Regularity for Partial Differential Equations: from De Giorgi-Nash-Moser Theory to Intrinsic Scaling

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1 A beautiful problem

In the academic year 1956-1957, John Nash had a visiting position at the Institute for Advanced Study (IAS) in Princeton, on a sabbatical leave from MIT, but he actually lived in New York City. The IAS at the time "was known to be about the dullest place you could find"[1] and Nash used to hang around the Courant Institute which was close to home and full of activity. That's how he came across a problem that mathematicians had been trying to solve for quite a while. The story goes that Louis Nirenberg, at the time a young professor at Courant, was the person responsible for the unveiling: "...it was a problem that I was interested in and tried to solve. I knew lots of people interested in this problem, so I might have suggested it to him, but I'm not absolutely sure", said Nirenberg recently in an interview to the Notices of the AMS (cf. [19]).

As so many other great questions of 20th century mathematics, it all started with one of Hilbert's problems presented on the occasion of the 1900 International Congress of Mathematicians in Paris, namely the 19th problem: *Are the solutions of regular problems in the calculus of variations always necessarily analytic?* A simple example of such a problem is, in modern terminology, the problem of minimizing a functional

$$\min_{w\in\mathcal{A}} \int_\Omega L(\nabla w(x))\,\mathrm{d}x$$

where $\Omega \subset \mathbf{R}^n$ is a bounded and smooth domain, the Lagrangian $L(\xi)$ is a smooth (possibly nonlinear) scalar function defined on $\mathbf{R}^n$ and $\mathcal{A}$ is a set of admissible functions (typically the elements of a certain function space satisfying a boundary condition like $w = g$ on $\partial\Omega$, for a given g). The question is to prove that, given the smoothness of L, the minimizer (assuming it exists) is also smooth.

[1] Cathleen Morawetz, quoted in [15].

Problems of this type are related to elliptic equations in that a minimizer u is a weak solution of the associated Euler-Lagrange equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} L_{\xi_i}(\nabla u(x)) = 0 \quad \text{in} \quad \Omega\ .$$

This equation can be differentiated with respect to x_k, to give that, for any $k = 1, 2, \ldots, n$, the partial derivative $\frac{\partial u}{\partial x_k} := v_k$ satisfies a linear PDE of the form

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\, \frac{\partial v_k}{\partial x_j}\Big) = 0\ , \qquad (1)$$

with coefficients $a_{ij}(x) := L_{\xi_i\xi_j}(\nabla u(x))$. The PDE is elliptic provided L is assumed to be convex.

In the 1950's, regularity theory for elliptic equations was essentially based on Schauder's estimates which, roughly speaking, guarantee that if $a_{ij} \in C^{k,\alpha}$ then the solutions of (1) are of class $C^{k+1,\alpha}$, for $k = 0, 1, \ldots$ So if it could be shown that $u \in C^{1,\alpha}$ then $a_{ij}(x) := L_{\xi_i\xi_j}(\nabla u(x))$ would belong to $C^{0,\alpha}$, v to $C^{1,\alpha}$ and u to $C^{2,\alpha}$; a bootstrap argument would then solve Hilbert's 19th problem.

Meanwhile, the existence theory had been developed through the use of direct methods: the minimization problem has a unique solution provided L, apart from satisfying natural growth conditions like

$$|L(\xi)| \le C\, |\xi|^p\ ,$$

is also coercive and uniformly convex. The notion of solution had to be conveniently extended and the admissible set $\mathcal{A}$ taken to be the set of functions that, together with their first weak derivatives, belong to L^p, i.e., that belong to the Sobolev space $W^{1,p}$.

So the existence theory gave a minimizer $u \in W^{1,p}$ and the missing step for the regularity problem to be solved was

$$u \in W^{1,p} \Longrightarrow u \in C^{1,\alpha}$$

i.e., from first derivatives in L^p to Hölder continuous first derivatives. In terms of the elliptic PDE (1), regularity theory worked if the leading coefficients were already somewhat regular (at least continuous) since it was based on perturbation arguments and comparison of the solutions with harmonic functions. Assuming only the measurability and the boundedness of the coefficients (together with the essential structural assumption of ellipticity) was insufficient, and nothing was known about the regularity of the solutions in this case.

The problem was solved by C.B. Morrey in 1938 for the special case $n = 2$ but the techniques he employed were typically two dimensional, involving complex analysis and quasi-conformal mappings. The n-dimensional problem remained open until the late 50's and that's exactly what Nirenberg told Nash about.

2 De Giorgi's breakthrough

The problem wouldn't resist the genius of John Nash and Ennio De Giorgi. The two men worked totally unaware of each other's progress and solved the problem using entirely different methods.

It was De Giorgi who did it first (actually for $p = 2$; the result would later be extended to any $p \in (1,\infty)$) and it is his proof that will now be analyzed. To really understand in full depth De Giorgi's ideas there is no way around the technicalities. In what follows I did my best to explain things in a clear way but the reader should not expect everything to be trivial or immediately understandable; so please grab a pencil and a piece of paper and be prepared to struggle a bit with inequalities and iterations.

Consider the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) = 0 \quad \text{in} \quad \Omega \qquad (2)$$

where $\Omega \subset \mathbf{R}^n$ is a smooth bounded domain and the coefficients a_{ij} are only assumed to be measurable and bounded, with

$$\|a_{ij}\|_{L^\infty} \leq \Lambda \ ,$$

and to satisfy the uniform ellipticity condition (for $\lambda > 0$)

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \ , \quad \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^n \ .$$

A weak solution of equation (2) is a function $u \in W^{1,2}(\Omega)$ which satisfies the integral identity

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} = 0 \qquad (3)$$

for all test functions $\varphi \in W_0^{1,2}(\Omega)$ (the elements of $W^{1,2}$ which vanish on the boundary $\partial\Omega$ in a suitable weak sense).

To simplify the writing we assume from now on that

$$\Omega = B_1 := \Big\{ x \in \mathbf{R}^n \ : \ |x| < 1 \Big\} \ .$$

Theorem 1 *Every weak solution of* (2) *is locally bounded.*

Proof. Let $k \geq 0$ and η be a smooth function with compact support in B_1. Put $v = (u-k)^+$ and take $\varphi = v\eta^2$ as test function in (3). The use of the assumptions and Young's inequality give

$$\int_{B_1} |\nabla v|^2\eta^2 \leq \frac{4\Lambda^2}{\lambda^2}\int_{B_1} |\nabla\eta|^2 v^2 \ . \qquad (4)$$

These Cacciopoli inequalities on level sets of u will be the building blocks of the whole theory and once they are obtained the PDE can be forgotten: the problem becomes purely analytic.

Next, by Hölder and Sobolev's inequalities (with $2^* = 2n/(n-2)$ being the Sobolev exponent),

$$\begin{aligned}\int_{B_1} (v\eta)^2 \quad &\leq \quad \Big(\int_{B_1} (v\eta)^{2^*}\Big)^{\frac{2}{2^*}} |\{v\eta \neq 0\}|^{1-\frac{2}{2^*}} \\ &\leq \quad c(n)\ |\{v\eta \neq 0\}|^{\frac{2}{n}} \int_{B_1} |\nabla(v\eta)|^2\end{aligned}$$

and since, due to (4),

$$\int_{B_1} |\nabla(v\eta)|^2 \leq \Big(\frac{4\Lambda^2}{\lambda^2}+1\Big)\int_{B_1} |\nabla\eta|^2 v^2$$

we arrive at

$$\int_{B_1} (v\eta)^2 \leq c(n,\lambda,\Lambda)\ |\{v\eta \neq 0\}|^{\frac{2}{n}} \int_{B_1} |\nabla\eta|^2 v^2 \ .$$

Now for fixed $0 < r < R < 1$, choose the cut-off function $\eta \in C_0^\infty(B_R)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B_r and $|\nabla\eta| \leq \frac{2}{R-r}$. Putting, for $\rho > 0$,

$$A(k,\rho) = \Big\{ x \in B_\rho \ : \ u(x) > k \Big\} \ ,$$

we obtain (with $C \equiv c(n,\lambda,\Lambda)$)

$$\int_{A(k,r)} (u-k)^2 \le \frac{C}{(R-r)^2} \; |A(k,R)|^{\frac{2}{n}} \int_{A(k,R)} (u-k)^2 .$$

For $h > k$ and $0 < \rho < 1$,

$$\int_{A(h,\rho)} (u-h)^2 \le \int_{A(k,\rho)} (u-k)^2$$

and

$$(h-k)^2 |A(h,\rho)| \le \int_{A(k,\rho)} (u-k)^2$$

so we have

$$\begin{aligned} &\int_{A(h,r)} (u-h)^2 \\ \le \quad & \frac{C}{(R-r)^2} \; |A(h,R)|^{\frac{2}{n}} \int_{A(h,R)} (u-h)^2 \\ \le \quad & \frac{C}{(R-r)^2} \, \frac{1}{(h-k)^{\frac{4}{n}}} \left(\int_{A(k,R)} (u-k)^2 \right)^{1+\frac{2}{n}} \end{aligned}$$

or, equivalently, with $\psi(s,\rho) = \|(u-s)^+\|_{L^2(B_\rho)}$

$$\psi(h,r) \le \frac{C}{R-r} \, \frac{1}{(h-k)^{\frac{2}{n}}} \, \psi(k,R)^{1+\frac{2}{n}} \;, \qquad (5)$$

for any $h > k > 0$ and $0 < r < R < 1$.

We are now ready to use the brilliant iteration scheme devised by De Giorgi. Define, for $m = 0, 1, 2, \ldots$

$$\begin{aligned} k_m &= k\,(1 - \frac{1}{2^m}) \\ r_m &= \frac{1}{2}\,(1 + \frac{1}{2^m}) \end{aligned}$$

where k is to be determined later. Due to (5), we then have, for $m = 0, 1, 2, \ldots$,

$$\psi(k_m, r_m) \le C \, \frac{2^{m+1+\frac{2m}{n}}}{k^{\frac{2}{n}}} \, \psi(k_{m-1}, r_{m-1})^{1+\frac{2}{n}} \qquad (6)$$

and can prove, by induction, that, for some $\gamma > 1$,

$$\psi(k_m, r_m) \le \frac{\psi(k_0, r_0)}{\gamma^m} \;, \; \forall\, m = 0, 1, 2, \ldots \qquad (7)$$

if k is chosen sufficiently large. In fact, it is trivial that it holds for $m = 0$; now suppose it holds for $m-1$ and write

$$\begin{aligned} \psi(k_{m-1}, r_{m-1})^{1+\frac{2}{n}} \quad &\le \quad \left\{ \frac{\psi(k_0, r_0)}{\gamma^{m-1}} \right\}^{1+\frac{2}{n}} \\ &= \quad \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{\gamma^{\frac{2m}{n} - (1+\frac{2}{n})}} \, \frac{\psi(k_0, r_0)}{\gamma^m} \;. \end{aligned}$$

From (6) we obtain

$$\psi(k_m, r_m) \le 2C\gamma^{1+\frac{2}{n}} \, \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{k^{\frac{2}{n}}} \, \frac{2^{m(1+\frac{2}{n})}}{\gamma^{\frac{2m}{n}}} \, \frac{\psi(k_0, r_0)}{\gamma^m}$$

and choose first $\gamma > 1$ such that $\gamma^{\frac{2}{n}} = 2^{1+\frac{2}{n}}$ and then k large enough so that

$$2C\gamma^{1+\frac{2}{n}} \, \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{k^{\frac{2}{n}}} \le 1 \;\Leftarrow\; k = C^* \, \psi(k_0, r_0)$$

where $C^* \equiv C^*(n, \lambda, \Lambda)$.

Finally let $m \to \infty$ in (7) to get $\psi(k, \frac{1}{2}) \le 0$, i.e.,

$$\|(u-k)^+\|_{L^2(B_{\frac{1}{2}})} = 0 \;.$$

Hence

$$\sup_{B_{\frac{1}{2}}} u^+ \le C^* \, \|u^+\|_{L^2(B_1)} \;.$$

Using a dilation argument, this estimate can be refined; indeed, for any $\theta \in (0,1)$ and $p > 1$, it holds

$$\sup_{B_\theta} u^+ \le \frac{C(n,\lambda,\Lambda)}{(1-\theta)^{n/p}} \, \|u^+\|_{L^p(B_1)} \;.$$

The same type of reasoning gives similar conclusions concerning u^- and the result follows. □

The basic general idea to obtain results concerning the continuity of a solution of a PDE at a point consists in estimating its oscillation in a nested sequence of concentric balls (cylinders in the parabolic case), centered at the point, and showing that it converges to zero as the balls shrink to the point. If this can be measured quantitatively it gives a modulus of continuity.

Denote the oscillation of a function u in B_r by $\mathrm{osc}(u, r)$. A further analysis, which uses the previous theorem, leads to

Theorem 2 *Let $u \in H^1(B_2)$ be a weak solution of* (2) *in B_2. There exists a constant $\gamma = \gamma(n, \lambda, \Lambda) \in (0,1)$ such that*

$$\mathrm{osc}\,(u, 1/2) \le \gamma \, \mathrm{osc}\,(u, 1) \;.$$

Thus (see below), there exists some constant $\alpha \in (0,1)$ such that, for $0 < r < R < 1$,

$$\mathrm{osc}\,(u, r) \le C \, \left(\frac{r}{R}\right)^\alpha \mathrm{osc}\,(u, R) \;,$$

which gives a Hölder modulus of continuity and

Theorem 3 (De Giorgi - Nash) *Every weak solution of* (2) *is Hölder continuous.*

3 Three papers... and a correction

In a series of three fundamental papers (and a correction) published in *Communications on Pure and Applied Mathematics*, the journal of the Courant Institute, Jürgen Moser made significant contributions to the theory. He first gave in [11] a new proof of De Giorgi's theorem, using the simple general principle that the estimates (4) hold for any convex function $f(u)$ of a solution u; the results were obtained by applying such estimates to powers $f(u) = |u|^p$, $p \geq 1$, and to the logarithmic function $\log^+ u^{-1}$. Then he proved Harnack's inequality for elliptic equations (cf. [12]):

Theorem 4 (Moser's Harnack inequality) *If u is a positive weak solution of* (2) *and K is a compact subset of Ω, then*

$$\max_K u \leq C \min_K u \ ,$$

where $C \equiv C(\Omega, K, \lambda, \Lambda)$.

The proof of the Harnack inequality made no use of the Hölder continuity of the solutions, which in turn is a simple consequence of that fact, as Moser showed in the paper. In fact, assume again that $\Omega = B_1$. Let, for $0 < r < 1$,

$$M(r) = \max_{\overline{B_r}} u \ , \qquad m(r) = \min_{\overline{B_r}} u \ ,$$

and apply Harnack's inequality to the domains B_r and $\overline{B_{\frac{r}{2}}}$ to get

$$\begin{aligned} \max_{\overline{B_{\frac{r}{2}}}} (M(r) - u) &= M(r) - m(r/2) \\ &\leq C \Big(M(r) - M(r/2) \Big) \\ &= C \min_{\overline{B_{\frac{r}{2}}}} (M(r) - u) \end{aligned}$$

and

$$M(r/2) - m(r) \leq C \Big(m(r/2) - m(r) \Big)$$

since $M(r) - u$ and $u - m(r)$ are positive solutions in B_r. Adding these two inequalities, we obtain

$$M(r/2) - m(r/2) \leq \frac{C-1}{C+1} \Big(M(r) - m(r) \Big)$$

or, with $\alpha = \frac{C-1}{C+1} < 1$,

$$\text{osc}\,(u, r/2) \leq \alpha \,\text{osc}\,(u, r) \ .$$

By induction,

$$\text{osc}\,(u, 2^{-k} r) \leq \alpha^k \,\text{osc}\,(u, r) \ ; \qquad k = 1, 2, \ldots$$

Now, for $\rho < r$, we can take k such that $2^{-k-1} r < \rho \leq 2^{-k} r$ to obtain

$$\text{osc}\,(u, \rho) \leq C \left(\frac{\rho}{r} \right)^\beta \text{osc}\,(u, r)$$

with $\beta = -\frac{\log \alpha}{\log 2} > 0$, and as a consequence the Hölder continuity of the function u.

Moser extended his results to parabolic equations, obtaining a Harnack inequality for nonnegative solutions of the parabolic analogue of (2), assuming only the boundedness of the coefficients and the condition corresponding to ellipticity (cf. [13] and [14]). Again his approach was essential *nonlinear* and contrasted dramatically with the approach via fundamental solutions that had been used by Hadamard and Pini to obtain Harnack estimates for solutions of the heat equation.

4 When Stanley met Livingstone

In 1958 the ICM would take place in Edinburgh and the deliberations on the Fields medalists were concluded early that year (the two medals were eventually awarded to Thom and Roth). Solving the regularity problem would probably be worth a medal. Nash in his own words [18]: "It seems conceivable that if either De Giorgi or Nash had failed on this problem (...) then the lone climber reaching the peak would have been recognized with mathematics' Fields medal." Nash solved the problem in the spring of 1957 using a *nonlinear* approach to attack *linear* equations. The main results would be announced in a note to the *Proceedings of the National Academy of Sciences* ([17]), submitted by Marston Morse of the IAS on October 6, 1957. By then, Nash had already found out, in late spring, about De Giorgi's proof: "(...) although I did succeed in solving the problem, I ran into some bad luck since, without my being sufficiently informed on what other people were doing in the area, it happened that I was working in parallel with Ennio De Giorgi of Pisa, Italy. And De Giorgi was first actually to achieve the ascent of the summit (...)." In fact, the seminal paper of De Giorgi was presented by Mauro Picone on April 24, 1957 to the Academy of Sciences of Torino and the results had been announced at the Congress of the Unione Matematica Italiana, which took place in Pavia in October, 1955. Some say that Nash was devastated when he learned about De Giorgi. That summer De Giorgi visited the Courant Institute and Peter Lax would say later about the meeting of the two men: "It was like Stanley meeting Livingstone."

The approach of Nash to the problem was totally different from De Giorgi's and some people think that his ideas were never fully understood (cf. [4]). He treated parabolic equations directly and obtained the results for elliptic equations as corollaries. The essence of his reasoning consisted of obtaining control of the properties of fundamental solutions of linear parabolic equations with variable coefficients. The crucial estimate is the moment bound

$$k_1 \sqrt{t} \leq \int |x| \, T(x,t) \, \mathrm{d}x \leq k_2 \sqrt{t} \ ,$$

which controls the moment of a fundamental solution T. About this result Nash wrote in [16]: "(...) it opens the door to the other results. We had to work hard to get (the bound), then the rest followed quickly."

Although the problem was "morally" solved, writing the paper proved to be technically very hard and Nash continued to work on it when he went back to MIT in the summer of 1957. A few steps in the proof were not clear and only a joint effort with such people as Lennart Carleson (who was visiting MIT on leave from Uppsala) and Elias Stein, both explicitly credited in the paper for some of the proofs, eventually led to his famous paper published in 1958 (Nash writes as an acknowledgement: "We are indebted to several persons" and then names eleven colleagues). There's a *petite histoire* about the publishing of the paper (cf. [15]): Nash first submitted it to *Acta Mathematica* through Carleson, who was an editor there, and made him know that he wanted the paper to be refereed quickly. Carleson gave it to Lars Hörmander (later a Fields medalist, together with John Milnor, in the Stockholm ICM in 1962) who did the job in two months and recommended the paper for publication. But Nash withdrew the paper, which would appear later in the *American Journal of Mathematics*. The reason for this might have been that, after "loosing" the Fields, Nash wanted the paper to be eligible for the AMS Bôcher Prize (awarded for a notable research memoir in analysis published during the previous five years in a recognized *North American* journal). The 1959 prize would be awarded to Louis Nirenberg for his work on partial differential equations.

Whether Nash's ideas were ever understood in full depth by anyone except himself remains unclear. The fact is that his work, although profusely cited, didn't give rise to much subsequent research. It was the more understandable approach of De Giorgi and Moser that the PDE community adopted and developed to full extent.

5 Intrinsic scaling

The work of De Giorgi, Moser and Nash concerned linear PDE's but the approach was essentially *nonlinear* since the linearity had no bearing in the proofs: it all stems out of the structure assumption on the differential operator.

In the elliptic case, this fact allowed for the extension to quasilinear equations of the type

$$\nabla \cdot \mathbf{a}(x,u,\nabla u) = b(x,u,\nabla u) \quad \text{in} \quad \Omega \ ,$$

where the principal part $\mathbf{a}$ satisfies the growth assumption

$$|\mathbf{a}(x,u,\nabla u)| \leq \Lambda \, |\nabla u|^{p-1} + \varphi(x)$$

and the ellipticity condition

$$\mathbf{a}(x,u,\nabla u) \cdot \nabla u \geq \lambda \, |\nabla u|^p - \varphi(x) \ ,$$

for constants $0 < \lambda \leq \Lambda$ and a bounded $\varphi \geq 0$; the prototype is the p-Laplacian equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \ .$$

Notice the nonlinear dependence on the partial derivatives and the nonlinear growth with respect to the gradient. The equation is degenerate if $p > 2$ and singular if $1 < p < 2$, since its modulus of ellipticity $|\nabla u|^{p-2}$ vanishes or blows up, respectively, at points where $|\nabla u| = 0$. Ladyzhenskaya and Ural'tzeva established the Hölder continuity of weak solutions (cf. [9], the bible of elliptic equations), extending De Giorgi's results, and Serrin [20] and Trudinger [21] obtained the Harnack inequality for nonnegative solutions following Moser's ideas.

Surprisingly enough, the theory didn't succeed so well in the parabolic case

$$u_t - \nabla \cdot \mathbf{a}(x,u,\nabla u) = b(x,u,\nabla u) \quad \text{in} \quad \Omega \times [0,T)$$

and Moser's proof could only be extended (by Aronson, Serrin and Trudinger) for the case $p = 2$, which corresponds to principal parts with a linear growth on $|\nabla u|$. The same happened with the methods of De Giorgi, which the Russian school extended to the parabolic case (always for $p = 2$), thus rediscovering Nash's results concerning the Hölder continuity (of solutions of parabolic equations) by entirely different methods. So, unlike the elliptic case, degenerate or singular equations like

$$u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \ ,$$

for which the principal part of the equation grows nonlinearly with $|\nabla u|$, seemed to behave differently, and the questions of regularity remained open until the 1980's.

To understand the difficulty, consider a parabolic cylinder

$$Q(\tau, R) := B_R \times (0, \tau) \ .$$

The use of Cacciopoli inequalities on level sets leads in this case to expressions of the form (compare with (4))

$$\sup_{0<t<\tau} \int_{B_R \times \{t\}} v^2 \eta^p + \int_0^\tau \int_{B_R} \left|\nabla v\right|^p \eta^p$$
$$\leq C \int_0^\tau \int_{B_R} |\nabla \eta|^p \, v^p \ .$$

The iterative argument of De Giorgi, as adapted by the Russian school to the parabolic case, required the equation to be nondegenerate ($p = 2$) so that the integral norms appearing in these estimates were homogeneous. This is not the case in the inequality above: the presence of the power p jeopardizes the homogeneity in the estimates and the recursive process itself. The key idea to overcome the difficulty presented by the inhomogeneity was introduced by DiDenedetto (cf. [3] for an account of the theory and an extensive list of references) and consists essentially of looking at the equation in its own geometry, i.e., in a geometry dictated by its degenerate structure. This amounts to re-scaling the standard parabolic cylinders by a factor depending on the oscillation of the solution. This procedure of *intrinsic scaling*, which somehow is an accommodation of the degeneracy, allows the recovering of the homogeneity in the energy estimates, written over these re-scaled cylinders, and the proof then follows more or less easily. One can say heuristically that *the equation behaves in its own geometry like the heat equation.*

Let's briefly describe the procedure for the degenerate case $p > 2$. Consider $R > 0$ such that $Q(R^{p-1}, 2R) \subset \Omega \times [0, T)$, define

$$\omega := \operatorname{osc}\,(u, Q(R^{p-1}, 2R))$$

and construct the cylinder

$$Q(a_0 R^p, R)\ , \qquad \text{with} \quad a_0 = \left(\frac{\omega}{A}\right)^{2-p}$$

where A depends only on the data. Note that for $p = 2$, i.e., in the nondegenerate case, we have $a_0 = 1$ and these are the standard parabolic cylinders that reflect the natural homogeneity of the space and time variables. Assume, without loss of generality, that $\omega < 1$ and also that

$$\frac{1}{a_0} = \left(\frac{\omega}{A}\right)^{p-2} > R$$

which implies that $Q(a_0 R^p, R) \subset Q(R^{p-1}, 2R)$ and the relation

$$\operatorname{osc}\,(u, Q(a_0 R^p, R)) \leq \omega\ . \tag{8}$$

This is in general not true for a given cylinder since its dimensions would have to be intrinsically defined in terms of the oscillation of the function within it; it is the starting point of the iteration process, in which the difficulties coming from the degenerate structure of the problem are overcome through the use of the re-scaled cylinders. The details are extremely technical and the interested reader can consult [3].

These ideas have been explored to obtain regularity results for other partial differential equations, like the porous medium equation or doubly nonlinear parabolic equations. I'll comment briefly on two extensions for which I am partly responsible.

The inclusion

$$\gamma(u)_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) \ \ni\ 0\ , \qquad p > 2,$$

where γ is a maximal monotone graph with a singularity at the origin, occurs as a model for the well-known two-phase Stefan problem when a nonlinear law of diffusion is considered, u being in that case the temperature and $\gamma(u)$ the enthalpy. As before, the equation is degenerate in the space part, but now it is also singular in the time part since "$\gamma'(0) = \infty$". In this case a further power appears in the energy estimates (the power one, which is due to estimating the singular term) and no re-scaling permits the compatibility of the three powers involved. The proof of the regularity in [22] uses the geometry of the nonsingular case to deal with the degeneracy but the price of a dependence on the oscillation in the various constants that are determined along the proof has to be paid. Owing to this fact, it is no longer possible to exhibit a modulus of continuity for the solution of the problem but only to define it implicitly. This is enough to obtain the continuity but the Hölder continuity, which holds in the nonsingular case, is lost.

Another example concerns the parabolic equation with two degeneracies

$$\partial_t u - \nabla \cdot (\alpha(u) \nabla u) = 0\ ,$$

where $u \in [0, 1]$ and $\alpha(u)$ degenerates for $u = 0$ and $u = 1$. An equation of this type is physically relevant since it shows up in a model describing the flow of two immiscible fluids through a porous medium and also in polymer chemistry and combustion. In [23] it is shown that u is locally Hölder continuous if α decays like a power at *both* degeneracies. A fine analysis of what happens near the two degeneracies leads to the construction of the cylinders used in the iterative process, with the appropriate geometry being once again dictated by the structure of the PDE.

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