Feature Article

Combining logic systems: Why, how, what for?

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1 Background

The practical significance of the problem of combining logics is widely recognized, namely in knowledge representation (within artificial intelligence) and in formal specification and verification of algorithms and protocols (within software engineering and security). In these fields, the need for working with several calculi at the same time is the rule rather than the exception. For instance, in a knowledge representation problem it may be necessary to work with temporal, spatial, deontic and probabilistic aspects (e.g., for reasoning with mixed assertions like "with probability greater than 0.99, sometime in the future smoking will be forbidden everywhere"). And in a software verification problem it may be necessary to mix equational, epistemic and dynamic logic features. That is, one needs, at least, to be able to develop theories with components in different logic systems, or, even better, to work with theories defined in the combination of those logic systems (where such mixed assertions are allowed).

Motivated by these applications that require the joint use of several deduction formalisms, the interest in combination of logic systems has recently been growing (as reflected in the series [9, 20, 2, 18, 26, 1]), but the topic is also of interest on purely theoretical grounds. For instance, one might be tempted to look at predicate temporal logic as resulting from the combination of first-order logic and propositional temporal logic. However, the approach will be significant only if general preservation results are available about the combination mechanism at hand. For example, if it has been established that completeness is preserved by a combination mechanism $\bullet$ and it is known that logic system $\mathcal{L}$ is given by $\mathcal{L}' \bullet \mathcal{L}''$, then the completeness of $\mathcal{L}$ follows from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. No wonder that much theoretical effort has been dedicated to establishing preservation results and/or finding preservation counterexamples about different combination mechanisms. For an early overview of the practical and theoretical issues see also [4].

Several forms of combination have been studied, like product [30, 21, 22, 23], fusion [38, 28, 29, 40, 19], temporalization [12, 13, 41, 14], parameterization [6], synchronization [33] and fibring [15, 16, 3, 17, 34, 42]. Fusion is the best understood combination mechanism. In short, the fusion of two modal systems leads to a bimodal system including the two original modal operators and common propositional connectives. Several interesting properties of logic systems (like soundness, weak completeness, Craig interpolation property and decidability) were shown to be preserved when fusing modal systems (see [28, 27]).

More recently, research has been directed at fibring, a more general combination mechanism proposed by Gabbay [15, 16]. Fibring can be applied beyond the universe of modal systems and captures fusion as a special case. Although well understood at the proof-theoretic level since it was proposed, fibring raised some difficulties at the semantic level [34].

For the sake of simplicity, we adopt here a basic universe of logic systems encompassing only propositional-based systems endowed with Hilbert calculi and ordered algebraic semantics. Nevertheless, this universe is rich enough to illustrate interesting features of fibring and to provide the basis for the combination of systems varying from intuitionistic to many-valued logics (including modal systems as special cases). Those interested in wider universes (encompassing first-order quantification, higher-order features, non truth-functional semantics, non Hilbert calculi, etc) where fibring can still be defined should look instead at [35, 36, 7, 5, 25, 31].

It is straightforward to define fibring in this basic universe. And, with respect to preservation results, we concentrate our attention on finding sufficient conditions only for the preservation of (strong global) completeness. Barring some examples and omitted proofs that can be found in the literature, the following presentation is self contained.

2 Logic systems

A *signature* C is an $\mathbb{N}$-indexed family of countable sets. The elements of each C_k are called *constructors* of arity k.

Let $T(C, \Xi)$ be the free algebra over C generated by Ξ. The *language* $L(C)$ is $T(C, \emptyset)$. We shall consider different signatures but we assume fixed once and for all a set Ξ of propositional variables. Fixed Ξ, the *schema language* $T(C, \Xi)$ is denoted by $sL(C)$.

A *rule* over C is a pair $r = \langle \Theta, \eta \rangle$ where $\Theta \cup \{\eta\} \subseteq sL(C)$. We shall work only with finitary rules, that is, we assume that the set Θ of premises is finite.

An *ordered algebra* over C is a tuple $\mathbf{A} = \langle A, \leq, \top, \cdot_{\mathbf{A}} \rangle$ where $\langle A, \leq, \top \rangle$ is a topped partial order and $\langle A, \cdot_{\mathbf{A}} \rangle$ is an algebra over C.

A *logic system* is a tuple $\mathcal{L} = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$ where C is a signature, $\mathcal{A}$ is a class of ordered algebras over C (the models of the system) and both R_ℓ and R_{g} are sets of rules over C. It is common to assume that the set of local rules R_ℓ is included in the set R_{g} of global rules.

As an example consider the following intuitionistic system. The signature contains the usual connectives. The class of models includes every ordered algebra induced by a Heyting algebra (with $a \leq b$ iff $a \wedge_{\mathbf{A}} b = a \sqcap b = a$). The local rules are the usual rules of a Hilbert calculus for intuitionistic propositional logic. Finally, there are no extra global rules. A detailed presentation of intuitionistic logic along these lines can be found in [32].

Consider also the example of the following modal system. The signature contains the usual basic propositional constants and connectives plus the modal operator $\Box$. The class of models includes every ordered algebra induced by a general Kripke structure $\langle W, \mathcal{B}, \rho, V \rangle$ as follows:

- $A = \mathcal{B}$; $a \leq b$ iff $a \subseteq b$; $\top = W$;
- $\pi_{\mathbf{A}} = V(\pi)$;
- $\neg_{\mathbf{A}}(a) = W \setminus a$;
- $\Rightarrow_{\mathbf{A}}(a, b) = (W \setminus a) \cup b$;
- $\Box_{\mathbf{A}}(a) = \{w \in W : w\rho v \text{ implies } v \in a \text{ for every } v \in W\}$.

(The notion of general Kripke structure was proposed in [39] in order to obtain a completeness theorem for modal logic.) A more direct approach would be to take as models the ordered algebras induced by modal algebras. The local rules include the classical propositional rules plus the normalization axiom

$$\langle \emptyset, (\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)) \rangle .$$

The unique extra global rule is the necessitation rule $\langle \{\xi_1\}, \Box\xi_1 \rangle$.

Many other interesting logics (even many-valued ones like Gödel's and Łukasiewicz's — see for instance [24]) are also logic systems in the sense given above.

Within the context of a logic system, the denotation $[\![\varphi]\!]^\alpha_{\mathbf{A}}$ of a schema formula φ on an ordered algebra $\mathbf{A}$ and for an assignment $\alpha : \Xi \to A$ is easily defined by induction on the structure of φ.

In any given logic system $\mathcal{L} = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}} \rangle$ we are able to define the following four consequence operators:

- global entailment: $\Gamma \vDash^{\mathrm{g}} \varphi$ iff, for every $\mathbf{A} \in \mathcal{A}$ and $\alpha : \Xi \to A$, if $\top \leq [\![\gamma]\!]^\alpha_{\mathbf{A}}$ for each $\gamma \in \Gamma$ then $\top \leq [\![\varphi]\!]^\alpha_{\mathbf{A}}$;
- local entailment: $\Gamma \vDash^{\ell} \varphi$ iff, for every $\mathbf{A} \in \mathcal{A}$, $\alpha : \Xi \to A$ and $a \in A$, if $a \leq [\![\gamma]\!]^\alpha_{\mathbf{A}}$ for each $\gamma \in \Gamma$ then $a \leq [\![\varphi]\!]^\alpha_{\mathbf{A}}$;
- global derivation: $\Gamma \vdash^{\mathrm{g}} \varphi$ iff φ can be derived from Γ using the rules in R_{g};
- local derivation: $\Gamma \vdash^{\ell} \varphi$ iff φ can be derived from Γ and theorems (formulae globally derived from an empty set of assumptions) using only the rules in R_ℓ.

Observe that in the modal system described above we can globally derive $(\Box\xi_1) \Rightarrow (\Box\xi_2)$ from $\xi_1 \Rightarrow \xi_2$ but we can not do so locally. The distinction between local and global reasoning appeared in the context of modal logic (local means carried out at a single world and global refers to reasoning about all worlds) but can be useful in other universes.

A logic system is said to be *strongly globally sound* when if $\Gamma \vdash^{\mathrm{g}} \varphi$ then $\Gamma \vDash^{\mathrm{g}} \varphi$. And it is said to be *strongly globally complete* when if $\Gamma \vDash^{\mathrm{g}} \varphi$ then $\Gamma \vdash^{\mathrm{g}} \varphi$. When we only consider $\Gamma = \emptyset$ we get the corresponding weak notions. Mutatis mutandis, we define the local versions.

3 Completeness theorem

A logic system is said to be *full* when $\mathcal{A}$ is composed of all ordered algebras over C that fulfill the rules in both R_ℓ and R_{g}. Therefore, every full logic system is (weakly and strongly, locally and globally) sound. A logic system has *verum* if its language contains a theorem that denotes $\top$ in every model.

A logic system is said to be *congruent* when for every Γ closed for global derivation, $c \in C_k$ and $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k \in sL(C)$:

$$\frac{\begin{array}{ll}\Gamma, \varphi_i \vdash^\ell \psi_i & \\ \Gamma, \psi_i \vdash^\ell \varphi_i & i = 1, \ldots, k\end{array}}{\Gamma, c(\varphi_1, \ldots, \varphi_k) \vdash^\ell c(\psi_1, \ldots, \psi_k)}$$

Theorem 3.1 Every full and congruent logic system with verum is strongly globally complete.

The proof is carried out using a Lindenbaum-Tarski construction. A syntactic ordered algebra $\mathbf{A}_\Gamma$ can be built as follows from each Γ closed for $\vdash^{\mathrm{g}}$. First we define a congruence relation over $sL(C)$: $\varphi \cong_\Gamma \psi$ iff $\Gamma, \varphi \vdash^\ell \psi$ and $\Gamma, \psi \vdash^\ell \varphi$. Then, we choose A to be $sL(C)/\cong_\Gamma$. The partial order is defined as follows: $[\varphi]_\Gamma \leq [\psi]_\Gamma$ iff $\Gamma, \varphi \vdash^\ell \psi$. The top $\top$ is the equivalence class of the verum. Finally, for each language constructor, $c_{\mathbf{A}_\Gamma}([\varphi_1]_\Gamma, \ldots, [\varphi_k]_\Gamma) = [c(\varphi_1, \ldots, \varphi_k)]_\Gamma$. Clearly, by construction, we infer that $[\![\varphi]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}_\Gamma} = \top$ iff $\varphi \in \Gamma$ and that $\mathbf{A}_\Gamma$ fulfills the rules of the logic system.

Assume $\Delta \not\vdash^{\mathrm{g}} \epsilon$. We have to show $\Delta \not\models^{\mathrm{g}} \epsilon$. It is sufficient to find an ordered algebra $\mathbf{A} \in \mathcal{A}$ such that $[\![\delta]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}} = \top$ for each $\delta \in \Delta$ and $[\![\epsilon]\!]^{\lambda\xi.[\xi]_\Gamma}_{\mathbf{A}_\Gamma} \neq \top$. Consider $\Gamma = \Delta^{\vdash^{\mathrm{g}}}$. Then, $\mathbf{A}_\Gamma$ globally satisfies each element of Δ (since $\Delta \subseteq \Gamma$) but $\mathbf{A}_\Gamma$ does not globally satisfy ϵ (since $\epsilon \notin \Gamma$). This concludes the proof of the completeness theorem.

Observe that the requirements for completeness are quite weak and usually fulfilled by commonly used logic systems (including those mentioned above as examples). Furthermore, any complete logic system can be made full without changing its entailments. And if verum is not present, it can be conservatively added to the language. But if the system at hand is not congruent, there is nothing we can do within the scope of the basic theory of fibring outlined here.

Note also that through a mild strengthening of the requirements of the theorem we can ensure finitary strong local completeness (see for instance [37]). A similar strong (local and global) completeness theorem is obtained in [42] without extra requirements for local reasoning but assuming a more complex semantics and using a Henkin construction.

4 Fibring

Consider signatures C and C' such that $C'_k \subseteq C_k$ for each $k \in \mathbb{N}$. Given an ordered algebra $\mathbf{A}$ over C, we denote by $\mathbf{A}|_{C'}$ the reduct $\langle A, \leq, \top, \cdot_{\mathbf{A}}|_{C'}\rangle$ of $\mathbf{A}$ by the inclusion (where $\cdot_{\mathbf{A}}|_{C'}$ is the restriction of $\cdot_{\mathbf{A}}$ to C'). Clearly, $\mathbf{A}|_{C'}$ is an ordered algebra over C'.

Given two logic systems $\mathcal{L}' = \langle C', \mathcal{A}', R_\ell{}', R_{\mathrm{g}}{}'\rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{A}'', R_\ell{}'', R_{\mathrm{g}}{}''\rangle$, their *fibring* $\mathcal{L}' \odot \mathcal{L}'' = \langle C, \mathcal{A}, R_\ell, R_{\mathrm{g}}\rangle$ is as follows:

- $C_k = C'_k \cup C''_k$ for each $k \in \mathbb{N}$;
- $\mathcal{A}$ is the class containing every ordered algebra $\mathbf{A}$ over C such that $\mathbf{A}|_{C'} \in \mathcal{A}'$ and $\mathbf{A}|_{C''} \in \mathcal{A}''$;
- $R_\ell = R_\ell{}' \cup R_\ell{}''$; $R_{\mathrm{g}} = R_{\mathrm{g}}{}' \cup R_{\mathrm{g}}{}''$.

This definition corresponds to the constrained version of fibring (as defined in [34]) since any symbols common to both logic systems will be shared. Unconstrained fibring can be obtained by making sure that no symbols are in both signatures. Fibring can appear as a universal construction in a suitable category of logic systems (as explored in [34] where the categorical approach was important in fine tuning the semantics of fibring).

As a first example of fibring, consider the combination of two modal systems while sharing the propositional connectives. This constrained fibring is equivalent to the fusion of the two given modal systems. The result is a bimodal system.

The combination of a modal system with a relevance system is similar from the point of view of fibring but beyond the scope of fusion. By sharing the propositional connectives we obtain a logic system with a modal box and a relevance implication. For details about relevance logic see for instance [11].

Note that, even when no symbols are shared, fibring may impose unexpected interactions between the logical operations from the two given logics. For instance, consider the unconstrained fibring of classical propositional logic and intuitionistic propositional logic. Unexpectedly, in the resulting logic system the intuitionistic implication collapses into classical implication. In short, in the resulting logic system we have two copies of classical logic. This first example of collapsing was first identified in [10]. Other examples are given in [37] where a relaxed notion of fibring is proposed in order to avoid such collapses.

5 Preservation results

We now turn our attention to transference results. We start by examining if soundness is preserved by fibring. Then we consider completeness. To this end we have to establish the preservation of other interesting properties, namely the metatheorem of deduction.

Theorem 5.1 Soundness is preserved by fibring.

It is straightforward to prove that (strong and weak, global and local) soundness is unconditionally preserved by fibring in the basic universe of logic systems considered here. However, in larger universes things can be more complicated. For instance, when fibring logic systems with quantifiers and using rules with side provisos (like, provided that term θ is free for variable x in formula ξ), soundness is not always preserved [36, 7].

Weak completeness is also not always preserved as shown in [42]. Herein we examine in detail if strong global completeness is preserved when fibring basic logic systems as defined above. Adapting the technique originally proposed in [42], we capitalize on the completeness theorem stated above about such logic systems. That is, when fibring two given logic systems that are full, congruent and with verum (and, therefore, strongly globally complete) we shall try to obtain the strong global completeness of the result by identifying the conditions under which fullness, congruence and verum are preserved by fibring.

Lemma 5.2 Fullness is preserved by fibring.

Lemma 5.3 The result of fibring has verum provided that at least one of the given logic systems has verum.

However, congruence is not always preserved by fibring. Consider the fibring of two logic systems $\mathcal{L}', \mathcal{L}''$ with the following signatures and rules:

$$C_0' = \{\pi_0, \pi_1, \pi_2\} \quad C_1' = \{c\} \quad C_k' = \emptyset \text{ for } k > 1$$
$$R_\ell{}' = \emptyset \quad R_{\rm g}{}' = \{\langle\{\xi\}, c(\xi)\rangle\}$$
$$C_0'' = \{\pi_0, \pi_1, \pi_2\} \quad C_k'' = \emptyset \text{ for } k > 0$$
$$R_\ell{}'' = R_{\rm g}{}'' = \{\langle\{\pi_0, \pi_1\}, \pi_2\rangle, \langle\{\pi_0, \pi_2\}, \pi_1\rangle\}$$

Clearly, both $\mathcal{L}'$ and $\mathcal{L}''$ are congruent, but their fibring $\mathcal{L} = \mathcal{L}' \odot \mathcal{L}''$ is not congruent. Indeed, consider $\Gamma = \{\pi_0\}^{\vdash^{\rm g}} = \{c^n(\pi_0) : n \geq 0\}$. So, from Γ, π_1 and π_2 are locally interderivable but, from Γ, $c(\pi_1)$ and $c(\pi_2)$ are not locally interderivable.

Fortunately, it is possible to establish a useful sufficient condition for the preservation of congruence by fibring. A logic system is said to have *implication* if its signature contains a binary connective $\Rightarrow$ fulfilling the following Metatheorem of Modus Ponens (MTMP)

$$\frac{\Gamma \vdash^\ell (\delta_1 \Rightarrow \delta_2)}{\Gamma, \delta_1 \vdash^\ell \delta_2}$$

and the following Metatheorem of Deduction (MTD):

$$\frac{\Gamma^{\vdash^{\rm g}}, \delta_1 \vdash^\ell \delta_2}{\Gamma^{\vdash^{\rm g}} \vdash^\ell (\delta_1 \Rightarrow \delta_2)}\,.$$

When fibring two logic systems with implication while sharing the implication symbol, it is straightforward to verify that the resulting logic system also has implication. Indeed:

Theorem 5.4 The result of fibring has MTMP provided that at least one of the given logic systems has MTMP and the implication symbol is shared.

Theorem 5.5 The result of fibring has MTD provided that both given logic systems have MTD and the implication symbol is shared.

The latter result is a direct corollary of the following fact:

Lemma 5.6 MTD holds in a logic system iff: (i) $\vdash^\ell (\xi \Rightarrow \xi)$; (ii) $\{\xi_1\}^{\vdash^{\rm g}} \vdash^\ell (\xi_2 \Rightarrow \xi_1)$; and (iii) $\{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\}^{\vdash^{\rm g}} \vdash^\ell (\xi \Rightarrow \gamma)$ for each local rule $\langle\{\gamma_1, \ldots, \gamma_k\}, \gamma\rangle$.

A logic system is said to have *equivalence* if it has implication and its signature contains a binary connective $\Leftrightarrow$ fulfilling the two Metatheorems of Biconditionality (relating implication with equivalence) and the Metatheorem of Substitution of Equivalents (MTSE).

Theorem 5.7 A logic system with equivalence is congruent.

When fibring two logic systems with equivalence while sharing the implication symbol as well as the equivalence symbol we obtain a logic system with equivalence. Therefore:

Theorem 5.8 The fibring while sharing implication and equivalence of full logic systems with equivalence and verum is strongly globally complete.

This preservation result is quite useful because many widely used logic systems do have equivalence in the sense above.

6 Final remarks

In this guided tour of the issues raised by the combination of logics we defined fibring in a very simple (yet useful) context and established some interesting transference results. As already mentioned, fibring can and has been defined and analyzed in much more complex situations. Current research is directed at widening the universe where fibring can be defined and at establishing other transference results like sufficient conditions for the preservation of interpolation properties, weak completeness and decidability. Concerning conditions for the preservation of weak completeness, it is still an open problem if the ghost symbol technique (used in [28] for proving the preservation of weak completeness by fusion) can be generalized in order to be used for fibring. With respect to the preservation of interpolation properties, the recent results in [8] seem to provide an appropriate context.

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