $\xi_t(x) = \eta_t(x + X(t))$. For this new process the position of the tagged particle becomes the number of shifts of the system, which can be written as a martingale plus an additive functional of the Markov Process, for details see [8]. For a transition rate $p(\cdot)$ homogeneous and regular of degree α, ie such that there exists a function $q : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ of class C^2 such that $p(x) = q(x)$ for any $x \in \mathbb{Z}\backslash\{0\}$ and such that $q(\lambda u) = \lambda^{1+\alpha}q(u)$ for any $\lambda \neq 0$ and $u \in \mathbb{R}\backslash\{0\}$ it was proved in the equilibrium case that:

Theorem 7.3 (Jara [8]). *Let $\eta.$ be the exclusion process with homogeneous regular transition rate of degree α, evolving on the time scale tN^α, starting from ν_ρ^* and let $X(tN^\alpha)$ be the position at time tN^α of the particle initially at the origin. Then*

$$\frac{X_{tN^\alpha}}{N} \xrightarrow[N\to+\infty]{} (1-\rho)\mathscr{Z}_t,$$

where $\mathscr{Z}_t$ is the Levy process whose characteristic function is given by $-\log E[\exp i\beta\mathscr{Z}_t] = t\psi(\beta)$ with

$$\psi(\beta) = \int_{\mathscr{R}}(1 - e^{i\beta u})q(u)du$$

and q as above.

In fact this theorem as stated holds for a general class of transition rates, which includes the jump-rate defined in the setting of this review, namely $p(x, y) = |y-x|^{-(1+\alpha)}$ with $\alpha \in (0, 2)$.

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