Mathematical Finance - a glimpse from the past challenging the future

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1 Introduction

Mathematical Finance is a flourishing area of modern science that was born in 1900 with Louis Bachelier's Ph.D thesis "Théorie de la speculation" [1]. This work is a result of his attempt to model stock prices on the Paris stock market. Since then, its importance has increased not only as the basis of the hectic financial activity of the modern world but also as a source of many interesting mathematical problems and theories.

Modelling of risky asset prices and modern option pricing techniques are often considered among the most mathematically complex of all applied areas of finance. Their roots rely on stochastic calculus and their development is intimately related to the history of stochastic integration. In fact, financial phenomena and instruments (bank accounts, bonds, stocks, options, rates, currencies, etc.) combine on the one hand a deterministic behaviour and on the other a degree of uncertainty due to time, risks and the (random) environment. That is why the theory of stochastic processes perfectly suits the needs of financial theory and strategy. What is commonly referred to as Mathematical Finance can be considered in a naïf way as the resultant of two vectors, stochastic integration and modelling of asset prices of financial markets operating under uncertainty.

In this text, given that it is impossible to give a panoramic or exhaustive view of the subject, we decided to focus on some key moments that in a pioneering way have determined the development of Mathematical Finance due to contributions either on the analysis and dynamics of financial markets or on the closely related mathematical theory.

2 Starting at the beginning - Bachelier and a work ahead of its time

Bachelier was the first person to model the dynamics of stock prices based on random walks and their limit cases. Combining probability techniques with Markov property, and using the fact that the Gaussian kernel gives the fundamental solution of the heat equation, he was able to model and study the market noise of the Paris Stock Market.

He proposed to regard the stock prices $S = (S_t)_{t\geq 0}$ as a random (stochastic) process

$$S_t = S_0 + \sigma W_t, \qquad t \geq 0,$$

where $W = (W_t)_{t\geq 0}$ is a stochastic term describing the *noise,* that is, the random component of the phenomenon that is called the *Brownian motion* or *Wiener process.*

In the following model, still designated as Bachelier's model, the stock prices $S = (S_t)_{t\leq T}$ follow a Brownian motion with *drift*, that is,

$$S_t = S_0 + \mu t + \sigma W_t, \qquad t \leq T. \tag{2.1}$$

In (2.1), it is considered that there is a (B, S) −market such that the bank account $B = (B_t)_{t\leq T}$ remains fixed, $B_t = 1$. In a differential form, Bachelier's model can be written as

$$dS_t = \mu.dt + \sigma dW_t.$$

In his work, Bachelier gave the price for a *European option.* Recalling the definition, *option* is generally defined as "a contract between two parties in which one party has the right but not the obligation to buy or sell some underlying asset, at an agreed strike price K, at an assigned time T, called maturity, while the second party, the writer, has the obligation to sell or buy it if the first party wants to exercise his right". *Call options* are contracts that give the option holder the right to buy a given asset, while *put options*, conversely, entitle the holder to sell. Having rights without obligations has financial value. So, option holders must purchase these rights, that is, must pay a *premium.* This type of contract derives its value from some underlying asset; so, they are called *derivative assets* or simply *derivatives.*

At expiry or maturity, a call option is worthless if $S_T < K$ but has the value $S_T - K$ if $S_T > K$. This means in financial terms that its *payoff* is

$$(S_T - K)^+ = \max(S_T - K, 0).$$

If, at maturity, $S_T > K$, then the option buyer obtains a profit equal to $(S_T - K)$ since he can buy the stock at the price K and sell it immediately at S_T. On the other hand, if at maturity $S_T < K$, then the buyer simply does not exercise his right and his loss is just the premium paid C_T. Starting from Brownian motion, Bachelier derived a formula for the expectation of the *payoff* $(S_T - K)^+$ of a call option

$$C_T = E(S_T - K)^+$$

which gives us the value of the *reasonable (fair) price* to be paid by the buyer to the writer of a call option at the moment of the contract, that is, as referred above, the *premium.* Considering the density function and the normal distribution, respectively,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{and} \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy, \tag{2.2}$$

the following formula

$$C_T = (S_0 - K)\, \Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\varphi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) \tag{2.3}$$

is called *Bachelier's formula* (which is in fact an updated version of several of Bachelier's results on options). It defines the price C_T of the standard *European call option* with pay-off function $(S_T - K)^+$ for the Bachelier model (2.1). The main interest of this model, besides of course the historical aspect, lies in the fact that it is both *arbitrage free* (does not allow riskless profits) and *complete* (is replicable) [15].

Correlations between price assets and options can be used by the investor to construct a portfolio in such a way that risk can be reduced – *hedging strategy.* So, valuing options becomes of great importance.

We have referred above to *Brownian motion.* Originally named after the biologist Robert Brown, this term has two meanings: the physical phenomenon that describes the random movement of small particles immersed in a fluid and the one of the mathematical models (used, for instance, to describe that movement). In the physical context, it was first modelled by Albert Einstein in 1905 [5]. At that time, the molecular nature of matter was still a controversial idea. Einstein observed that, if the kinetic theory of fluids was right, every small particle of water would receive a random number of impacts of random strength and from random directions in any short period of time. This random bombardment would explain the jittering motion of small particles exactly in the way described by Brown.

However, five years before, in 1900, Louis Bachelier had already given a mathematical theory of Brownian motion in his doctoral thesis, using a stochastic process as a model for the price and relying on his belief in the power of the law of probability to explain the stock market

"*Si, a l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation a ceux de la théorie, ce n' eétait pas pour vérifier des formules etablies par les méthodes mathématiques, mais pour montrer seulement que le marché, a son insu, obéit a une loi qui le domine: la loi de la probabilité*".

Bachelier's work had little impact for a long time, in spite of the favourable report of his mentor, Henri Poincaré. His mathematical reasoning was not rigorous; and could not be, since the mathematical techniques used later to make it rigorous, that is, measure theory and axiomatic probability, had not been developed yet. But his results were basically correct. However, the "taste and goals" of the scientific elite of that time were not sensitive to mathematical applications for economics problems. Although there are some references to Bachelier results in later works of Kolmogorov, Doob and Itô, for example, it was only fifty years later that his thesis came to the limelight after having been "discovered" in the MIT library by the economist Paul Samuelson, Nobel Laureate in Economics in 1970. The impact of Bachelier's work in Samuelson's opinion can be clearly seen in his remark "*Bachelier seems to have had something of a one-track mind. But what a track*" [13] (see also [14]).

3 The classical Black-Scholes model

Bachelier's analytical valuation for options exhibited however a weakness as far as financial instruments were concerned, since the prices in the model could take negative values. In the 1960s, in order to overcome that weakness, Samuelson suggested modelling prices using what is now designated as geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \sigma^2/2\right)t} \tag{3.1}$$

and which, from Itô's calculus, can be written in the differential form

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right). \tag{3.2}$$

This suggestion provided a workable model for asset prices and anticipated the central result of modern finance, the Black-Scholes option-pricing formula. It is assumed that the bank account $B = (B_t)_{t\geq 0}$ evolves according to the formula

$$dB_t = rB_t dt,$$

where r is the interest rate, whereas the price of the risky asset evolves according to the stochastic differential equation (3.2) whose solution with initial condition S_0 is (3.1). The coefficients are constant: μ measures the global behaviour of S while the coefficient σ, called the *volatility,* measures the importance of the *noise,* that is, of the influence of the Brownian motion. The larger σ is, the greater the influence of Brownian motion. Expression (3.1) shows that $S(t) > 0$.

Assuming that the function $C = C(t, S)$ is sufficiently smooth, Fisher Black and Miron Scholes [2] and Robert Merton [12], working independently, obtained as model for the dynamics of a European call option the so-called *fundamental equation*

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{3.3}$$

with the final condition

$$C(T, S) = \max(S - K, 0). \tag{3.4}$$

An explicit solution can be determined using methods of partial differential equations that involve transforming this problem into the heat equation with an adequate condition. The solution, that is, the price of a call option at time t is given by

$$C(t, S) = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$

where Φ is defined in (2.2) and

$$d_\pm = \frac{\ln\frac{S}{K} + (T-t)\left(r \pm \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}},$$

and Black-Scholes Option Pricing Formula is

$$\begin{aligned} C_T \quad = \quad & S_0\Phi\left(\frac{\ln\frac{S_0}{K} + T\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}\right) \\ & - Ke^{-rT}\Phi\left(\frac{\ln\frac{S_0}{K} + T\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}\right). \end{aligned}$$

This result can be derived by a so-called *martingale* proof but using the solution of the fundamental equation was, in fact, the original proof established by Black, Scholes and Merton. If we look at Bachelier's formula, it can be easily recognized as a forerunner of the Black-Scholes formula.

On account of the above achievement, the Nobel prize in Economics was awarded to R. Merton and M. Scholes in 1997, thus also honoring F. Black (who died in 1995).

The above models concern continuous time. In 1976, three years after the Black-Scholes-Merton model, a model for discrete time was developed: the Cox-Ross-Rubinstein binomial model [3]. It assumed a risk-neutral world, that is, it recognized that investor risk preferences did not interfere in the pricing of derivatives. This model was simple, flexible and suitable for pricing American as well as European options. In fact, Black-Scholes-Merton gave an exact solution for European options, that could be exercised only at maturity, but was not able to provide values for American options that could be exercised before expiry. In general, solutions for American options can only be obtained numerically. This explains the important role played by numerical analysis and computational techniques in option pricing.

Asset and option pricing are fundamental elements in Portfolio Theory. This theory concerns the construction of portfolios, taking into account the benefits of diversification, so that expected returns may be optimized for a given level of market risk. Pioneering work concerning the Modern Portfolio Theory was carried out by Harry Markowitz [11], Nobel Laureate in Economics in 1990.

4 From finance to stochastic analysis

From the above paragraphs it is clear that financial phenomena and risky asset modelling are a wonderful playground for Mathematics.

A rigorous mathematical theory of Brownian motion was developed by Norbet Wiener in 1923 [16] by combining new results on measure theory with Fourier analysis. On account of those studies, Brownian motion is commonly referred to as *Wiener process.*

In the 1930s, a fundamental role was played by Kolmogorov. Among numerous major contributions made in a whole range of different areas of Mathematics, either pure or applied, he built up probability theory in a rigorous way, providing the axiomatic foundation on which the subject has been based ever since, and laid the foundations of the theory of Markov processes [10].

The concept of martingales in probability theory was introduced by Paul Pierre Lévy in the late 1930s. It was developed extensively by Doob who published a fundamental work on the subject [4] in 1953.

Considered as the father of stochastic integration, Kiyosi Itô published his first paper on the subject in 1944 [8]. Reflecting on the studies of Wiener and Kolmogorov and attempting to study the connection between partial differential operators, such as the heat operator, and Markov processes, he constructed a stochastic differential equation of the form

$$dX_t = \sigma (X_t) \, dW_t + \mu (X_t) \, dt$$

where W represents a standard Wiener process. This formula created two problems: the first one was to give sense to $\sigma (X_t) \, dW_t$ and the second one was to relate his work with Kolmogorov's results on Markov processes. He gave a positive answer to those problems. Namely, he developed a new calculus to solve the problem arising from the fact that Riemann-Stieljes integration was no longer valid. This new differential/integral calculus was named after his work as *Itô calculus.*

By 1980, arbitrage pricing theory had become well understood. A close link between nonexistence arbitrage opportunities and martingales was established in the so-called *Fundamental Theorem of Asset Pricing.* This theorem is due to Harrison, Kreps and Pliska [6, 7] and became a pioneering result after which many contributions appeared to improve the understanding of the subject. It points out that stochastic integration is extremely well suited to the study of stochastic processes arising in finance.

5 Conclusion

The financial world is fast-changing and needs constant updating in order to operate financial resources where new financial instruments and strategies are always appearing. Determination of opportunities is becoming more and more reliant on complex mathematics, which drives studies into new areas.

Among all the possible directions, presently most of them are related to incomplete markets, in which Black-Scholes style replication is impossible. Risk neutral world is no longer assumed and any pricing formula must involve some balance of the risks involved. Moving to incomplete markets means that mathematical finance must inevitably demand new approaches and lead to new developments in mathematical research.

In the above paragraphs we have tried to give the reader a combination of a slight flavour of some financial concepts and instruments that concern Financial Theory, together with a quick look at the Mathematics involved. Above all, we aim to provide a stimulating glance at the huge complexity and multidisciplinary features of the subject, as well as at the reciprocal challenges continuously appearing between practical and theoretical aspects.

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