Warp Drive With Zero Expansion

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Introduction

As everyone knows, Einstein's Relativity forbids all material objects (or even signals) to travel faster than light. What is sometimes ignored is that this is a *local* statement: speed with respect to an observer can only be defined in a neighborhood of this observer. For instance, it is well known that the universe is expanding, all galaxies (on average) speeding away from each other. An analogy which is particularly well suited is the surface of an expanding balloon, with the galaxies as points on this surface. Although the galaxies are not moving with respect to the balloon's surface, the distance between them is increasing; if they are sufficiently far apart (i.e., if the balloon is large enough), then the distance will increase faster than 300000 kilometers per second. So in a sense they will be moving faster than the speed of light with respect to each other. This is indeed what happens with galaxies at the edge of the visible universe. The "thou shall not travel faster than light" commandment in this analogy simply forbids objects to travel faster than light with respect to the balloon's surface. (Incidentally, this analogy also shows that there is no "center of the universe" where the Big Bang occurred; the Big Bang simply means the epoch where the balloon was very small and very hot - in a sense it happened in all points of space).

These ideas were used by Miguel Alcubierre ([Alc94]) to construct (in theory) a "warp drive", allowing a spaceship to travel faster than light, by deforming space in the following manner: take a ball containing your spaceship (the "warp bubble"), and keep it undeformed; contract space in front of the bubble, expand space behind it. Since there is no *a priori* constraint on the speed of contraction/expansion, it is possible to move the bubble from one point to another as quickly as one wishes.

In what follows we will explain exactly how this is done within the mathematical framework General Relativity, show how it can be generalized and see how this attempt at circumventing Einstein's prohibition is doomed to fail.

General Relativity

General Relativity is the physical theory of space, time and gravitation. It states that spacetime is a 4-dimensional Lorentzian manifold (i.e., a pseudo-Riemannian manifold (M, g) for which the metric g has signature $(-, +, +, +)$), satisfying the *Einstein equation*

$$G = 8\pi T,$$

where the *Einstein tensor* G is just the trace-reversed Ricci tensor,

$$G = R - \frac{\operatorname{tr} R}{2} g,$$

and the *energy-momentum tensor* T describes the matter content of the spacetime. Thus any 4-dimensional Lorentzian manifold can be thought of as a spacetime containing the matter described by

$$T = \frac{1}{8\pi} G.$$

However, an arbitrary choice is almost certain to generate an unphysical energy-momentum tensor.

A nonzero tangent vector $v \in TM$ is said to be *timelike*, *lightlike* or *spacelike* according to whether $g(v, v) < 0$, $g(v, v) = 0$ or $g(v, v) > 0$ (the zero vector is by definition spacelike). A curve $c : \mathbb{R} \to M$ whose tangent vector $\dot{c}$ remains in one of the above classes is given the same name. Timelike curves are interpreted as possible histories of test particles with nonvanishing rest mass (which must travel slower than light); the length

$$\tau = \int_{t_0}^{t_1} |g(\dot{c}(t), \dot{c}(t))|^{\frac{1}{2}} dt$$

is then the time measured by the particle between the events $c(t_0)$ and $c(t_1)$. Lightlike curves are interpreted as possible histories of test particles with vanishing rest mass (which must travel at the speed of light, e.g., photons or neutrinos).

If c is a geodesic, i.e.,

$$\nabla_{\dot{c}}\dot{c} = 0,$$

then

$$\frac{d}{dt}\left[g(\dot{c},\dot{c})\right] = 2g\left(\dot{c},\nabla_{\dot{c}}\dot{c}\right) = 0.$$

Thus geodesics are always curves of a given type. Timelike geodesics are interpreted as the histories of free-falling test particles with nonzero rest mass; the fact that they are geodesics means that free-falling particles measure *more* time between any two (sufficiently close) events than any other particle. Lightlike geodesics are interpreted as the histories of free-falling test particles with zero rest mass; the extremality property in this case is that no other massive or massless particle can travel between two (sufficiently close) events on the lightlike geodesic.

Unlike Newtonian mechanics, General Relativity provides no canonical way of splitting spacetime into space plus time. A possible choice is to take an arbitrary *spacelike hypersurface*, i.e., a hypersurface $\Sigma \subset M$ whose orthogonal vector field is timelike, and consider its evolution along the orthogonal geodesics. A local chart (x^1, x^2, x^3) on Σ can therefore be extended to a local chart (t, x^1, x^2, x^3) on M (from this point on called an *Eulerian chart*), where t is the arclength (time) measured along the orthogonal (timelike) geodesic. In these local coordinates, the metric is just

$$g = -dt \otimes dt + \gamma(t),$$

where

$$\gamma(t) = g_{ij}(t, x^1, x^2, x^3)dx^i \otimes dx^j$$

must be a Riemannian metric on Σ (we are using the summation convention on the indices $i, j = 1, 2, 3$). This allows us to interpret General Relativity as describing the evolution of a Riemmannian metric $\gamma(t)$ on the 3-dimensional manifold Σ. This metric yields the distances measured between nearby Eulerian observers.

The Einstein equation can then be formulated in terms of γ and the *extrinsic curvature*

$$K = \frac{1}{2}\frac{\partial\gamma}{\partial t}.$$

It implies

$$\frac{\partial}{\partial t}(\operatorname{tr} K) - \operatorname{tr}(K^2) = -8\pi\left(T_{00} - \frac{1}{2}\operatorname{tr} T\right);$$
$$\operatorname{tr} R + (\operatorname{tr} K)^2 - \operatorname{tr}(K^2) = 16\pi T_{00},$$

where $R = R(t)$ is now the Ricci tensor of (Σ, γ) (not (M, g)).

It is also possible to show that

$$\operatorname{tr} K = \frac{1}{(\det\gamma)^{\frac{1}{2}}}\frac{\partial}{\partial t}\left[(\det\gamma)^{\frac{1}{2}}\right].$$

In other words, $\operatorname{tr} K$ measures the fractional variation of the volume element for Eulerian observers: $\operatorname{tr} K < 0$ in some region means that the volume of that region is decreasing.

Most models of matter are described by energy-momentum tensors satisfying both the *strong energy condition*, which implies

$$T_{00} - \frac{1}{2}\operatorname{tr} T \geq 0,$$

and the *weak energy condition*, which implies

$$T_{00} \geq 0$$

(confusingly the strong energy condition does not imply the weak energy condition). If T satisfies the strong energy condition and $\operatorname{tr} K$ does not vanish at some point then our Eulerian chart must break down at some value of t: indeed, in this case the Einstein equation implies that

$$\frac{\partial}{\partial t}(\operatorname{tr} K) - \operatorname{tr}(K^2) \leq 0.$$

Using the inequality

$$(\operatorname{tr} A)^2 \leq n \operatorname{tr}(A^2)$$

(which holds for any real $n \times n$ symmetric matrix A) one can easily prove that starting from a nonzero value $\operatorname{tr} K$ must blow up in finite time. This breaking down of the Eulerian chart can either be a coordinate singularity or a genuine geometric singularity (meaning that M is geodesically incomplete); indeed this can be thought of as a primitive version of the famous Penrose-Hawking singularity theorems.

Warp Drive Spacetimes

We will now describe a class of spacetimes which can be understood simply by studying a (time-dependent) vector field in Euclidean 3-space. These will then be used to construct our warp drives. If you find what follows a bit too technical you can turn to the short summary in the beginning of section 4.

Definition 1. *A* warp drive spacetime (M, g) *is defined by taking* $M = \mathbb{R}^4$ *with the usual Cartesian coordinates* $(t, x, y, z) \equiv (t, x^i)$ *and*

$$g = -dt \otimes dt + \sum_{i=1}^{3}(dx^i - X^i dt) \otimes (dx^i - X^i dt)$$

for three unspecified bounded smooth functions $(X^i) \equiv (X, Y, Z)$.

The Riemannian metric γ induced in the spacelike hypersurfaces $\{dt = 0\}$ is just the ordinary Euclidean flat metric. A warp drive spacetime is completely defined by the vector field

$$\mathbf{X} = X^i \frac{\partial}{\partial x^i} = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z},$$

which we can think of as a (time-dependent) vector field defined in Euclidean 3-space.

Notice carefully that the chart (t, x^i) is *not* an Eulerian chart; Eulerian observers' histories are integral curves of the unit normal vector to the $\{dt = 0\}$ hypersurfaces,

$$n = \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} + \mathbf{X}.$$

Proposition 2. *The extrinsic curvature tensor is*

$$K = \frac{1}{2}\left(\partial_i X^j + \partial_j X^i\right) dx^i \otimes dx^j.$$

Proof. The extrinsic curvature tensor is given by

$$K = \frac{1}{2} \pounds_n \gamma = \frac{1}{2} \pounds_{\left(\frac{\partial}{\partial t} + \mathbf{X}\right)} \gamma.$$

Now

$$\pounds_{\frac{\partial}{\partial t}} \gamma = \pounds_{\frac{\partial}{\partial t}} \delta_{ij} dx^i \otimes dx^j = \frac{\partial \delta_{ij}}{\partial t} dx^i \otimes dx^j = 0$$

(where δ_{ij} is the Kronecker delta). On the other hand, since $\mathbf{X}$ is tangent to the spacelike hypersurfaces, we can use the usual formula for the Lie derivative of the metric,

$$\pounds_{\mathbf{X}} \gamma = \left(\delta_{kj} D_i X^k + \delta_{ik} D_j X^k\right) dx^i \otimes dx^j =$$

$$\left(D_i X^j + D_j X^i\right) dx^i \otimes dx^j,$$

where D stands for the Levi-Civita connection determined by γ. Since γ is just the flat Euclidean metric, $D = \partial$ and we get the formula above. □

Corollary 3. *The expansion of the volume element associated with the Eulerian observers is given by* $\nabla \cdot \mathbf{X}$.

Proof. We just have to notice that

$$\operatorname{tr} K = K^i{}_i = \partial_i X^i.$$

□

Corollary 4. *A warp drive spacetime is flat wherever* $\mathbf{X}$ *is a Killing vector field for the Euclidean metric (irrespective of time dependence). In particular, a warp drive spacetime is flat wherever* $\mathbf{X}$ *is spatially constant.*

Proof. Since the spacelike surfaces are flat, all curvature comes from the extrinsic curvature. Thus the spacetime will be flat wherever the extrinsic curvature is zero, *i.e.*, wherever $\pounds_{\mathbf{X}} \gamma = 0$. □

In particular, the Einstein equation implies that there is no matter in these regions. Also there is no geodesic deviation, and hence no tidal forces.

Theorem 5. *Non flat warp drive spacetimes violate the weak or the strong energy condition.*

Proof. We already know that if the strong energy condition holds and $\operatorname{tr} K \neq 0$ at some event, then $\operatorname{tr} K$ blows up in finite time. Since $\nabla \cdot \mathbf{X}$ is finite, the strong energy condition can only hold if $\operatorname{tr} K \equiv 0$. However, it follows from the Einstein equation that

$$T_{00} = \frac{1}{16\pi}\left(\operatorname{tr} R + (\operatorname{tr} K)^2 - \operatorname{tr}(K^2)\right)$$

where $R = 0$ is the Ricci tensor of the flat Cauchy surfaces $dt = 0$. Thus if $\operatorname{tr} K = 0$ we have $T_{00} \leq 0$, and $T_{00} = 0$ *iff* $K \equiv 0$. Consequently if the spacetime does not violate neither the strong nor the weak energy conditions it must be flat. □

Warp Drive With Zero Expansion

We have seen in the previous section that given a time-dependent smooth bounded vector field $\mathbf{X}$ in Euclidean 3-space we can construct a Lorentzian manifold with a global chart $\{t, x^i\}$ having the following properties:

1. The space sections $\{dt = 0\}$ are just Euclidean 3-space;
2. The free-fall Eulerian observers move in this Euclidean 3-space with velocity $\mathbf{X}$;
3. The fractional volume variation of these observers is (unsurprisingly) $\nabla \cdot \mathbf{X}$;
4. There exists no matter nor tidal forces wherever $\mathbf{X}$ is spatially constant;
5. Unfortunately, the strong or weak energy conditions are always violated.

If

$$\dot{c} = \dot{t} \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i}$$

is the tangent vector to a timelike geodesic, we must have

$$g(\dot{c}, \dot{c}) < 0 \Leftrightarrow -\dot{t}^2 + \sum_{i=1}^{3} (\dot{x}^i - X^i \dot{t})^2 < 0 \Leftrightarrow \left\| \frac{d\mathbf{x}}{dt} - \mathbf{X} \right\| < 1,$$

where

$$\mathbf{x} = x^i \frac{\partial}{\partial x^i} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}.$$

This can be readily interpreted as meaning that the speed of any test particle with nonvanishing rest mass with respect to the Eulerian observers must be smaller than the speed of light (which is normalized to 1). However, *there is no a priori limit for the speed* $\mathbf{X}$ *of the Eulerian observers themselves.* This was used by Miguel Alcubierre ([Alc94]) to construct the following example of a spacetime in which superluminal travel is possible:

Example 6. *Choose*

$$X = v_s f(r_s);$$
$$Y = Z = 0,$$

with

$$v_s(t) = \frac{dx_s(t)}{dt};$$
$$r_s = \left[(x - x_s)^2 + y^2 + z^2\right]^{\frac{1}{2}},$$

where $x_s(t)$ *is arbitrary and* $f : [0, +\infty) \to [0, 1]$ *is a smooth function such that* $f' \leq 0$, $f = 1$ *in a neighborhood of the origin and* $f = 0$ *in a neighborhood of infinity. Let us call the region* $f(r_s) = 1$ *the* interior of the warp bubble *and the region* $f(r_s) = 0$ *the* exterior of the warp bubble*. In both these regions* $\mathbf{X}$ *is spatially constant, and hence they contain no matter and generate no tidal forces; nevertheless, Eulerian observers inside the warp bubble move with arbitrary speed* v_s *with respect to Eulerian observers outside the warp bubble (there is no reason why* v_s *should be smaller than* 1*).*

The expansion of the volume element associated with the Eulerian observers in this example is

$$\operatorname{tr} K = \partial_x X = v_s f'(r_s) \frac{x - x_s}{r_s}.$$

Since $f' \leq 0$*, we see that volume is decreasing in front of the bubble and increasing behind it. This compression/expansion was thought to be a fundamental ingredient in the warp drive mechanism; we will presently see that it's not. Alcubierre also found that the energy conditions were violated at the bubble's wall (i.e., the region where* $f' \neq 0$*), as we now know to be unavoidable.*

It is convenient to replace the x *coordinate with*

$$\xi = x - x_s(t).$$

This effectively corresponds to replacing X *with* $X - v_s$*, so that the Eulerian observers inside the bubble stand still whereas the Eulerian observers outside the bubble move with speed* v_s *in the negative* ξ*-direction. Obviously* $\operatorname{tr} K$ *retains its value, but now*

$$r_s = \left(\xi^2 + y^2 + z^2\right)^{\frac{1}{2}}$$

does not depend on the coordinate t*.*

Definition 7. *The vector field* $\mathbf{X}$ *is said to* generate a warp bubble with velocity $\mathbf{v}_s(t)$ *if* $\mathbf{X} = \mathbf{0}$ *for small* $\|\mathbf{x}\|$ *(the* interior of the warp bubble*) and* $\mathbf{X} = -\mathbf{v}_s(t)$ *for large* $\|\mathbf{x}\|$ *(the m exterior of the warp bubble)*

To construct a warp drive with zero expansion all one has to do then is to find a divergenceless field generating a warp bubble with velocity $v_s(t)\frac{\partial}{\partial x}$ (see [Nat02] for details on how to do this).

The Alcubierre warp drive can be pictured as contracting space in front of the warp bubble and expanding it behind; a zero expansion warp drive can be thought of as sliding the warp bubble through normal space.

Lightlike geodesics and horizons

Besides violating energy conditions, warp drive spacetimes have much more serious problems, namely horizons. To see evidence of this, let us consider the case of a vector field $\mathbf{X}$ generating a warp bubble with velocity $v_s \frac{\partial}{\partial x}$ satisfying

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{0} \ \left(\Rightarrow \frac{dv_s}{dt} = 0 \right).$$

Since null geodesics must satisfy

$$g(\dot{c}, \dot{c}) = 0 \Leftrightarrow dt^2 = \sum_{i=1}^{3} (dx^i - X^i dt)^2 \Leftrightarrow \left\| \frac{d\mathbf{x}}{dt} - \mathbf{X} \right\| = 1,$$

we see that a flash of light outside the warp bubble can be pictured in the Euclidean 3-space as a spherical wavefront which is simultaneously expanding with speed 1 and moving in the direction of $\mathbf{X}$ with speed $\|\mathbf{X}\| = v_s$. Thus it is clear that if $v_s > 1$ then events inside the warp bubble cannot causally influence events outside the warp bubble at large positive values of x, as no particle emitted from inside the bubble can reach those points. Assuming cylindrical symmetry about the x-axis, there will be a point on the positive x-axis where $\|\mathbf{X}\| = 1$; the cylindrically symmetric surface through this point whose angle α with $\mathbf{X}$ is given by

$$\sin \alpha = \frac{1}{\|\mathbf{X}\|}$$

is a horizon, in the sense that events inside the warp bubble cannot causally influence events on the other side of this surface (see figure 1). Notice that away from the warp bubble we have

$$\sin \alpha = \frac{1}{v_s}$$

which is the familiar expression for the Mach cone angle. Also notice that the interior of the warp bubble is causally disconnected from part of the bubble's wall. This is the so-called you-need-one-to-make-one problem with the warp drive: the warp bubble wall, where your (unphysical) matter fields live, cannot be generated from inside the bubble. You'd need someone who was already traveling faster than light to generate it for you.

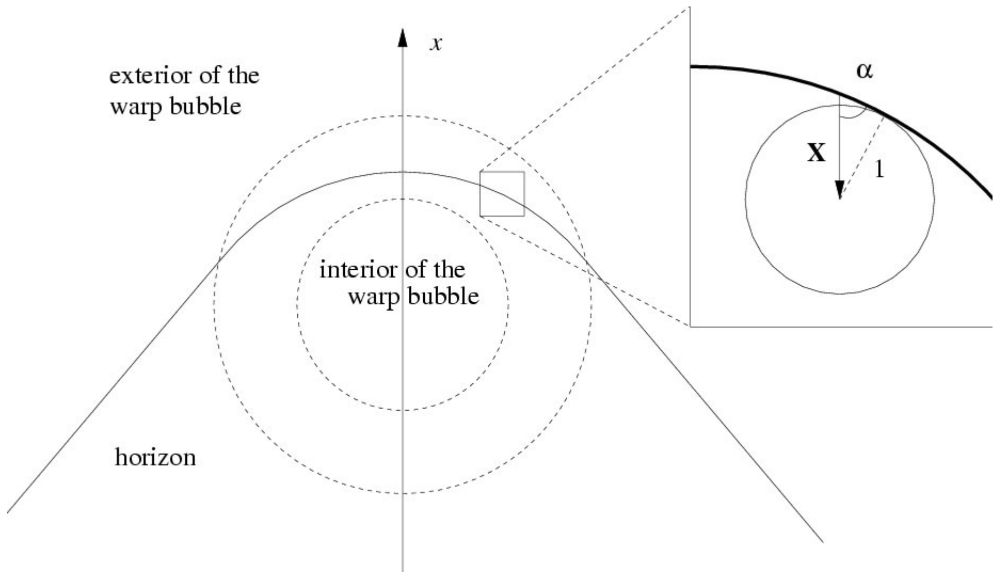

Figure 1: Computing the horizon.

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