Math in the Media

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"Samurai mathematician".

Seki Takakazu (Seki Kowa). Image from Tensai no Eiko to Zasetsu by Masahiko Fujiwara (Shincho-sha, Tokyo, 2002), used with permission.

"Samurai Mathematician Set Japan Ablaze With Brief, Bright Light" is the title of a History of Science News-Focus piece in the October 10 2008 Science. Dennis Normile reports from Tokyo, site of a history of mathematics conference last August dedicated to the memory of Seki Takakazu. Seki (c. 1642-1708, also Seki Kowa) was in fact born into a samurai family; his portrait includes the the two swords (daisho) attesting his warrior status. That he "set Japan ablaze" may be an overstatement, but the "bright light" is appropriate: Seki (an almost exact contemporary of Leibniz, but working in complete isolation from European mathematics) "devised new notation for handling equations with several variables and developed solutions for equations with an unknown raised to the fifth power". Furthermore, according to Normile, "His most significant work focused on determinants ... , a field he pioneered a year or two ahead of ... Leibniz". The word "brief" is, unfortunately, also correct. Seki was ahead of his time, and soon after his death, even though his works were gathered and preserved by his students, "the Japanese mathematical tradition hit a dead end". Normile quotes the science historian Hikosaburo Komatsu: Seki's more erudite work "was too difficult for people to pick up and carry forward". Only now are his most innovative contributions being recognized. Namely, his "discovery around the year 1680 of a general theory of elimination, a method of solving simultaneous equations by whittling down the number of unknown quantities one by one". According to Komatsu, the work had been overlooked because it led to calculations "almost beyond human capabilities".

The diagonalization of physics. Cantor's diagonal argument occurs in his (second, 1891) proof of the uncountability of the real numbers. As Wikipedia tells us, "it demonstrates a powerful and general technique, which has since been reused many times in a wide range of proofs, also known as diagonal arguments ... The most famous examples are perhaps Russell's paradox, the first of Gödel's incompleteness theorems, and Turing's answer to the [Halting Problem]". Now this technique has been extended to the real world, or at least to our understanding of it. Philippe Binder reports in a Philosophy of Science *News & Views* piece in *Nature* (October 16, 2008) on work of the physicist/generalist David Wolpert published earlier this year (*Physica D* **237** 1257-1281). According to Binder, Wolpert has demonstrated "that the entire physical Universe cannot be fully understood by any single inference system that exists within it". How does he get there? In Binder's telling, Wolpert "introduces the idea of inference machines – physical devices that may or may not involve human input – that can measure data and perform computations, and that model how we come to understand and predict nature". These machines process U, "the space of all world-lines (sequences of events) in the Universe that are consistent with the laws of physics". Wolpert defines strong inference as "the ability of one machine to predict the total conclusion function of another machine for all possible set-ups". And then he uses diagonalization to prove:

- Let C_1 be any strong inference machine for U. There is another machine, C_2, that cannot be strongly inferred by C_1.
- No two strong inference machines can be strongly inferred from each other.

"The two statements together imply that, at best, there can be only a 'theory of almost everything'." Binder goes on to give some smaller-scale possible instantiations of the phenomenon.

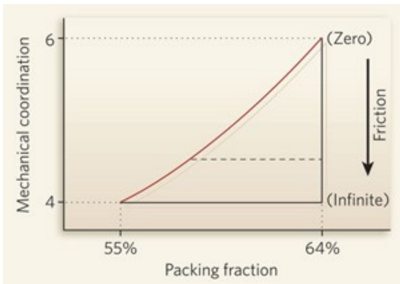
Solzhenitsyn mathematician. *Paris-Match* ran a photo essay on Alexander Solzhenitsyn (died August 3, 2008) in its August 7-13 issue. It included this picture of the author tutoring his children in mathematics.

Solzhenitsyn teaching his children the derivation of the quadratic formula.

Paris-Match's caption reads: "1982: pour ses trois fils il retrouve le tableau noir du professeur de maths qu'il a été". For more details we can turn to his 1970 Nobel Prize Autobiography.

- "I wanted to acquire a literary education, but in Rostov such an education that would suit my wishes was not to be obtained. ... I therefore began to study at the Department of Mathematics at Rostov University, where it proved that I had considerable aptitude for mathematics. But although I found it easy to learn this subject, I did not feel that I wished to devote my whole life to it. Nevertheless, it was to play a beneficial role in my destiny later on, and on at least two occasions, it rescued me from death. For I would probably not have survived the eight years in camps if I had not, as a mathematician, been transferred to a so-called sharashia, where I spent four years; and later, during my exile, I was allowed to teach mathematics and physics, which helped to ease my existence and made it possible for me to write...."
- "In 1941, a few days before the outbreak of the war, I graduated from the Department of Physics and Mathematics at Rostov University. At the beginning of the war, owing to weak health, I was detailed to serve as a driver of horsedrawn vehicles during the winter of 1941-1942. Later, because of my mathematical knowledge, I was transferred to an artillery school ..." [He is arrested in 1945 for having written "certain disrespectful remarks about Stalin" in letters to a friend, and sentenced to eight years in a detention camp.]
- "In 1946, as a mathematician, I was transferred to the group of scientific research institutes of the MVD-MOB (Ministry of Internal Affairs, Ministry of State Security). I spent the middle period of my sentence in such "SPECIAL PRISONS" (The First Circle)". [A month after serving out his sentence, he is exiled for life to Kok-Terek (southern Kazakhstan). "This measure was not directed specially against me, but was a very usual procedure at that time". Stalin dies in 1953 but Solzhenitsyn's exile lasts until June, 1956.]
- "During all the years of exile, I taught mathematics and physics in a primary school and during my hard and lonely existence I wrote prose in secret ..."

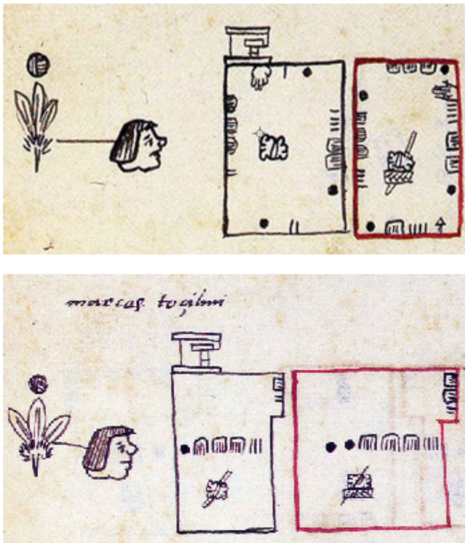
"Mathematics of the spheres". The most efficient way to pack equal-sized spheres in three dimensions involves placing them in layers along a hexagonal tiling of the plane and fitting adjacent layers together so that each sphere in one layer fits into the dimple determined by 3 adjacent spheres in the layer below. In these arrangements each sphere touches 12 others, and the average density, or *packing fraction* (volume of spheres)/(volume of ambient space) is approximately 0.74. Thomas Hayes' 1998 proof that these packings are in fact optimal (the Kepler Conjecture) is now generally accepted. But suppose spheres are dumped into a container without being carefully stacked. What density can such a *random packing* expect to achieve? Experimentally it ranges between 55% (*random loose packing*) and 64% (*random close packing*). Clearly friction will play a role, but how? In the May 29 2008 *Nature*, a CCNY-Fortaleza team (Chaoming Song, Ping Wang, Hernán A. Makse) use statistical mechanics to describe a phase space for packings, and to give an intelligible model for these questions.



The phase space for sphere packings. For each value of the friction coefficient the dashed line represents the possible packing fractions that can be realized by a stable sphere packing. The mechanical coordination number, the average number of adjacent spheres that contribute to holding a given sphere in place, varies monotonically with friction. This image is from Francesco Zamponi's "News and Views" analysis of the Song-Wang-Makse paper in the same issue: *Nature* **453** 606-607, and is used with permission.

"Mathematics of the spheres" is the way these items were characterized in the Nature "Editor's Summary".

Aztec area algorithms. The Aztec numbering system is pretty well understood: it is a base-20 place-value system with a symbol for zero. But some extra numerical symbols ("arrow", "hand", "heart", "bone") appear on land surveys, where they seem to represent quantities smaller than 1. An explanation for these mysterious symbols was recently (April 4, 2008) published in Science. Barbara Williams (Wisconsin-Rock County) and Maria del Carmen Jorge y Jorge (UNAM) exploited the data from the *Codex Vergara* (in the Bibliothèque Nationale) and the *Codice de Santa Maria Asuncion* (Fondo Reservado de la Biblioteca Nacional de Mexico, UNAM), where a number of plots are recorded with their side measurements *and their areas*. The areas are invariably whole numbers of the squared unit; by a process of trial and error they were able to reconstruct some of the algorithms the Aztecs had used to calculate the areas, and work backwards to figure out values for the unknown symbols.



Part of the holdings of a sixteenth-century Mexican landowner, shown with linear side measurements and with areas. Details of two pages from the *Codice de Santa Maria Asuncion* (Fondo Reservado de la Biblioteca Nacional de Mexico, UNAM). The sides of the plots are labeled with their lengths in Aztec numeration: Solid dot = 20, vertical line = 1, grouped by 5s. The areas are written in Aztec place-notation with the 20s place in the center and the 1s place in a tab on the upper right. Some of the notation for "fractional" length measurements appears in this chart: the arrow and the hand. The dimensions of the right-most plot are, clockwise from the top, 35, 34+hand, 29+arrow, 39; its area is given as $59 \times 20 + 12$ or 1192 square units. In the left-most plot the 17 is presumably a copying error for 37; otherwise area 767 is impossible. Images courtesy of Prof. Maria del Carmen Jorge.

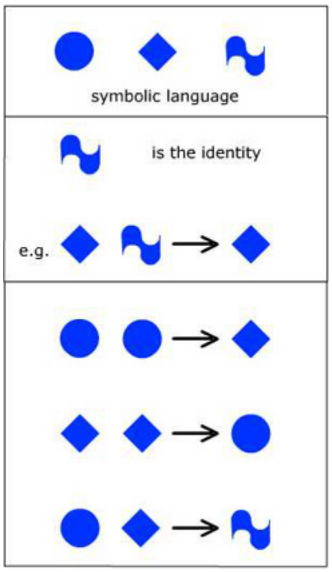
- Starting with the simplest example: the *Codex Vergara* contains many examples of plots with length 20, width 10 and area 200. It also has an example of a plot with length = 20, width = 10+ $\rightarrow$, and area = 210. The authors infer that in all these cases the Aztecs used "area = length times width", and that consequently $\rightarrow$ is worth 1/2 the unit.

- In a more complex example, the plot is a near-trapezoid with sides 26, 32, 30, 10 and area 588. They infer that the Aztecs are using the surveyor's rule: "area = the product of the averages of the opposite sides", because this algorithm is simple and gives that exact answer: $(26+30)/2 \times (32+10)/2 = 588$. Then they consider the quadrilateral with sides 36, 12, 37, 12 and area 438. The only plausible explanation they find is that the surveyor's rule (here identical with the trapezoid rule) was used, with $(36+37)/2 = 36+ \rightarrow$ and $12 \times (36+ \rightarrow) = 432 + 12 \rightarrow = 438$. This implies that the Aztecs used $\rightarrow$ in calculation, and not only in measurement.

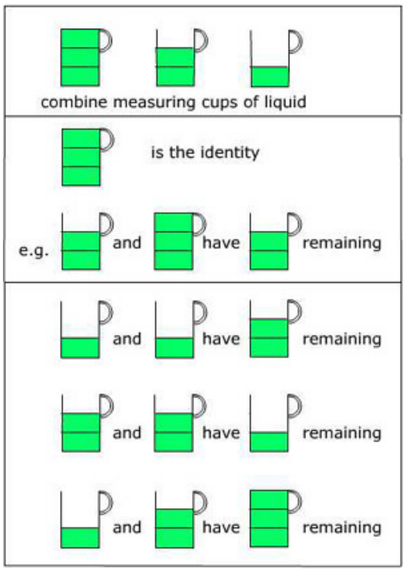
- Other algorithms are similarly inferred. For example the quadrilateral with sides 24, 16, 25, 24 and area 492 was most plausibly calculated with the "triangle rule" (taking it as two right triangles joined at the hypothenuse): $(24 \times 16)/2 + (25 \times 24)/2 = 492$.

As the authors remark, the explanations for the hand (3/5 unit), the heart (2/5 unit) and the bone (1/5 unit) are mathematically less compelling. [In fact, in every case that I checked the author's "calculated value" involves arbitrary approximations and/or roundings before matching with the "recorded area" from the Codex. For example their plot $03 - 030$ has sides 42, 20, 47, 23 and area 1005, for which they use the rule "average of one pair of opposite sides times an adjacent side" as follows, using the arrow ($\rightarrow = 1/2$) and the heart ($(=) = 2/5$): $(23+20)/2 = 21+ \rightarrow \sim 21 + (=)$, $47(21+(=)) = 987 + 47(=) = 987 + 9(5(=)) + 2(=) = 987 + 9X2 + 2(=) = 1005 + 2(=)$ rounds down to 1005. The problem is that there are too many unknowns. We have no clue as to the exact shape of many of the plots. We do not even know whether the surveyors always used a formula going from side lengths to areas or whether other estimates or measurements were involved. But the examples spelled out above do give very strong evidence that sixteenth-century Aztecs had good algorithms for computing areas of rectangles, trapezoids and triangles, and that they could and did calculate with quantities less than the unit. -TP] This article was featured in `ScienceNOW` ("How Aztecs Did the Math"), and picked up in the `Los Angeles Times` ("Aztec math finally adds up"), the `Scientific American` ("Aztec Math Used Hearts and Arrows") and `National Geographic News` ("Aztec Math Decoded, Reveals Woes of Ancient Tax Time").

Teaching math with concrete examples? "Abstract knowledge, such as mathematical knowledge, is often difficult to acquire and even more difficult to apply to novel situations. It is widely believed that a successful aproach to this challenge is to present the learner with multiple concrete and highly familiar examples of the to-be-learned concept". This is the start of "The Advantage of Abstract Examples in Learning Math" by Jennifer Kaminsky, Vladimir Sloutsky and Andrew Heckler (Center for Cognitive Science, OSU), an Education Forum report in the April 25 2008 Science. As can be surmised from the title, the authors present evidence that this widely shared belief is wrong. symbolic instantiation



Generic Instantiation



Concrete Instantiation

Two ways of presenting the concept commutative group with three elements. In the generic example, the concept is presented as a set of rules (—¿) linking pairs of objects to a third object.

In the concrete example, "participants were told that they needed to determine a remaining amount when different measuring cups of liquid are combined".

In one of their experiments, "undergraduate college students learned one or more instantiations of a simple mathematical concept. They were than presented with a transfer task that was a novel instantiation of the learned concept". The instantiations in question were a "generic" instantiation (top figure above) and three different "concrete" instantiations, one of which is illustrated above (the others involved slices of pizza or tennis balls in a container). They authors report that "all participants successfully learned the material" but that when transfer was tested participants who had been taught the generic condition "performed markedly higher than participants in each of the three concrete conditions". The authors also investigated the advantage of teaching a concrete instantiation and then a generic one, and found that "participants who learned only the generic instantiation outperformed those who learned both concrete and generic instantiations". They conclude that "grounding mathematics deeply in concrete contexts can potentially limit its applicability". [It is curious that the authors did not investigate the standard paradigm: abstract definition followed by concrete example, which has the advantage of showing students an example of how to transfer. -TP]

Non-verbal number acuity counts

An article published online September 7, 2008 by Nature bears the title "Individual differences in non-verbal number acuity correlate with maths achievement". The authors, a Johns Hopkins team led by Justin Halberda, elaborate in the Abstract: "Our results show that individual differences in achievement in school mathematics are related to individual differences in the acuity of an evolutionarily ancient, unlearned approximate number sense". What is this ancient unlearned number sense? There turns out to be an "approximate number system," or ANS, which is "shared by adults, infants and non-human animals". These groups "can all represent the approximate number of items in visual or auditory arrays without verbally counting, and use this capacity to guide everyday behaviour such as foraging". The authors set out to investigate whether this ancestral ability is uniform among humans, and if not whether it correlates with other, more symbolic, mathematical talent.

They studied a group of 64 14-year-olds and measured their "ANS acuity" by trials in which "subjects saw spatially intermixed blue and yellow dots presented on a computer screen too rapidly (200 ms) to serially count. Subjects indicated which colour was more numerous by key press and verbal response". dot pattern

Images like this one were flashed on a screen too rapidly for the dots to be counted. Subjects were asked to estimate which color was more numerous. Here there are 8 yellow dots and 6 blue; ratios varied randomly from 1:2 to 7:8.

The authors discovered a "surprisingly large variation in the ANS acuity". Some subjects could detect excesses as relatively small as 10 over 9 with 75

This research was picked up by Natalie Angier in the September 16 2008 New York Times under the headline "Gut Instinct's Surprising Role in Math".

Midge dynamics in Lake Myvatn.

50 generations of midge population in Lake Myvatn. The solid line represents observations, the dashed line output from the mathematical model with nine tuned parameters. Image courtesy of Anthony Ives.

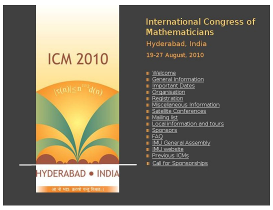
"Mathematics Explains Mysterious Midge Behavior" is the title of an article by Kenneth Chang in the March 7 2008 *New York Times*. At Myrvatn ("Midge Lake") in northern Iceland, during mating season, the air can be thick with male midges (*Tanytarsus gracilentus*), billions of them. Chang quotes Anthony Ives (Wisconsin) "It's like a fog, a brown dense fog that just rises around the lake." And yet in other years, at the same time, there are almost none. Ives was the lead author on a report in *Nature* (March 6 2008) that gave an explanation for this boom-and-bust behavior in which, as Chang describes it, "the density of midges can rise or fall by a factor of a million within a few years." In the *Nature* report ("High-amplitude fluctuations and alternative dynamical states of midges in Lake Myvatn"), Ives and his co-authors characterize the midge ecology as one "driven by consumer-resource interactions, with midges being the consumers and algae/detritus the resources" and they set up a system of three coupled nonlinear difference equations, one each for midges, algae and detritus, to model it. The dynamics of this system include a stable state as well as a stable high-amplitude cycle; small variations in parameters can drive the system from one of those attractors to the other.

Alternative stable states of the midge-algae-detritus model. In the panel on the left, the plane is tangent to the manifold containing the cyclic component of the dynamics around the stationary point. The white region in the plane shows the domain of attraction to the invariant closed set, whereas the region in grey gives the domain of attraction to the outer stable cycle. The red lines give two examples of trajectories that converge to the outer stable cycle. The panel on the right shows the plane in more detail to illustrate the fine structure of the domain of attraction to the invariant closed set. The blue pentagon shows the unstable period 5 cycle that makes up part of the boundary between domains of attraction to the inner invariant closed set and the outer stable cycle. Image courtesy of Anthony Ives.

19-27 August 2010: International Congress of Mathematicians - ICM 2010,

Hyderabad, India.

http://www.icm2010.org.in/



July 18–22, 2011: 7th International Congress on Industrial and Applied Mathematics - ICIAM 2011,

Vancouver, BC, Canada.

http://www.iciam2011.com/