Dynamics with Coquaternions

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1 Introduction

Coquaternions, also known in the literature as split quaternions, are elements of a four-dimensional hypercomplex real algebra generalising complex numbers. This algebra was introduced in 1849 by the English mathematician James Cockle [4], only six years after the famous discovery by Hamilton of the algebra of quaternions [12].

Although coquaternions are not as popular as quaternions, in recent years one can observe an emerging interest among mathematicians and physicists on the study of these hypercomplex numbers. In fact, they have been considered in several papers by different authors and various applications have been developed; see e.g. [1, 3, 6, 7, 8, 9, 10, 11, 16, 17, 18, 21].

The dynamics of the quadratic map in the complex plane has been intensively studied in the last decades and can now be considered a well-established theory. This map exhibits a rich dynamical behaviour and has given birth to extraordinarily beautiful pictures which have passed into the popular domain.

In this note, we give a first insight into the world of coquaternions, reflecting the recent interests of the authors. In particular, we recall some results on the zeros of coquaternionic polynomials [8] and discuss several aspects of the dynamics of one family of quadratic maps on coquaternions [6].

The nature of the algebra under consideration leads to results which can be considered as even richer and more interesting than the ones obtained in the complex or quaternionic cases.

2 The algebra of coquaternions

2.1 Basic results

The algebra of real coquaternions is an associative but non-commutative algebra over $\mathbb{R}$ defined as the set $\mathbb{H}_{\text{coq}} = \{q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, with the operations of addition and scalar multiplication defined component-wise and where the so-called imaginary units $\mathsf{i}, \mathsf{j}, \mathsf{k}$ satisfy

$$\mathsf{i}^2 = -1, \ \ \mathsf{j}^2 = \mathsf{k}^2 = 1, \ \ \mathsf{ijk} = 1.$$

The expression for the product of two coquaternions follows easily from the above multiplication rules; in particular, $\mathsf{ij} = -\mathsf{ji} = \mathsf{k}$, $\mathsf{jk} = -\mathsf{kj} = -\mathsf{i}$, $\mathsf{ki} = -\mathsf{ik} = \mathsf{j}$ and therefore for $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ we have

$$\mathsf{q}^2 = q_0^2 - q_1^2 + q_2^2 + q_3^2 + 2q_0\left(q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}\right).$$

Given a coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} := q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k}$; the number q_0 is called the *real part* of q and denoted by $\operatorname{Re}\mathsf{q}$ and the *vector part* of q, denoted by $\operatorname{Vec}\mathsf{q}$, is given by $\operatorname{Vec}\mathsf{q} := q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$.

We identify the set of coquaternions with null vector part with the set $\mathbb{R}$ of real numbers. For geometric purposes, we also identify the coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ with the element (q_0, q_1, q_2, q_3) in $\mathbb{R}^4$.

It is easy to see that the algebra of coquaternions is isomorphic to $\mathcal{M}_2(\mathbb{R})$, the algebra of real 2×2 matrices, with the map $\mathbb{H}_{\text{coq}} \to \mathcal{M}_2(\mathbb{R})$ defined by

$$\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \ \mapsto Q = \begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}$$

establishing the isomorphism. Keeping this in mind, we call *trace* of q, which we denote by $\operatorname{tr}\mathsf{q}$, to the quantity given by $\operatorname{tr}\mathsf{q} := 2q_0 = 2\operatorname{Re}\mathsf{q} = \mathsf{q} + \overline{\mathsf{q}}$ and call *determinant* of q to the quantity, denoted by $\det\mathsf{q}$, given by

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$\det \mathrm{q} := q_0^2 + q_1^2 - q_2^2 - q_3^2 = \mathrm{q}\bar{\mathrm{q}}$. The result contained in the following lemma can be shown by a simple verification.

LEMMA 1.— For any coquaternion $\mathrm{q} \in \mathbb{H}_{\mathrm{coq}}$, we have

$$\mathrm{q}^2 = (\mathrm{tr}\,\mathrm{q})\mathrm{q} - \det \mathrm{q}.$$

Naturally, some of the results for coquaternions can be established by invoking the aforementioned isomorphism and making use of known results for matrices. For example, one can use this approach to conclude that, unlike $\mathbb{C}$ and $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$ is not a division algebra. In fact, a coquaternion q is invertible if and only if $\det \mathrm{q} \neq 0$. In that case, we have $\mathrm{q}^{-1} = (\det \mathrm{q})^{-1}\bar{\mathrm{q}}$.

For our future purposes it is useful to recall now the following concept: we say that a coquaternion q is *similar* to a coquaternion p, and write $\mathrm{q} \sim \mathrm{p}$, if there exists an invertible coquaternion h such that $\mathrm{p} = \mathrm{h}\mathrm{q}\mathrm{h}^{-1}$. This is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$, partitioning $\mathbb{H}_{\mathrm{coq}}$ in the so-called *similarity classes*. As usual, we denote by [q] the similarity class containing q. The following result can be easily proved (see [6] and [16]).

LEMMA 2.— Let $\mathrm{q} = q_0 + \mathrm{Vec}\,\mathrm{q}$ be a coquaternion and let $r = \det(\mathrm{Vec}\,\mathrm{q})$. If q is real, then $[\mathrm{q}] = \{q_0\}$; if q is non-real, then $[\mathrm{q}] = [\mathrm{q}_s]$, where

$$\begin{aligned} \mathrm{q}_s &= q_0 + \sqrt{r}\,\mathrm{i}, && \text{if} \quad r > 0, && \text{(1a)}\\ \mathrm{q}_s &= q_0 + \sqrt{-r}\,\mathrm{j}, && \text{if} \quad r < 0, && \text{(1b)}\\ \mathrm{q}_s &= q_0 + \mathrm{i} + \mathrm{j}, && \text{if} \quad r = 0. && \text{(1c)} \end{aligned}$$

Since similar coquaternions have the same determinant, the previous lemma completely characterizes the similarity classes in $\mathbb{H}_{\mathrm{coq}}$. This means that two non-real coquaternions p and q are similar if and only if

$$\mathrm{Re}\,\mathrm{p} = \mathrm{Re}\,\mathrm{q} \quad \text{and} \quad \det(\mathrm{Vec}\,\mathrm{p}) = \det(\mathrm{Vec}\,\mathrm{q}). \qquad (2)$$

The coquaternion q_s will be referred to as the *standard representative* of [q]. Lemma 2 says that the standard representative of [q] is either a *complex number*, a *perplex number* (number of the form $a+b\mathrm{j}$) or a *dual number* (number of the form $a+b(\mathrm{i}+\mathrm{j})$). Associated with these numbers we will consider three important subspaces of dimension two of $\mathbb{H}_{\mathrm{coq}}$, the so-called *canonical planes* or *cycle planes*: the complex plane $\mathbb{C}$, the *Minkowski plane* $\mathbb{P}$ of perplex numbers and the *Laguerre plane* $\mathbb{D}$ of dual numbers.

Two coquaternions p and q (whether or not real) satisfying (2) are called *quasi-similar*. Naturally, quasi-similarity is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$; the corresponding equivalence class of q, i.e. the set

$$\{x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k} \,:\, x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = r\},$$

is called the *quasi-similarity class* of q and denoted by $[\![\mathrm{q}]\!]$. Here, as before, q_0 and r denote respectively the real part and the determinant of the vector part of q. We can identify $[\![\mathrm{q}]\!]$ with an hyperboloid in the hyperplane $x_0 = q_0$, which will be:

- a hyperboloid of one sheet or a hyperboloid of two sheets, if $r < 0$ or $r > 0$, respectively; in such cases $[\![\mathrm{q}]\!] = [\mathrm{q}]$;
- a degenerate hyperboloid (i.e. a cone), if $r = 0$; in this case, $[\![\mathrm{q}]\!] = [q_0 + \mathrm{i} + \mathrm{j}] \cup \{q_0\}$.

2.2 SOME REMARKS ON COQUATERNIONIC POLYNOMIALS

In contrast to the case of quaternionic polynomials, the problem of finding the zeros of polynomials defined over the algebra $\mathbb{H}_{\mathrm{coq}}$ only drew the attention of researchers quite recently; see [5, 8, 13, 14, 15, 19].

A complete characterisation of the zero set of left unilateral polynomials over coquaternions, i.e. of polynomials whose coefficients are coquaternions located on the left-hand side of the variable, can be found in [8]. In particular, it is proved that the zeros of monic polynomials of degree n belong to, at most, $n(2n-1)$ quasi-similarity classes; each of these classes can either contain a unique zero (*isolated zero*) or be totally made up of zeros (*hyperboloidal zero*) or contain a straight line of zeros (*linear zero*). We point out that there is no analogue of the Fundamental Theorem of Algebra, as there are coquaternionic polynomials with no zeros.

To offer a glimpse of the diversity of behaviours that the zero sets of coquaternionic polynomials may have, we now present some examples. An algorithm to compute and classify all the zeros of a coquaternionic polynomial is available in [8] and can be used to check the following statements:

1. $P(x) = x^2 - \mathrm{j}$ has no zeros;
2. $P(x) = x^2 + (3+\mathrm{i}+\mathrm{j}+\mathrm{k})x + 3 + \mathrm{i} + \mathrm{j} + 3\mathrm{k}$ has only one isolated zero, $\mathrm{z} = -1 + \frac{1}{2}\mathrm{i} - \mathrm{j} - \frac{1}{2}\mathrm{k}$;
3. $P(x) = x^2 - \mathrm{j}x - 1 - \mathrm{i}$ has six isolated zeros (the maximum number of zeros a quadratic polynomial can have), namely

$$\begin{aligned} \mathrm{z}_1 &= \mathrm{k},\\ \mathrm{z}_2 &= \mathrm{j} + \mathrm{k},\\ \mathrm{z}_{3,4} &= \pm\left(\tfrac{1+\sqrt{2}}{2} + \tfrac{1}{2}\mathrm{i}\right) + \tfrac{1}{2}\mathrm{j} + \tfrac{1-\sqrt{2}}{2}\mathrm{k},\\ \mathrm{z}_{5,6} &= \pm\left(\tfrac{1-\sqrt{2}}{2} + \tfrac{1}{2}\mathrm{i}\right) + \tfrac{1}{2}\mathrm{j} + \tfrac{1+\sqrt{2}}{2}\mathrm{k}; \end{aligned}$$

4. $P(x) = x^2 + 1$ has two isolated zeros, $\mathrm{z}_{1,2} = \pm 1$, and the hyperboloidal zero, $H = [\![\mathrm{j}]\!]$ (which can be identified with an hyperboloid of one sheet in the hyperplane $x_0 = 0$);
5. $P(x) = x^2 - \mathrm{j}x - 1 - \mathrm{j}$ has two isolated zeros, $\mathrm{z}_1 = -1$ and

$z_2 = 1 + \mathrm{j}$, and two linear zeros,

$$L_1 = \left\{-\tfrac{1}{2} + \alpha \mathrm{i} - \tfrac{1}{2}\mathrm{j} + \alpha \mathrm{k} \,:\, \alpha \in \mathbb{R}\right\},$$
$$L_2 = \left\{\tfrac{1}{2} + \alpha \mathrm{i} + \tfrac{3}{2}\mathrm{j} - \alpha \mathrm{k} \,:\, \alpha \in \mathbb{R}\right\}$$

(which can be identified with straight lines).

It is worth mentioning that a *Mathematica* notebook to classify and determine the nth roots of a coquaternion q (the zeros of $x^n - \mathrm{q}$) is available at the webpage `http://w3.math.uminho.pt/CoquaternionsRoots`, where one can also obtain a *Mathematica* add-on application implementing the algebra of coquaternions.

3 Coquaternionic Quadratic Map

We now consider the quadratic map

$$\begin{array}{rccc} \mathrm{f}_\mathrm{c} : & \mathbb{H}_{\mathrm{coq}} & \to & \mathbb{H}_{\mathrm{coq}} \\ & \mathrm{q} & \mapsto & \mathrm{q}^2 + \mathrm{c} \end{array}$$

where c is a fixed parameter in $\mathbb{H}_{\mathrm{coq}}$.

When the parameter $\mathrm{c} \in \mathbb{C}$, we will use f_c to denote the complex map obtained by restricting f_c to the complex plane, i.e. $f_\mathrm{c} \coloneqq \mathrm{f}_\mathrm{c}|_{\mathbb{C}}$.

3.1 Preliminary results

We first recall several basic definitions and present some results which will play an important role in the remaining part of the paper.

For $k \in \mathbb{N}$, we shall denote by f_c^k the k-th iterate of f_c, inductively defined by $\mathrm{f}_\mathrm{c}^0 = \mathrm{id}_{\mathbb{H}_{\mathrm{coq}}}$ and $\mathrm{f}_\mathrm{c}^k = \mathrm{f}_\mathrm{c} \circ \mathrm{f}_\mathrm{c}^{k-1}$. For a given initial point $\mathrm{q}_0 \in \mathbb{H}_{\mathrm{coq}}$, the *orbit* of q_0 under f_c is the sequence $\left(\mathrm{f}_\mathrm{c}^k(\mathrm{q}_0)\right)_{k \in \mathbb{N}_0}$. A point $\mathrm{q} \in \mathbb{H}_{\mathrm{coq}}$ is said to be a *periodic point of* f_c *with period* $n \in \mathbb{N}$, if $\mathrm{f}_\mathrm{c}^n(\mathrm{q}) = \mathrm{q}$, with $\mathrm{f}_\mathrm{c}^k(\mathrm{q}) \neq \mathrm{q}$ for $0 < k < n$; in this case, we say that the set $\mathcal{C} = \{\mathrm{q}, \mathrm{f}_\mathrm{c}(\mathrm{q}), \dots, \mathrm{f}_\mathrm{c}^{n-1}(\mathrm{q})\}$ is an *n-cycle* for f_c, usually written as $\mathcal{C} : \mathrm{q}_0 \overset{\mathrm{f}_\mathrm{c}}{\to} \mathrm{q}_1 \overset{\mathrm{f}_\mathrm{c}}{\to} \cdots \overset{\mathrm{f}_\mathrm{c}}{\to} \mathrm{q}_{n-1}$ with $\mathrm{q}_i = \mathrm{f}_\mathrm{c}^i(\mathrm{q})$. Periodic points of period one are called *fixed points*.

It follows from the result in Lemma 1 that the orbit of any coquaternion q lies in the subspace $\mathrm{span}_{\mathbb{R}}(1, \mathrm{q}, \mathrm{c})$ of $\mathbb{H}_{\mathrm{coq}}$. The following result is also simple to establish.

Lemma 3.— For any invertible coquaternion h, let ϕ_h be the map defined by $\phi_\mathrm{h}(\mathrm{q}) = \mathrm{h}^{-1}\mathrm{q}\mathrm{h}$. Then, the dynamical system $(\mathbb{H}_{\mathrm{coq}}, \mathrm{f}_\mathrm{c})$ is dynamically equivalent to the dynamical system $(\mathbb{H}_{\mathrm{coq}}, \mathrm{f}_{\phi_\mathrm{h}(\mathrm{c})})$.

As a consequence of the two previous lemmas, we immediately conclude that to study the dynamics of the quadratic map $\mathrm{f}_\mathrm{c}(\mathrm{q}) = \mathrm{q}^2 + \mathrm{c}$ there is no loss of generality in assuming that c is either real or has one of the standard forms (1).

3.2 Fixed points of f_c

Let $\mathrm{q} = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ and $\mathrm{c} = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$ be coquaternions. From Lemma 1 we see that q is a fixed point of f_c if and only if it satisfies the equation

$$(2q_0 - 1)\mathrm{q} - \det \mathrm{q} = -\mathrm{c}. \tag{3}$$

Next, we consider separately $\mathrm{q}_0 \neq 1/2$ and $\mathrm{q}_0 = 1/2$.

3.2.1 Case $q_0 \neq 1/2$

We first note that it follows from (3) that, if $q_0 \neq 1/2$, then $\mathrm{q} \in \mathrm{span}_{\mathbb{R}}(1, \mathrm{c})$. In particular, if c is chosen in one of the cycle planes, then q belongs to the same plane.

(i) For $\mathrm{c} = c_0 + c_1\mathrm{i}$, with $c_1 \geq 0$, we are simply considering the case of the complex quadratic map f_c; hence, the fixed points of f_c are, as is well-known, given by

$$\mathrm{q}_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\mathrm{c}}\right).$$

Note that, for $\mathrm{c} = c_0 \in \mathbb{R}$, $c_0 \geq 1/4$, the corresponding fixed points do not satisfy the condition we are considering here, $q_0 \neq 1/2$.

(ii) For $\mathrm{c} = c_0 + c_2\mathrm{j}$, with $c_2 > 0$, the dynamics is restricted to the cycle plane $\mathbb{P}$. Here, it is convenient to use the so-called *dual basis* $(\mathrm{e}_1, \mathrm{e}_2)$ with $\mathrm{e}_1 = (1 + \mathrm{j})/2$ and $\mathrm{e}_2 = (1 - \mathrm{j})/2$, which satisfies

$$\mathrm{e}_1^2 = \mathrm{e}_1, \quad \mathrm{e}_2^2 = \mathrm{e}_2 \quad \text{and} \quad \mathrm{e}_1\mathrm{e}_2 = \mathrm{e}_2\mathrm{e}_1 = 0.$$

Expressing q and c in this basis, we get $\mathrm{q} = x\mathrm{e}_1 + y\mathrm{e}_2$ and $\mathrm{c} = a\mathrm{e}_1 + b\mathrm{e}_2$, where

$$x = q_0 + q_2,\ y = q_0 - q_2,\ a = c_0 + c_2,\ b = c_0 - c_2.$$

Hence,

$$\mathrm{q}^2 + \mathrm{c} = (x^2 + a)\mathrm{e}_1 + (y^2 + b)\mathrm{e}_2.$$

This shows that f_c has fixed points if and only if $c_0 + c_2 < 1/4$ and $c_0 - c_2 < 1/4$, which are

$$\mathrm{q}_{1,2} = \frac{1}{2} \pm \frac{1}{4}\left(A + B + (A - B)\mathrm{j}\right),$$
$$\mathrm{q}_{3,4} = \frac{1}{2} \pm \frac{1}{4}\left(A - B + (A + B)\mathrm{j}\right),$$

where A and B are given by

$$A = \sqrt{1 - 4(c_0 + c_2)} \text{ and } B = \sqrt{1 - 4(c_0 - c_2)}.$$

(iii) For $\mathrm{c} = c_0 + \mathrm{i} + \mathrm{j}$, we have $\mathrm{q} = q_0 + a(\mathrm{i} + \mathrm{j})$ and so

$$\mathrm{q}^2 + \mathrm{c} = (q_0^2 + c_0) + (2q_0 a + 1)(\mathrm{i} + \mathrm{j}).$$

Thus, f_c has fixed points if and only if $c_0 < 1/4$, which are given by

$$\mathrm{q}_1 = \frac{1}{2}\left(1 - \sqrt{1 - 4c_0}\right) + \frac{1}{\sqrt{1-4c_0}}(\mathrm{i} + \mathrm{j}),$$
$$\mathrm{q}_2 = \frac{1}{2}\left(1 + \sqrt{1 - 4c_0}\right) - \frac{1}{\sqrt{1-4c_0}}(\mathrm{i} + \mathrm{j}).$$

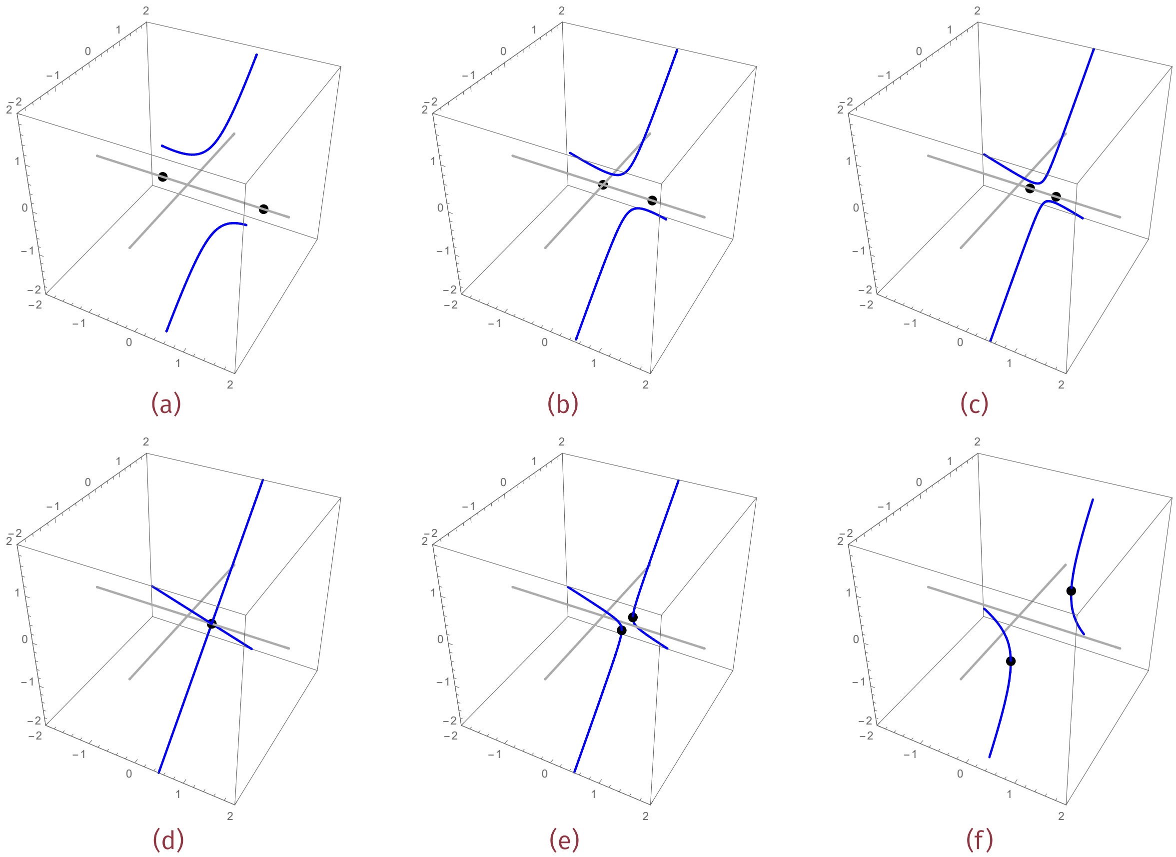

Figure 1.—Plots, in the hyperplane $q_3 = 0$, of the sets of fixed points corresponding to the case $\mathsf{c} = c_0 \in \mathbb{R}$; the real and complex fixed points are identified with black points and the fixed points not in $\mathbb{C}$ form the blue lines; the gray lines represent the real and imaginary axes. (a) $c_0 = -0.8$; (b) $c_0 = -0.0$; (c) $c_0 = -0.18$; (d) $c_0 = -0.25$; (e) $c_0 = -0.3$; (f) $c_0 = -1.7$

3.2.2 *Case $q_0 = 1/2$*

In this case, Eq. (3) reduces to $\det \mathsf{q} = \mathsf{c}$. Since $\det \mathsf{q}$ is a real number, we conclude that $\mathsf{f_c}$ has no fixed points, unless $\mathsf{c} = c_0 \in \mathbb{R}$.

The map f_{c_0} has one real fixed point $\mathsf{q} = q_0 = 1/2$, for $c_0 = 1/4$. We now discuss the non-real fixed points of f_{c_0}. Since a real number commutes with any coquaternion, we have, for any invertible $\mathsf{h} \in \mathbb{H}_{\rm coq}$,

$$\begin{aligned} \mathsf{h}^{-1}\mathsf{f}_{c_0}(\mathsf{q})\mathsf{h} &= \mathsf{h}^{-1}\mathsf{q}^2\mathsf{h} + \mathsf{h}^{-1}c_0\mathsf{h} \\ &= (\mathsf{h}^{-1}\mathsf{q}\mathsf{h})^2 + c_0 = \mathsf{f}_{c_0}(\mathsf{h}^{-1}\mathsf{q}\mathsf{h}). \end{aligned}$$

Hence,

$$\begin{aligned} \mathsf{f}_{c_0}(\mathsf{q}) = \mathsf{q} &\iff \mathsf{h}^{-1}\mathsf{f}_{c_0}(\mathsf{q})\mathsf{h} = \mathsf{h}^{-1}\mathsf{q}\mathsf{h} \\ &\iff \mathsf{f}_{c_0}(\mathsf{h}^{-1}\mathsf{q}\mathsf{h}) = \mathsf{h}^{-1}\mathsf{q}\mathsf{h} \end{aligned}$$

which shows that to determine the non-real fixed points of the coquaternionic map f_{c_0} we only have to identify the fixed points of this map with any of the three special forms (1) and to construct the corresponding similarity classes.

As it is well-known, there is only one fixed point of the form (1a), which occurs for $c_0 > 1/4$, the point $\mathsf{q_s} = 1/2 + (\sqrt{4c_0 - 1}/2)\mathsf{i}$. Also, it is simple to verify that the only fixed point of f_{c_0} of the form (1b) is given by $\mathsf{q_s} = 1/2 + (\sqrt{1-4c_0}/2)\mathsf{j}$, for $c_0 < 1/4$, whereas $\mathsf{q_s} = 1/2 + \mathsf{i} + \mathsf{j}$ is the only fixed point of the form (1c) and occurs when $c_0 = 1/4$. In summary, we have the following three sets of fixed points, depending on the value of c_0:

$$\begin{aligned} \mathcal{F}_1 &= \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4c_0}\,\mathsf{j}\right], && \text{if } c_0 < 1/4, \\ \mathcal{F}_2 &= \left[\tfrac{1}{2} + \mathsf{i} + \mathsf{j}\right] \cup \left\{\tfrac{1}{2}\right\}, && \text{if } c_0 = 1/4, \\ \mathcal{F}_3 &= \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{4c_0 - 1}\,\mathsf{i}\right], && \text{if } c_0 > 1/4. \end{aligned}$$

Having in mind the relation between similarity and quasi-similarity classes referred to in Sec. 2.1, it is clear that any of the above sets can be identified with an hyperboloid in the hyperplane $q_0 = 1/2$.

In Fig. 1 we present plots obtained by fixing $q_3 = 0$, and considering several values of the parameter c_0. The known fixed points of the dynamics in $\mathbb{C}$ are identified with black points and the fixed points not in $\mathbb{C}$ are given by blue lines (hyperbolas resulting from the intersection of the hyperboloids $\mathcal{F}_i$ with the hyperplane $H_3 = \{(q_0, q_1, q_2, q_3) \in \mathbb{R}^4 : q_3 = 0\}$); the real and imaginary axis are identified with gray lines.

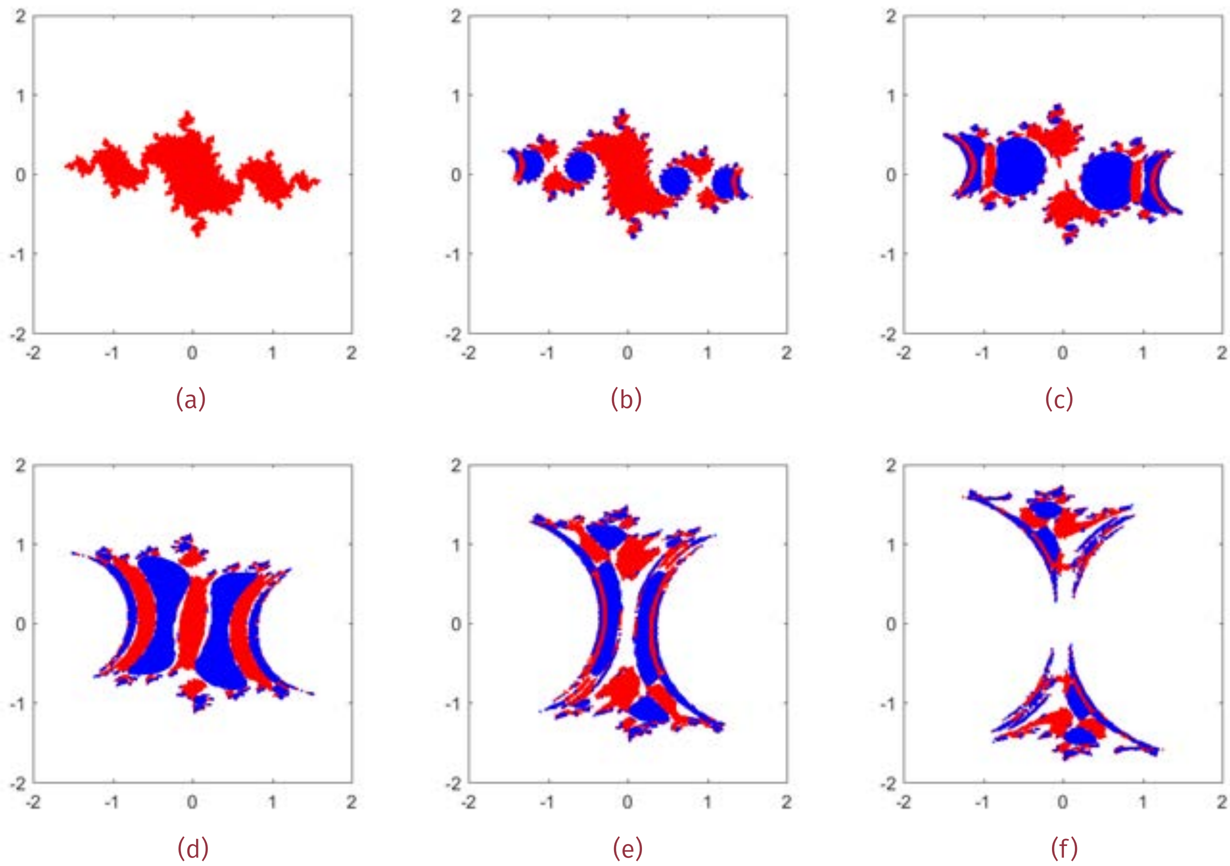

Figure 2.—Plots, in different planes parallel to the complex plane, of the basin of attraction of the 2-cycle C_2 (red) and the (1,2)-set cycle $C_{1,2}$ (blue). (a) $q_2 = 0$; (b) $q_2 = 0.2$; (c) $q_2 = 0.4$; (d) $q_2 = 0.8$; (e) $q_2 = 1.25$; (f) $q_2 = 1.55$

3.3 Cycles of sets

The results of the previous section show that we now have a situation not present in the classical case of the real/complex quadratic maps: the existence of fixed points forming sets of non-isolated points. The same may be true for periodic points of other periods; see, e.g. [6]. This motivates us to introduce a definition of cycles of sets.

Definition 4.— We say that the sets $\mathcal{S}_0, \dots, \mathcal{S}_{m-1}$ form an *(m,n)-set cycle* $\mathcal{C}_{m,n}$ for the map $\mathrm{f_c}$, and write

$$\mathcal{C}_{m,n} : \mathcal{S}_0 \overset{\mathrm{f_c}}{\to} \mathcal{S}_1 \overset{\mathrm{f_c}}{\to} \cdots \overset{\mathrm{f_c}}{\to} \mathcal{S}_{m-1},$$

if:

(i) each of the sets $\mathcal{S}_i$; $i = 0, \dots, m-1$, is formed by periodic points of period n of $\mathrm{f_c}$;

(ii) $\mathcal{S}_i = \mathrm{f_c}(\mathcal{S}_{i-1})$, $i = 1, \dots, m-1$, and $\mathrm{f_c}(\mathcal{S}_{m-1}) = \mathcal{S}_0$;

(iii) the sets $\mathcal{S}_0, \dots, \mathcal{S}_{m-1}$ are pairwise separated by ε-neighborhoods.

Note that if $\mathcal{C}_{m,n}$ is an (m,n)-set cycle, then n must be a multiple of m. When $m = n$, we simply call the cycle an n-set cycle and denote it by $\mathcal{C}_n$.

As shown in [6], for $\mathrm{c} = c_0 + c_1\mathrm{i}$, with $c_1 > 0$ and c_0, c_1 satisfying $c_1^2 > 4c_0 + 3$, the set

$$\mathcal{P} = \left\{ -\frac{1}{2} + \frac{c_1}{2}\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} \;\middle|\; q_2^2 + q_3^2 = \frac{c_1^2 - 4c_0 - 3}{4} \right\} \tag{4}$$

is made up of periodic points of period two of the map $\mathrm{f_c}$ and, if $\mathrm{q} \in \mathcal{P}$, then $\mathrm{q} = \mathrm{f_c}(\mathrm{p})$ where $\mathrm{p} = -1/2 + (c_1/2)\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k} \in \mathcal{P}$. Hence,

$$\mathcal{C}_{1,2} : \mathcal{P} \circlearrowleft_{\mathrm{f_c}} \tag{5}$$

is a $(1,2)$-set cycle.

Other examples of set cycles for the quadratic coquaternionic map can be found in [6].

We would like to remark that some results for the quadratic map on the algebra $\mathcal{M}_2(\mathbb{R})$—which can naturally, be translated to the coquaternionic formalism—were obtained in [2] and [20].

3.4 Basins of attraction

Due to the appearance of set cycles, we now have to adapt the usual notion of basin of attraction. We propose to use the following definition.

Definition 5.— Let $\mathcal{C}_{m,n} : \mathcal{S}_0 \overset{\mathsf{f_c}}{\to} \mathcal{S}_1 \overset{\mathsf{f_c}}{\to} \cdots \overset{\mathsf{f_c}}{\to} \mathcal{S}_{m-1}$ be an (m,n)-set cycle for $\mathsf{f_c}$. The *basin of attraction* of $\mathcal{C}_{m,n}$, denoted by $\mathcal{B}(\mathcal{C}_{m,n})$, is given by

$$\mathcal{B}(\mathcal{C}_{m,n}) = \bigcup_{\ell=0}^{m-1} \mathcal{B}(\mathcal{S}_\ell),$$

where

$$\mathcal{B}(\mathcal{S}_\ell) := \{\mathsf{q} \in \mathbb{H}_{\mathrm{coq}} : \lim_k d(\mathsf{f_c}^{km}(\mathsf{q}), \mathcal{S}_\ell) = 0\}$$

and d is a distance function.

Naturally, when a set cycle reduces to a cycle of isolated points, we recover the usual definition of basin of attraction of that cycle.

As an illustrative example, we consider now two different cycles for the map $\mathsf{f_c}$: the 2-cycle of isolated complex points

$$\mathcal{C}_2 : \mathsf{q}_1 \overset{\mathsf{f_c}}{\to} \mathsf{q}_2,$$

where

$$\mathsf{q}_{1,2} = \frac{1}{2}(1 \pm \sqrt{-3-4\mathsf{c}}),$$

and the (1,2)-set cycle $\mathcal{C}_{1,2}$ defined by (5) with $\mathcal{P}$ the set given by (4), for a particular choice of the parameter c, the complex number $\mathsf{c} = -0.95 + 0.2\mathsf{i}$.

In Fig. 2 we present plots of the basins of attraction of these two cycles. The representations are two-dimensional plots obtained by assuming $q_3 = 0$ and considering different values for q_2, i.e., all the pictures correspond to plots in planes parallel to the complex plane. In the plots, the points in the basin of attraction of the cycle $\mathcal{C}_2$ are colored in red and the points in the basin of attraction of the cycle $\mathcal{C}_{1,2}$ are colored in blue.

The plot on the top-left of Fig. 2 corresponds to $q_2 = 0$, i.e., is a plot in the complex plane, and we immediately recognize the picture associated with the dynamics of the quadratic complex map f. As the value of q_2 increases, the two coquaternionic basins of attraction appear, showing an interesting intertwined structure.

4 Conclusions

As it is well-known, to study the dynamics of complex quadratic maps we only have to consider the particular family of maps of the form $f_c(x) = x^2 + c$, since any quadratic map may be converted, by conjugacy, to a member of this family. In the coquaternionic case, the situation is totally different.

Due to the non-commutativity of the product of coquaternions, the sum of two mth degree monomials $\mathsf{a}_0 x \mathsf{a}_1 x \cdots \mathsf{a}_{m-1} x \mathsf{a}_m$ and $\mathsf{a}'_0 x \mathsf{a}'_1 x \cdots \mathsf{a}'_{m-1} x \mathsf{a}'_m$ can not be written simply in the form $\mathsf{A}_0 x \mathsf{A}_1 x \cdots \mathsf{A}_{m-1} x \mathsf{A}_m$ and hence, the general expression of a quadratic coquaternionic polynomial is

$$\sum_{j=1}^{n} \mathsf{a}_0^j x \mathsf{a}_1^j x \mathsf{a}_2^j + \sum_{j=1}^{k} \mathsf{b}_0^j x \mathsf{b}_1^j + \mathsf{c}, \quad n, k \in \mathbb{N},$$

with a_i^j, b_i^j and c coquaternions. Not surprisingly, contrary to what happens in the commutative case, no conjugacy equivalence of a quadratic coquaternionic polynomial to a simple form is available.

The important differences from the complex setting already observed for the simple coquaternionic quadratic family $\mathsf{f_c}(\mathsf{q}) = \mathsf{q}^2 + \mathsf{c}$ and the interesting results obtained for the zeros of unilateral coquaternionic polynomials lead us to believe that coquaternions — in particular the study of more general coquaternionic quadratic maps and of more general polynomials — are an area worth exploring.

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