Feature Article

Poisson vs. Symplectic Geometry

Rui Loja Fernandes[1]

Departamento de Matemática, Instituto Superior Técnico
1049-001 Lisboa, PORTUGAL

rfern@math.ist.utl.pt

Abstract

We survey the relationship between symplectic and Poisson geometries, emphasizing the construction of the symplectic groupoid associated with a Poisson manifold.

1 Introduction

Poisson and symplectic geometries are usually referred to as the geometries underlying classical mechanics. Let us recall briefly why this is so. In every introductory course in mechanics one learns Hamilton's equations describing the motion of a mechanical system with Hamiltonian H:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \qquad (i = 1, \dots, n) \tag{1.1}$$

where $(q_1, \dots, q_n)$ are the position and $(p_1, \dots, p_n)$ are the momenta, which together form coordinates on the phase space of the system.

Symplectic geometry originates from the following interpretation of equations (1.1). Introduce the closed 2-form:

$$\omega := \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{1.2}$$

Since this 2-form is non-degenerate, every smooth function H determines a vector field X_H by the requirement:

$$i_{X_H}\omega = \mathrm{d}H.$$

Now (1.1) is just the equation for the integral curves of this vector field. More generally, one defines a **symplectic manifold** to be a manifold M equipped with a symplectic form, i.e., a closed, non-degenerate, 2-form ω. Then every smooth function $H : M \to \mathbb{R}$ determines a **Hamiltonian vector field** X_H by exactly the same procedure. Darboux's theorem (see [9]) states that, around any point, there exist local coordinates (q_i, p_i) such that ω takes the form (1.2), so locally we recover the standard formulation.

A slightly different interpretation of equations (1.1) leads to Poisson geometry. One defines a bilinear, skew-symmetric bracket of functions on the phase space by setting for any pair of functions F and G:

$$\{F, G\} := \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right) \tag{1.3}$$

and observes that Hamilton's equations can be written in the form:

$$\begin{cases} \dot{q}_i = \{q_i, H\} \\ \dot{p}_i = \{p_i, H\} \end{cases} \qquad (i = 1, \dots, n).$$

More geometrically, any smooth function H determines a **Hamiltonian vector field** X_H by:

$$X_H(\cdot) = \{\cdot, H\},$$

and Hamilton's equations are just the equations for the integral curves of this vector field. Another justification for the introduction of the Poisson bracket is the study of first integrals of the system: if F and G are two first integrals, then their Poisson bracket $\{F, G\}$ is also a first integral. This is because, for any triple of functions F, G and H, we have the Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

[1]Supported in part by FCT/POCTI/FEDER and by grants POCI/MAT/55958/2004 and POCI/MAT/57888/2004.

All this motivates defining a **Poisson manifold** to be a manifold M equipped with a **Poisson bracket** $\{\ ,\ \}$, i.e., a Lie bracket on the algebra of smooth functions $C^\infty(M)$ which satisfies the Leibniz identity:

$$\{F, GH\} = \{F, G\}H + G\{F, G\}.$$

Then any smooth function $H : M \to \mathbb{R}$ determines a **Hamiltonian vector field** X_H by the procedure above. But now, contrary to the symplectic situation, locally we may not recover anymore the standard form (1.3) of the Poisson bracket. In fact, we have the following theorem, which maybe consider as the first significant result in Poisson geometry:

Theorem 1.1 (Weinstein [10]). *Let $(M, \{\ ,\ \})$ be a Poisson manifold. For every $x_0 \in M$ there exists coordinates $(q_1, \dots, q_n, p_1, \dots, p_n, y_1, \dots, y_l)$ centered at x_0 such that:*

$$\{F, G\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right) + \sum_{j,k=1}^{l}\pi_{jk}(y)\frac{\partial F}{\partial y_j}\frac{\partial G}{\partial y_k},$$

where $\pi_{jk}(y) = -\pi_{kj}(y)$ are certain functions of the y_j's alone which vanish at 0.

Therefore, contrary to symplectic geometry, where there are no local invariants, in Poisson geometry it is important to understand the local structure as well.

More generally, what can be said about the relationship between symplectic and Poisson geometry? In one direction we have that every symplectic manifold is a Poisson manifold: to every symplectic form ω on a manifold M, one associates a Poisson bracket by setting:

$$\{F, G\} := \omega(X_F, X_G).$$

Conversely, the Darboux-Weinstein theorem above implies that a Poisson manifold is (singular) foliated by symplectic (immersed) submanifolds.

However, there is a much more subtle and deeper relation between Poisson and symplectic geometry: to every Poisson manifold one can associate a canonical symplectic object. Moreover, its properties encode both the local and global behavior of a Poisson manifold. In the remainder of this paper we will explain how this object arises naturally, and we will discuss briefly its relevance in the study of both local and global properties of a Poisson manifold.

2 Contravariant geometry

Let $(M, \{\ ,\})$ be a Poisson manifold. What kind of paths should one consider in M which take into account the Poisson geometry? Because M is foliated into symplectic submanifolds, paths in M should preserve this foliation. However, this is a singular foliation and to take care of this we must give some "internal geometry" to the paths. More precisely, let $\pi : T^*M \to M$ be the cotangent bundle and consider the the bundle map defined by:

$$\# : T^*M \to TM, \quad \mathrm{d}H \mapsto X_H.$$

It is easy to check that the symplectic leaves of M are in fact the integral leaves of the distribution $\mathrm{Im}\# \subset TM$.

Definition 2.1. A **cotangent path** is a path $a : [0,1] \to T^*M$ such that:

$$\#a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\pi(a(t)).$$

The space of cotangent paths will be denoted by $P_\Pi(M)$.

Note that the base path $\gamma(t) = \pi(a(t))$ of a cotangent path lies in a symplectic leaf. These kind of paths where introduce first by A. Weinstein, and they show up in virtual every global construction in Poisson geometry. For example, in [7] one studies connections in Poisson geometry and shows that parallel transport is defined along cotangent paths.

There is a *general principle* in Poisson geometry that every construction in standard (covariant) geometry can be dualized to a (contravariant) construction in Poisson geometry. The cotangent paths we have introduced is just one instance of this principle. Another instance is Poisson cohomology. Recall that de Rham cohomology of a manifold is the cohomology of the complex of differential forms $(\Omega^\bullet(M), \mathrm{d})$, where the differential of a r-form is given by the usual formula:

$$\begin{aligned}\mathrm{d}\omega(X_0, \dots, X_r) = \\ \sum_{k=0}^{r}(-1)^{k+1}X_k(Q(X_0, \dots, \widehat{X}_k, \dots, X_r)) + \\ \sum_{k<l}(-1)^{k+l+1}\omega([X_k, X_l], X_0, \dots, \widehat{X}_k, \dots, \widehat{X}_l, \dots, X_r),\end{aligned} \tag{2.1}$$

where $X_0, \dots, X_r \in \mathfrak{X}(M)$ are vector fields, $[\ ,\ ]$ denotes the usual Lie bracket of vector fields, and the hat over a factor means omitting that factor. Following the general principle above, in Poisson geometry one considers the dual objects to differential forms, i.e., the multivector fields $\mathfrak{X}^r(M)$, and defines a contravariant exterior differential $\mathrm{d}_\Pi : \mathfrak{X}^r(M) \longrightarrow \mathfrak{X}^{r+1}(M)$ by:

$$\begin{aligned}\mathrm{d}_\Pi Q(\alpha_0, \dots, \alpha_r) = \\ \sum_{k=0}^{r}(-1)^{k+1}\#\alpha_k(Q(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_r)) + \\ \sum_{k<l}(-1)^{k+l+1}Q([\alpha_k, \alpha_l]_\Pi, \alpha_0, \dots, \widehat{\alpha}_k, \dots, \widehat{\alpha}_l, \dots, \alpha_r),\end{aligned} \tag{2.2}$$

where $\alpha_0, \dots, \alpha_r \in \Omega^1(M)$. Here $[\ ,\ ]$ is a Lie bracket on 1-forms induced from the Poisson bracket, which on exact 1-forms is given by

$$[\mathrm{d}F, \mathrm{d}G]_\Pi = \mathrm{d}\{F, G\},$$

and extends to any pair of 1-forms by requiring that

$$[\alpha, F\beta]_\Pi = F[\alpha, \beta]_\Pi + \#\alpha(F)\beta.$$

It is easy to see that d_Π is indeed a differential: $\mathrm{d}_\Pi^2 = 0$. Hence it defines the **Poisson cohomology** $H^\bullet_\Pi(M)$ of the Poisson manifold. It is not hard to see that for a symplectic manifold the Poisson cohomology is isomorphic to the usual de Rham cohomology. However, in general, the Poisson cohomology is quite hard to compute.

Observe that the defining equation of a Hamiltonian vector field can be written in the form $X_H = \mathrm{d}_\Pi H$. It follows that $\mathrm{d}_\Pi X_H = 0$ for any Hamiltonian vector field. More generally, any vector field X such that $\mathrm{d}_\Pi X = 0$ is called a **Poisson vector field**. It is easy to check that X is a Poisson vector field iff its flow preserves Poisson brackets. The first Poisson cohomology group is just the quotient of the Poisson vector fields by the Hamiltonian vector fields.

In geometry one integrates 1-forms over curves. Dually, in Poisson geometry one integrates vector fields over cotangent paths: if $X \in \mathfrak{X}(M)$ is a vector field and $a \in P_\Pi(M)$ is a cotangent path with base path γ, then one defines:

$$\int_a X := \int_0^1 \langle X(\gamma(t)), a(t)\rangle \mathrm{d}t.$$

The usual integral of *closed* 1-forms is invariant under homotopy and depends only on the end-points of the curve provided the form is exact. In Poisson geometry there is also a notion of **cotangent homotopy** between cotangent paths (the precise definition can be found in [6]), and we have:

Proposition 2.2 ([6]). *The integral of a* Poisson *vector field is invariant under cotangent homotopies. For a Hamiltonian vector field the integral depends only on the end-points of the cotangent path.*

In ordinary topology one defines the fundamental group $\pi_1(M, x_0)$ of a pointed space (M, x_0) to be the loops based at x_0 modulo homotopies, where the group multiplication arises from concatenation of paths. If M is connected, changing the base point leads to isomorphic fundamental groups. If one considers (not necessarily closed) paths modulo homotopy then we do not get a group anymore because we cannot always multiply two paths. We get instead a groupoid $\Pi_1(M) \rightrightarrows M$: there are source and target maps

$$\mathbf{s}([\gamma])) = \gamma(0), \quad \mathbf{t}([\gamma])) = \gamma(1),$$

and the product $[\gamma] \cdot [\tau]$ is defined provided $\mathbf{s}([\gamma]) = \mathbf{t}([\tau])$.

In Poisson geometry we consider the analogous **Poisson fundamental groupoid** $\Sigma(M) \rightrightarrows M$ formed by cotangent paths modulo cotangent homotopies:

$$\Sigma(M) := P_\Pi(M)/\sim$$

Of course we can consider only cotangent paths whose base paths are loops based at x_0, and these form the **isotropy group**

$$\Sigma(M, x_0) = \mathbf{s}^{-1}(x_0) \cap \mathbf{t}^{-1}(x_0),$$

which should be thought of as the *Poisson fundamental group* based at x_0. The Poisson fundamental groups at different base points are isomorphic provided the base points lie in the *same* symplectic leaf of M (otherwise, they may be non-isomorphic). Also, contrary to the fundamental group of a space, Poisson fundamental groups are usually non-discrete topological groups. This is because they contain information about the local behavior at x_0 of our Poisson structure. More generally, the groupoid $\Sigma(M)$ provides both local and global information about the Poisson structure. But before we turn into that we need to study its geometry.

3 Symplectic Groupoids

The Poisson fundamental groupoid $\Sigma(M)$ is a topological groupoid since it is a quotient of the Banach manifold $P_\Pi(M)$. Moreover, $\Sigma(M)$ has at most one smooth structure compatible with the quotient topology for which the projection $P_\Pi(M) \to \Sigma(M)$ is submersion. Whenever this smooth structure exists $\Sigma(M)$ becomes a Lie groupoid (i.e., the groupoid structure is compatible with the smooth structure) and we say that M is an **integrable** Poisson manifold. The obstructions to integrability where determined recently in [6, 5], solving a long standing problem in Poisson geometry (and Lie groupoid theory). Let us explain briefly how they arise.

First of all, for each $x \in M$ there exists attached to (M, π) a certain Lie algebra $\mathfrak{g}_x$. As a vector space, we have $\mathfrak{g}_x := \mathrm{Ker}\,\#_x \subset T^*_x M$ and the Lie bracket is the restriction of the Lie bracket on 1-forms. We call $\mathfrak{g}_x$ **isotropy Lie algebra** at x. Now, if $\Sigma(M)$ is smooth, each isotropy Lie group $\Sigma(M, x)$ is a Lie group with Lie algebra $\mathfrak{g}_x$ which, in general, is neither connected nor simply connected. If we denote by G_x the 1-connected Lie group with Lie algebra $\mathfrak{g}_x$, the connected component of the identity $\Sigma(M, x)^0$ is isomorphic to $G_x/\mathcal{N}_x$ where $\mathcal{N}_x \subset G_x$ is a certain normal discrete subgroup called the **monodromy group** at x. One can show that the monodromy group can also be described as the image of a certain homomorphism $\partial : \pi_2(S,x) \to G_x$, where S denotes the symplectic leaf through x. Moreover, this description is still valid in the non-integrable case, but now the monodromy groups $\mathscr{N}_x = \mathrm{im}(\partial)$ need not be discrete subgroups anymore. In fact, the main theorem of [5] implies:

Theorem 3.1. *A Poisson manifold is integrable iff the monodromy groups $\mathscr{N}_x \subset G_x$ are uniformly discrete as $x \in M$ varies.*

From now on we will assume that (M,π) is an integrable Poisson manifold so that $\Sigma(M)$ is a Lie groupoid. We will show now that $\Sigma(M)$ is a symplectic manifold and that the symplectic form ω is *compatible* with the groupoid multiplication, i.e., that

$$m^*\omega = \pi_1^*\omega + \pi_2^*\omega \tag{3.1}$$

where $m : \mathscr{G}^{(2)} \to \mathscr{G}$ is the multiplication in $\mathscr{G}$, defined on the space $\mathscr{G}^{(2)} \subset \mathscr{G} \times \mathscr{G}$ of composable arrows, and $\pi_1, \pi_2 : \mathscr{G}^{(2)} \to \mathscr{G}$ are the (restrictions of the) projections to the first and second factors.

In order to understand how the symplectic form appear we recall an alternative construction of $\Sigma(M)$ due to Cattaneo and Felder [1], and which is related with the Poisson-sigma model of string theory. Let us denote by $P(T^*M)$ the set of all paths in the cotangent bundle, so that $P_\Pi \subset P(T^*M)$. Since $P(T^*M) \simeq T^*P(M)$ is the cotangent bundle of the manifold of paths in M, it carries a natural symplectic form ω_{can}. Now the results in [1] (see the explanations in [6]) show that there exists a Lie algebra action

$$P_0\Omega^1(M) \to \mathfrak{X}(P(T^*M))$$

where $P_0\Omega^1(M)$ denotes the Lie algebra of time-dependent 1-forms α_t satisfying $\alpha_0 = \alpha_1 = 0$, with Lie bracket $[\ ,\ ]_\Pi$. The cotangent paths $P_\Pi \subset P(T^*M)$ form an invariant submanifold and two cotangent paths lie in the same orbit iff they are cotangent homotopic.

Now observe that the space of cotangent paths is precisely the level set $J^{-1}(0)$ of the map $J : P(T^*M) \to P_0\Omega^1(M)^*$ given by:

$$\langle J(a), \eta\rangle = \int_0^1 \langle \frac{d}{dt}\pi(a(t)) - \pi^\sharp a(t), \eta(t,\gamma(t))\rangle dt.$$

We have the following result due to Cattaneo and Felder [1]:

Theorem 3.2. *The Lie algebra action of $P_0\Omega^1(M)$ on $P(T^*M)$ is Hamiltonian, with equivariant moment map $J : P(T^*M) \to P_0\Omega^1(M)^*$.*

Hence the groupoid $\Sigma(M)$ can be described alternatively as a Marsden-Weinstein reduction:

$$\Sigma(M) = P(T^*M)//P_0\Omega(M). \tag{3.2}$$

We deduce:

Corollary 3.3. *If $\Sigma(M)$ is smooth, then it admits a symplectic form which turns $\Sigma(M)$ into a symplectic groupoid.*

Proof. We only need to check the compatibility of the symplectic form with the product. First note that we have the following explicit formula for the symplectic form ω_{can} in $P(T^*M)$:

$$\omega_{\mathrm{can}}(U_1,U_2)_a = \int_0^1 \overline{\omega}_{can}(U_1(t),U_2(t))dt,$$

for all $U_1, U_2 \in T_aP(T^*M)$, where $\overline{\omega}_{can}$ is the canonical symplectic form on T^*M. The additivity of the integral shows that that condition (3.1) holds at the level of $P(T^*M)$, hence it must hold also on the reduced symplectic space $\Sigma(M)$. □

4 Local vs. Global properties

The symplectic groupoid $\Sigma(M)$ encodes both local and global properties of a Poisson manifold, which otherwise would be difficult or impossible to understand. We will illustrate these with two examples.

Let $(M, \{\ ,\ \})$ be a Poisson manifold and assume that the Poisson bracket vanishes at a point x_0. This means that in the local form given by Theorem 1.1 there is only the second term (no (p,q) coordinates), so for x close to x_0 we have:

$$\{y_i, y_j\}(x) = c_{ij}^k y_k + \cdots$$

where the dots represent higher order terms. Here the c_{ij}^k are just the structure constants of the isotropy Lie algebra $\mathfrak{g}_{x_0} = T^*_{x_0}M$ in the basis $\{\mathrm{d}_{x_0}y_1, \ldots, \mathrm{d}_{x_0}y_m\}$. The *linearization problem* asks for new local coordinates where the higher order terms vanish (see [8] for a survey of this problem). We have the following deep theorem:

Theorem 4.1 (Conn [2, 3]). *If the isotropy Lie algebra is semisimple of compact type then there exists linearizing coordinates.*

The original proof due to Conn is a hard analysis proof based on the Nash-Moser method. He constructs successive coordinate systems which give better approximations to the linearizing coordinates and which do converge to the linearizing coordinates. Using the symplectic groupoid $\Sigma(M)$, Crainic and the author gave a soft geometric proof of this result, along the following lines:

- The hypothesis of the theorem is equivalent to the isotropy Lie group $\Sigma(M, x_0)$ being a compact, 1-connected Lie group.

- By Reeb stability, the s-fibers are compact and $\mathbb{Q}$-homological 2-connected. Hence the groupoid $\Sigma(M)$ is a proper 2-groupoid with 2-connected fibers.

- By the vanishing cohomology theorem of Crainic [4], the differentiable groupoid cohomology of $\Sigma(M)$ vanishes. By the Van Est Theorem (see [4]), it follows that the 2nd Poisson cohomology group $H^2_\Pi(M)$ also vanishes.

- The vanishing of $H^2_\Pi(M)$ allows one to apply a contravariant version of Moser's Path Method (see [9]) to obtain the linearizing coordinates.

Notice how this proof makes clear the relevance of the assumption, while in the original proof the assumption is hidden in the analysis (it is used to build certain norms necessary for the Nash-Moser method to work).

The previous example was about local properties of a Poisson manifold. Let us give a different example where both global and local properties are present. We claim that the following result holds:

Theorem 4.2. *If a Poisson manifold $(M, \{\ ,\ \})$ integrates to a compact symplectic groupoid $\Sigma(M)$ then the Poisson bracket cannot vanish at any point.*

Proof. Let us assume that the Poisson bracket vanishes at some point x_0. Just like we observed in the proof above, it follows that $\Sigma(M)$ is a compact groupoid with 2-connected fibers, and from that we conclude that $H^2_\Pi(M)$ vanishes. Now the Poisson bracket always defines a class $[\Pi] \in H^2_\Pi(M)$, and hence this class must be trivial. At the level of the groupoid, this means that the cohomology class $[\omega] \in H^2(\Sigma(M))$ of the symplectic form in $\Sigma(M)$ is trivial. But this is not possible since, by assumption, $\Sigma(M)$ is a compact manifold. □

As examples of Poisson manifolds (M, π) with $\Sigma(M)$ compact, we can take compact symplectic manifolds with finite fundamental group. I don't know of any other examples, and either they do not exist or they will provide an extremely interesting class of Poisson manifolds. So I believe it is important to solve the following:

Open Problem. Are there (non-symplectic) Poisson manifolds which integrate to a compact symplectic groupoid $\Sigma(M)$?

References

[1] A.S. Cattaneo and G. Felder, *Poisson sigma models and symplectic groupoids*, Quantization of singular symplectic quotients, Progr. Math., vol. 198, Birkhäuser, Basel, 2001, pp. 61–93.

[2] J. Conn, Normal forms for analytic Poisson structures, *Annals of Math.* **119** (1984), 576–601.

[3] ———, Normal forms for smooth Poisson structures, *Annals of Math.* **121** (1985), 565–593.

[4] M. Crainic, Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, *Comment. Math. Helv.* **78** (2003), no. 4, 681–721.

[5] M. Crainic and R.L. Fernandes, Integrability of Lie brackets, *Ann. of Math. (2)* **157** (2003), no. 2, 575–620.

[6] ———, Integrability of Poisson brackets, *J. Differential Geom.* **66** (2004), no. 1, 71–137.

[7] R.L. Fernandes, Connections in Poisson geometry. I. Holonomy and invariants, *J. Differential Geom.* **54** (2000), no. 2, 303–365.

[8] R.L. Fernandes and P. Monnier, Linearization of Poisson brackets, *Lett. Math. Phys.* **69** (2004), 89–114.

[9] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford Mathematical Monographs. Oxford University Press, New York, 1998.

[10] A. Weinstein, The local structure of Poisson manifolds, *J. Differential Geometry* **18** (1983), 523–557.