Applications of Mathematics in Fluid Dynamics

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1 General considerations

Fluid dynamics [1, 2, 3] represents one of the very few fields where, in the framework of classical physics, a full comprehension of the problem is still far from being achieved, and therefore constitutes a vast subject for ongoing and future research. Even if relativistic [4] and quantum [5] hydrodynamics have their own importance, the laws of classical physics and of continuum mechanics are implemented in almost the entirety of hydrodynamic investigations.

The analytical approach faces the obstacles of non-linearity and non-locality of the problem, and typically of non-ideality of the initial and boundary conditions or of the forcing terms. The computational approach is nowadays very common, but numerical simulations of fluid flows usually have to deal with the very huge number of active—and non-trivially interacting—degrees of freedom to be described. Experiments can only reproduce part of the interesting problems, and heavily rely on the (Bertrand-Vaschy-Buckingham) π theorem [6, 7, 8] for the appropriate geometric scaling and the introduction of nondimensional numbers. Among these latter, the best renowned is the one associated with Osborne Reynolds' famous experiment [9]:

$$\text{Reynolds number} = \mathrm{Re} \equiv \frac{LU}{\nu}.$$

Here, L and U are characteristic length and speed scales of the flow under consideration, and the kinematic viscosity is defined as the ratio between dynamic viscosity and mass density: $\nu \equiv \mu/\rho$.

Two main descriptions of the analytical problem are possible [10]. One is Lagrangian: a small region of fluid (*particle* or *parcel*) is ideally identified, isolated and followed along its evolution. All its properties are thus represented by physical quantities which are only functions of time, and obey ordinary differential equations. The other is Eulerian, where partial differential equations are derived for fields depending on space and time. The two descriptions are complementary and both relevant, and related by the fundamental statement that the velocity of a fluid particle equals by definition the local and instantaneous velocity field:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t). \tag{1}$$

The equation for the fluid velocity is due to Claude-Louis-Marie-Henri Navier [11] and George Gabriel Stokes [12], and in its incompressible form—when free from thermal effects—reads:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \boldsymbol{u}, \tag{2}$$

endowed with appropriate initial and (Dirichlet/Neumann/Robin) boundary conditions, and possibly modified with the appearance of a forcing term on the right-hand side.

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Here $p(\boldsymbol{x}, t)$ is the pressure field, and is the source of non-locality of the problem. Indeed, the continuity equation for incompressible flows reads as a solenoidality property, $\nabla \cdot \boldsymbol{u} = 0$, and pressure is required to satisfy a Poisson equation. Therefore, even if (2) is in principle evaluated locally at one single point, actually it contains a term which represents a contribution coming from a spatial integral on the whole domain, as the propagation velocity of any disturbance is infinite. Despite this difficulty, incompressibity is a scheme widely used for the simplifications it brings about, and is usually abandoned only when compressibility effects cannot be neglected, most notably because the velocities into play are not negligible with respect to the sound speed [13]. In this latter case, the mass density varies. Also thermal effects can come into play and modify the parameters, in which case also the evolution of the temperature field must be taken into account, along with a suitable equation of state. Even more problematically, also viscosity can be different from a constant, and then the fluid under consideration is dubbed as non-Newtonian and described by a different equation.

Apart from the incompressible scheme, common simplifications in the resolution process hold if the flow is parallel (only pointing in one same direction always and everywhere), or plane (independent of at least one direction), or potential (i.e. irrotational: $\nabla \times \boldsymbol{u} = \boldsymbol{0}$). Other intrinsic properties of (2) are its non-linearity and its time irreversibility (with energy dissipation), due to the second term on the left and on the right-hand sides respectively. Three main analogues of the Navier-Stokes equation are worth mentioning: the Burgers one [14], where the pressure term is dropped and which gives rise to compressibility shocks; the Euler one [15], where the viscous contribution is neglected leading to finite-time singularities, and which is often an excellent approximation of the problem in the bulk of the fluid but requires matching asymptotic techniques developed by Ludwig Prandtl [16] to be extended to boundary layers near walls; and the Stokes one, where the left-hand side of (2) is negligible and which describes creeping flows. Dropping only one term on left-hand side is also common: the first, when one looks for steady solutions of the full equation; the second, when linearizing around a mean flow with small-amplitude fluctuations.

When Re grows past a certain threshold, the flow loses its laminar character, undergoes a series of transitions and becomes fully turbulent [17, 18, 19, 20, 21]. Fluid turbulence is a problem of paramount importance and difficulty (sometimes referred to as "the last mystery of classical physics"), as admitted by Richard Feynman and underlined by two famous historical quotes. One is from Peter Bradshaw [22]: "Turbulence was probably invented by the devil on the seventh day of creation, when the Good Lord wasn't looking". An older one, "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic", is attributed to Horace Lamb, even if a very similar version—with *relativity* as the former matter—was reportedly pronounced by Werner Heisenberg.

It is absolutely astonishing to note the degree of uncertainty related to turbulent flows, for instance in comparison to the one typical of astronomy. Not unusual are calculations of the trajectory of a poorly-known asteroid for many decades to come, and safe conclusions that it will barely miss an impact with the Earth in more than one century, even if all we know about it comes from few possible measurements from such a far distance. On the other hand, in principle we can measure as much information as we want in our low atmosphere, but weather forecast is limited to very few days, with the practical impossibility of specifying the exact hour and location of some kind of precipitation or disruption.

Turbulent flows can simply be seen as general solutions of the Navier-Stokes equation (2) lacking any property of regularity characteristic of laminar flows. Turbulence is a phenomenon very far from equilibrium, with typically no small parameter in which to expand around a known state. Turbulent velocity fields greatly enhance mixing and dispersion, and are intermittent and usually ergodic. They are self-organized and made up of coherent structures, the so-called *eddies*, but they are chaotic [23]. Therefore—despite being far from random—they cannot be described deterministically, and a statistical approach is common [24, 25, 26, 27, 28], with the turbulent quantities (or often their deviations from the mean) considered as stochastic variables. This implies that also many tools of statistical mechanics are employed, along with several techniques borrowed from theoretical physics, such as renormalization-group formalism, diagrammatic representation, path-integral formulation, second-quantization algorithm, non-linear Schrödinger and Ginzburg-Landau equations [29, 30].

Without entering the details of this field, we leave the interested reader to the vast literature on the subject, and we only point out the enormous difference in phenomenology between two-dimensional and three-dimensional turbulence (see [31, 32] and bibliographical references therein).

The number of mathematical tools employed in the analytic investigation of fluid mechanics is immense [33, 34], which justifies the fact that this subject is often studied in centers of applied mathematics. They range from the most common ones—SO-d decomposition [35], Fourier/Laplace/Legendre transforms—to more sophisticated counterparts, among which we just mention a few here: functional analysis and Furutsu-Novikov-Donsker theorem [36, 37, 38], multifractals, multiplicative model, refined large-deviation theory, steepest-descent method, Lyapunov exponents and Cramér function, telegraph-noise model and degenerate perturbation theory [39], (time-ordered) matrix exponentials and Cayley-Hamilton theorem [40], Hermitianization [41], Ornstein-Uhlenbeck process [42], multiple-scale technique [43], variational formulation, adjoint method, continued fractions and Heun equation [44].

Of course, describing the velocity field is the principal objective in fluid mechanics. However, there are a lot of related problems which deserve the same attention and importance, also because they may appear easier for some aspects but harder for others. First of all, one should mention transport phenomena.

From a Lagrangian perspective, one can think of replacing a fluid particle with a tracer one, i.e., a particle which has the same exact properties of the fluid replaced (and therefore evolves in the same way and does not alter the effects on the surrounding fluid) but that simply acts as a marker and can be followed individually in its evolution. More complex cases arise when these inclusions are inertial particles—the subject of the next section—or polymers. These latter denote particles with an internal structure, which can be described by different models (e.g. Oldroyd-B, FENE-P, etc. [45, 46]) and produce non-trivial feedbacks on the carrier fluid, the most important of which is probably the drag reduction that can be achieved in oil ducts with efficiency improvement of even 80% by simply adding a few parts per million in mass.

From the Eulerian viewpoint, the transported quantity is a field. This latter can be a scalar or a vector, and may be passive or active depending on whether it has a feedback on the advecting velocity—by appearing as a source term on the right-hand side of (2) and thus fully coupling the system. Passive-scalar advection [47], e.g., for the concentration field of a tracer, is a paradigmatic case because, despite the linearity and locality of the problem, many aspects are reminiscent of the kinetic-energy cascade picture [48, 49, 50, 51, 52, 53, 54, 55]. In the realm of active-vector turbulence, magnetohydrodynamics still constitutes a formidable problem [56].

Large-Eddy Simulations (LES) consist in a computational resolution of these problems, different from the Reynolds-Averaged (RANS) and Direct (DNS) Numerical counterparts [57] in the sense that a coarse-graining procedure is implemented [58, 59], whose analytical foundations are still being studied. The basic difficulty relies on a closure problem, due to the fact that any filtering operation aimed at focusing on the sole large scales does not commute with the non-linear multiplicative term in (2), thus requiring a suitable parameterization of the small scales which cannot just be neglected [60, 61].

Back to the investigation of the velocity field itself, the different types of instabilities [62] represent a huge research theme. Among them, we can mention those named as Rayleigh-Bénard (fluid cooled from above and heated from below), Taylor-Couette (fluid between two counter-rotating cylinders), Kelvin-Helmholtz (fluid with internal layers moving in relative shear) and Von Kármán-Strouhal (fluid in the wake of an obstacle with vortex street).

Finally, control theory [63] plays a crucial role. By means of studies of structural sensitivity, the aim is to identify which changes in the boundary conditions or in the forcing terms are the most suitable to obtain some desirable or desired result (such as the reduction of the aerodynamic drag on cars or of acoustic noise on airplanes), in the sense that they maximize the kinetic-energy gain or engage/delay some transition.

1.1 Inertial particles

Particles that have a different mass density (σ) with respect to the surrounding fluid, and whose size—let us say radius R in the range μm $\div$ mm—is small but not tiny, cannot simply be described as point tracers and have a finite relative inertia. Their trajectory thus deviates from the underlying fluid one, which can lead to preferential concentration and even clustering. Common examples are droplets in gases, bubbles in liquids and aerosols in fluids. Let us consider the simplest realistic model, where the particles are spherical and isolated, or belonging to a very dilute suspension, in order to neglect any interaction with boundaries and other particles, and to take into account their feedback on the flow in an effective simplified fashion.

Equation (1) now becomes a second-order dynamical system of the Langevin type (Itô or Stratonovich) for the particle position $\boldsymbol{x}(t)$ and velocity $\boldsymbol{v}(t)$, because Newton's law can be recast as [64, 65]:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t) \\ \dot{\boldsymbol{v}}(t) = \beta \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{x}(t), t) - \dfrac{\boldsymbol{v}(t) - \boldsymbol{u}(\boldsymbol{x}(t), t)}{\tau} \\ \qquad + \dfrac{\sqrt{2\kappa}}{\tau}\boldsymbol{\eta}(t) + (1-\beta)\boldsymbol{g} \ . \end{cases} \tag{3}$$

The four terms on the right-hand side of the equation for the acceleration represent the four basic components of the force acting on the particle. The first is proportional to the non-dimensional coefficient

$$\beta \equiv \frac{3\rho}{\rho + 2\sigma} \in [0, 3] \tag{4}$$

and expresses the added-mass effect, i.e. the fact that any motion of the particle implies a movement of fluid around it: this contribution vanishes for very heavy particles ($\sigma \gg$

$\rho \Rightarrow \beta \simeq 0$) and is maximum for very light ones ($\sigma \ll \rho \Rightarrow \beta \simeq 3$), because there all inertia lies in the particle or in the fluid, respectively; for tracers ($\sigma = \rho \Rightarrow \beta = 1$) of course this acceleration is the same as if a fluid particle were there.[1] The second is the linear viscous drag for small relative slip velocity, it means that the particle relaxes towards the local and instantaneous flow configuration with a typical response time $\tau \equiv R^2/3\nu\beta$ ($\to 0$ for tracers and $\to \infty$ for ballistic objects), from which the Stokes number—a measure of the inertia-driven delay—can be constructed:

$$\mathrm{St} \equiv \frac{\tau}{L/U}. \tag{5}$$

The third takes thermal noise into account via the Brownian diffusivity κ, coupled through the standard vectorial white noise $\boldsymbol{\eta}(t)$, and gives rise to the Péclet number:

$$\mathrm{Pe} \equiv \frac{LU}{\kappa}. \tag{6}$$

The fourth represents buoyancy, parallel to gravity acceleration $\boldsymbol{g}$ for heavy particles ($\beta < 1$) and anti-parallel for light ones ($\beta > 1$), and leads to the definition of the Froude number:

$$\mathrm{Fr} \equiv \frac{U}{\sqrt{Lg}}. \tag{7}$$

Some corrective terms have been neglected in (3), namely those due to Basset–Boussinesq (time integration for memory effects), Faxén (spatial expansion for finite particle size), Oseen (non-linearity for finite relative slip velocity) and Saffman (side lift in case of rotation).

From (3), one derives the generalized Fokker-Planck equation for the phase-space density $p(\boldsymbol{x}, \boldsymbol{v}, t)$, which serves as a basis to compute the quantities of physical relevance. Typically, in the presence of a localized particle source—such as a chimney for pollutants in the atmosphere, or a syringe for powders in microchannels—one is interested in the temporal evolution of the physical-space concentration. If no source is present, the most important quantities are the particle transport properties, such as: the average terminal velocity (or more precisely its deviation with respect to the asymptotic bare value in still fluids) describing how even a zero-mean flow can modify the sedimentation process and Stokes' drift; and the effective *eddy* diffusivity, whose value also tells one whether, in the frame of reference moving according to the ballistic component, the diffusion process is standard or anomalous. All this information can be obtained from p, either by simply integrating on the velocity degree-of-freedom, or by dealing with this latter in some appropriate way in order to obtain advection-diffusion-like equations known as auxiliary cell problem [66].

It is known that these phenomena critically depend on the interplay of several control quantities, i.e., the full details of both flow and particles contribute to establish whether, e.g., the activation of a fluid velocity field increases or decreases the settling of a suspended particle. Among the key factors is the list of non-dimensional numbers β (4), St (5), Pe (6) and Fr (7), to which we must add a few others. Most importantly, the compressibility degree (an analogue of the Mach number)

$$\mathscr{P} \equiv \frac{\langle (\nabla \cdot \boldsymbol{u})^2 \rangle}{\langle ||\nabla \boldsymbol{u}||^2 \rangle} \in [0, 1] , \tag{8}$$

where the average can be on the space-time periodicity, or on the statistical ensemble for random flows. Then, the space dimension d, where only the two- and three- dimensional cases can be investigated if the flow is incompressible, but also $d = 1$ if $\mathscr{P} \neq 0$.

Also the geometric and temporal details of the flow are extremely relevant, and to fix the ideas let us focus on two examples. First, a laminar 2D incompressible flow with a cellular structure,

$$\boldsymbol{u} = U \begin{pmatrix} \sin(2\pi k x_1/L) \cos 2\pi[x_2/L + \sin(\omega t)] \\ -k \cos(2\pi k x_1/L) \sin 2\pi[x_2/L + \sin(\omega t)] \end{pmatrix},$$

where k is the vertical-to-horizontal aspect ratio, and ω is the angular frequency of synchronous vertical oscillation of the cells. Second, a zero-mean, stationary, homogeneous, isotropic, Gaussian random flow, with two-point correlation

$$\langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{0}, 0) \rangle = U^2 f_{ij}(\mathscr{P}) e^{-x^2/2L^2} e^{-t^2/2T^2} \cos(\omega t)$$

(the tensorial structure f_{ij} simply enforces the desired compressibility degree); here, T is the characteristic life time of turbulent vortices and ω is an angular frequency which takes into account the presence of recirculation— i.e., areas with negatively-correlated velocity—that lead to the definition of two additional non-dimensional numbers:

$$\text{Kubo number} = \mathrm{Ku} \equiv \frac{T}{L/U}, \tag{9}$$

$$\text{Strouhal number} = \mathrm{Sr} \equiv \frac{\omega}{2\pi U/L}. \tag{10}$$

It can be shown that also the parameters $\mathscr{P}$ (8), Ku (9), Sr (10), along with d and k, have a huge impact on the transport properties, not only directly, but indirectly too, by changing or even reversing the direct role of other parameters.

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[1] It is debated whether the material derivative d/dt should be computed along the particle trajectory as $\partial/\partial t + \boldsymbol{v}(t) \cdot \nabla$ or along the corresponding fluid path as $\partial/\partial t + \boldsymbol{u}(\boldsymbol{x}(t), t) \cdot \nabla$

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