Geometric Aspects of Modern Dynamics

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The conference *Geometric Aspects of Modern Dynamics* was held at the Department of Mathematics of the Faculty of Sciences of the University of Porto from 11 through 15 January 2016. The event was partially supported by the following institutions: Centro de Matemática da Universidade do Porto (CMUP), Centro Internacional de Matemática (CIM), Fundação Luso-Americana para o Desenvolvimento (FLAD), Fundação para a Ciência e a Tecnologia (FCT), Institut de Mathématiques de Toulouse (IMT) and Reitoria da Universidade do Porto (UP).

The conference brought together more than 70 experts in dynamical systems coming from various countries and including several field leaders for a program consisting of 24 talks. The scientific and organizing committees for the conference consisted of M. Abate (*University of Pisa*, Italy), A. Glutsyuk (*ENS-Lyon*, France and *HSE-Moscow*, Russia), M. Lyubich (*Stony Brook*, US), J. Raissy (*University of Toulouse*, France), J. Rebelo (*University of Toulouse*, France) and H. Reis (*University of Porto*, Portugal).

Broadly speaking, dynamical systems has to do with determining the asymptotic behavior of systems that *evolve with time*. The beginning of the theory is generally ascribed to Poincaré's investigations of the qualitative behavior of solutions of differential equations. The point of view of dynamics, however, was gradually enlarged to encompass the iterations of a diffeomorphism/endomorphism, more general finitely generated (semi) group actions, foliations and so on. It is the fact that the nature of these maps, differential equations and so on can be extremely varied that accounts for the existence of some many different trends in dynamical systems of which hyperbolic dynamics, conservative dynamics, group actions, and complex dynamics are examples.

Complex dynamics

Complex dynamics can roughly be described as the part of dynamics where the system under study has a holomorphic nature, regardless of it is a differential equation, a diffeomorphism or an endomorphism. It then just natural that the techniques and specific phenomena lying in the scope of complex analysis become an all important tool in the area. Basic examples of dynamics belonging to the universe covered by *complex dynamics* include most classical differential equations such as Gauss hypergeometric equation, Halphen systems, Riccati equations, Painlevé equations. It also includes the iteration of rational fractions on the Riemann sphere and questions about convergence of root-finding algorithms for polynomial equations.

Dynamical systems is an area strongly represented in the Portuguese mathematical community and the country counts on recognized groups working on (partially) hyperbolic dynamics, strange attractors, and conservative dynamics to name only a few. In contrast, there are very few Portuguese mathematicians working on *Complex dynamics* although some of them have acquired international recognition. Time seems then ripe to make the area of Complex dynamics better known not only to our students but also to our colleagues working in other branches of dynamics.

A conference in dynamical systems emphasizing complex dynamics

The conference *Geometric Aspects of Modern Dynamics* featured a wide range of classical topics in dynamics such as those stemming from classical hyperbolic dynamics (SRB measures, topologically/smoothly equivalent systems and so on), real analytic geometry and Hilbert's problem, counting problems and lattices in spaces of negative curvature. A definite emphasis was, however, put on complex dynamics as around 2/3 of the program was devoted to topics connected with complex dynamics in a large sense. As a matter of fact, the approach to complex dynamics taken by the organizers was precisely to try to work out a *bigger picture* for the field and, ultimately, this attempt at considering complex dynamics in a broader sense constituted a distinctive trait of our conference with respect to most conferences devoted to complex dynamics. This point of view deserves to be further elaborated here.

The origin of complex dynamics goes back to the nineteenth century and to the study of classical differential equations. Landmarks in this direction are Riemann's theory on Gauss hypergeometric equation and Poincare's work on Riccati equation which led Poincare to study basic properties of Kleinian groups. Around the same time E. Schröder studied the convergence of Newton's method and obtained some fundamental results concerning the iteration of rational fractions on the Riemann sphere. The study of complex differential equations was then continued by Painlevé while Julia and Fatou have laid the foundations of the iterative theory of rational fractions. After the works of Painlevé and of Fatou-Julia, the subject of *complex dynamics* has developed into two main strands, namely *iterative dynamics*, where the main object of study is the dynamics arising from one single holomorphic automorphism/endomorphism, and *holomorphic foliations* whose aim is to understand the dynamics of the holomorphic foliation associated with a (possibly meromorphic) differential equation. Although each of these strands has evolved into a large body of results and methods, it is perhaps a bit disappointing to realize that communication between experts in *iterative theory* and *holomorphic foliations* has not been very effective.

Over the past twenty five years or so, the community formed by mathematicians working in complex dynamics has hoped for a more unified theory where each of the above mentioned main strands in the field would be able to benefit from progresses made in the other. Establishing connections between two topics in Mathematics is always useful and often sparks some accelerate progress. For example, in recent years, the *iterative side* of complex dynamics has benefitted immensely from its connections with Kleinian groups while techniques of *holomorphic foliations* have found new applications in complex algebraic geometry. Yet, and despite of the efforts of some outstanding mathematicians such as Dennis Sullivan and Etienne Ghys, examples of situations where relevant interactions between these two *sides* of complex dynamics have occurred are still not numerous.

In the conference of Porto some concerted effort was done in terms of stimulating collaboration between *iterative theory* and *holomorphic foliations*. Common features between the problems studied in *iterative theory* and in *holomorphic foliations* have long been identified but, somehow, the differences between them have out-weighted the similarities and provided major obstacles to a unified framework. Probably the simplest and greatest obstacle to bring the themes together can be summarized as follows: in *iterative theory* we study one single and globally defined map whereas the dynamics of a *holomorphic foliation* is encoded by a collection of maps that are only locally defined. Very recently, however, some very subtle but promising directions to develop deeper collaborations between the two sides have emerged and some of them were highlighted in the talks.

Interactions between different branches of dynamics

As mentioned above, a significant part of the conference program was devoted to promising directions for developing a more unified theory in complex dynamics. In more general terms, it is also true that the idea of breeding new connections between the several themes in dynamics represented in the conference was ubiquitous in the designing of the program.

It is well-known that the abstract setting of ergodic theory often provides a useful language in which problems from different areas of dynamics can be formulated in a unified way. In fact, when it comes to complex dynamics, the developing of a suitable ergodic theory for foliations is widely viewed as a fundamental step to build effective bridges with the *iterative theory*. Ergodic theory for holomorphic foliations was the subject of talks by J. Rebelo and by N. Sibony. The organizers were pleased to see one expert from *iterative theory* and one expert in *holomorphic foliations* giving two very closely related talks about the same topic.

Another interesting aspect was J.-P. Ramis talk which focused on producing new relevant examples of *iterative dynamics* out of Painlevé equations. In turn, R. Roeder discussed examples of endomorphisms of the complex projective plane without invariant foliations. Similarly M. Abate talked about the existence of parabolic curves for germs of diffeomorphisms tangent to the identity: a topic in which the local theory of foliations provides important tools.

On a different direction, A. Guillot and H. Reis have talked about complex differential equations with uniform solutions along with some applications to problems of complex geometry. Concerning (pseudo) differential geometry, M. Pollicott and N. Tholozan discussed lattices in negative curvature and some related problems with ergodic theoretic nature.

Apart from *holomorphic foliations*, there was also a number of talks on Henon maps a topic where *iterative theory* in complex dynamics comes close to *real* dynamics problems related to SRB measures and strange attractors, subjects well represented in the University of Porto, and J.-F. Alves, E. Bedford, M. Martens, A. Pinto, and M. Yampolsky have all given talks connected with this circle of ideas.

Conclusion and sources for students and newcomers

The literature on dynamical systems is vast and continuously increasing as the area remains very active. The collection of *Handbook of Dynamical Systems* offers a good overview of many aspects of what might be called *real* dynamics, as opposed to complex dynamics in the sense described above. Yet the literature on complex dynamics is huge as well. People interested in holomorphic foliations may benefit largely from classical material including Painlevé's *Leçons sur la théorie analytiques des Équations Différentielles* or P. Ince's classic *Ordinary Differential Equations*. Modern expositions of the theory are given in [3], [4], [6]. For the *iterative side* of complex dynamics, two good introductions are provided by the books of Carleson-Gamelin and of Milnor [2], [5]. For higher dimensional theory, we refer the reader to [1]. Additional useful information, including announcements for several past and upcoming conferences in the field, can be obtained from the Dynamical Systems web-page of Stony Brook: http://www.math.stonybrook.edu/dynamical-systems.

References

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