Representations of commutative rings via the prime spectrum

by **Jorge Vitória***

Given a commutative noetherian ring R we discuss its representations, i.e. its R-modules. The prime spectrum of R plays a fundamental role, controlling much of the structure of the category of R-modules. We illustrate this in two instances, surveying a parametrisation of localising subcategories and a parametrisation of flat ring epimorphisms.

1 Introduction

Representation theory studies actions of rings on abelian groups or vector spaces. The objects of study are, therefore, R-modules, where R is a ring. Here are some examples of representations.

1. Let $(X, +)$ be an abelian group. Then X admits a natural $\mathbb{Z}$-action, setting an integer n to act on $x \in X$ by iterating $|n|$ times the operation $+$ on x if $n \geq 0$ or on $-x$ if $n < 0$.
2. Let V be a vector space over a field $\mathbb{K}$. The field $\mathbb{K}$ naturally acts on V through the multiplication of vectors by scalars. This turns V into a $\mathbb{K}$-module. If, furthermore, we choose a linear endomorphism $f\colon V \longrightarrow V$, then we can define an action of the polynomial ring $\mathbb{K}[x]$ on V, setting the action of x^n on a vector v as $f^n(v)$. This yields a $\mathbb{K}[x]$-module structure on V.
3. Let G be a finite group acting on a $\mathbb{K}$-vector space V. Then V can be regarded as a $\mathbb{K}G$-module, where $\mathbb{K}G$ is the ring whose elements are formal linear combinations of elements of G over $\mathbb{K}$ (multiplication is defined by the operation in G). For example, if G is the (dihedral) group D_4 of symmetries of a square and $V = \mathbb{R}^2$, then D_4 acts naturally on $\mathbb{R}^2$. This turns V into a module over $\mathbb{R}D_4$, where the multiplication extends linearly the composition of symmetries in D_4.
4. Let $\mathfrak{g}$ be a complex Lie algebra with an action on a complex vector space V. Then V is naturally a module over $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra. For example, if $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebra of 2×2 matrices with zero trace, then $\mathfrak{g}$ acts naturally on $V = \mathbb{C}^2$. This action turns V into an $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$-module, where $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ can be shown to be isomorphic to the ring $\mathbb{C}\langle x,y,z\rangle/(xy - yx + 2y, xz - zx - 2z, yz - zy + x)$ with non-commuting variables x, y and z.

A traditional aim of representation theory is to classify all representations of a given ring. This is, in general, a very difficult, if not impossible, task. One might, however, try to classify families of modules satisfying common properties. This macroscopic approach to representation theory has been very successful over the past decades, shifting the focus from individual modules to collections (or *subcategories*) of modules, with the help of category theory and homological algebra.

2 Categories of modules and some special subcategories

Given a ring R, consider the category $\mathrm{Mod}(R)$ whose objects are all (right) R-modules and whose morphisms are the R-linear homomorphisms of abelian groups, i.e. those that preserve the R-action.

Example 1.— The category $\mathrm{Mod}(\mathbb{Z})$ is (equivalent to) the category of abelian groups. Similarly, the category $\mathrm{Mod}(\mathbb{K})$ for a field $\mathbb{K}$ is (equivalent to) the category of $\mathbb{K}$-vector spaces. The category $\mathrm{Mod}(\mathbb{K}[x])$ is (equivalent to)

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the category whose objects are pairs (V, f), where V is a $\mathbb{K}$-vector space and f is a $\mathbb{K}$-linear endomorphism of V. The morphisms in this category between two pairs (V, f) and (W, g) are those linear maps $\alpha : V \longrightarrow W$ such that $g \circ \alpha = \alpha \circ f$.

In this note, by subcategory of $\mathsf{Mod}(R)$ we mean a strict and full subcategory (see [9] for basic terminology). This means that a *subcategory* is completely described by its collection of objects. We aim to classify some subcategories of $\mathsf{Mod}(R)$, and the ones we are particularly interested in are determined by *closure conditions*. A subcategory $\mathcal{U}$ of $\mathsf{Mod}(R)$ is said to be closed under

- (co)products if for any (set-indexed) family of R-modules lying in $\mathcal{U}$, its (co)product lies in $\mathcal{U}$.
- (co)kernels if for any map of R-modules $f: M \longrightarrow N$, we have that the (co)kernel of f lies in $\mathcal{U}$.
- extensions if for any short exact sequence of R-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

if X and Z lie in $\mathcal{U}$, then so does Y.

The following definition sets up the kind of subcategories we will be interested in.

Definition 1.— Let R be a ring. A subcategory $\mathcal{X}$ of $\mathsf{Mod}(R)$ is said to be a **Serre subcategory** if for any short exact sequence of R-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

X and Z lie in $\mathcal{U}$ if and only if Y lies in $\mathcal{U}$. Moreover, we say that $\mathcal{X}$ is **localising** if it is a Serre subcategory which is closed under coproducts and we say that $\mathcal{X}$ is **bireflective** if $\mathcal{X}$ is closed under products and coproducts, kernels and cokernels.

Example 2.— In the category of abelian groups, $\mathsf{Mod}(\mathbb{Z})$, consider the subcategory Tors, formed by all abelian groups for which every element has finite order, and the subcategories $\mathcal{U}_n$ $(n > 1)$, formed by all abelian groups annihilated by n (i.e. abelian groups for which the order of every element divides n).

1. Given an abelian group M in Tors, every subgroup of M and every quotient group of M also lies in Tors. The same conclusion can easily be reached for extensions between two groups in Tors, and for coproducts of such groups. Therefore, Tors is a localising subcategory of $\mathsf{Mod}(\mathbb{Z})$.
2. The subcategory Tors is not, however, closed under products. Let M be the product indexed by the natural numbers of the abelian groups $\mathbb{Z}/n\mathbb{Z}$. The elements of M are the sequences $(a_n)_{n\in\mathbb{N}}$ with each a_n being an element in $\mathbb{Z}/n\mathbb{Z}$. It is easy to see that the sequence given by $a_n = 1$ does not have finite order and, thus, M does not lie in Tors.
3. It is easy to see that $\mathcal{U}_n$ is closed under kernels and cokernels and given any family of abelian groups annihilated by n, both its product and coproduct will be annihilated by n. As such, $\mathcal{U}_n$ is a bireflective subcategory of $\mathsf{Mod}(\mathbb{Z})$. However, the short exact sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{1\mapsto n} \mathbb{Z}/n^2\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

shows that it is not extension-closed (since $\mathbb{Z}/n^2\mathbb{Z}$ does not lie in $\mathcal{U}_n$). Hence, $\mathcal{U}_n$ is not a Serre subcategory and, as such, it is also not a localising subcategory.

In this note we will provide a way of classifying all localising subcategories and all extension-closed bireflective subcategories of $\mathsf{Mod}(R)$, when R is a commutative noetherian ring. In the remainder of this section, we want to explain why these subcategories are relevant in representation theory.

2.1 Categorical localisations

In order to study the structure of a category such as $\mathsf{Mod}(R)$, one technique is to consider its localisations, or Serre quotient categories. It follows from [4] that for any Serre subcategory $\mathcal{S}$ of $\mathsf{Mod}(R)$, there is an abelian category $\mathsf{Mod}(R)/\mathcal{S}$ and an exact functor

$$q_{\mathcal{S}} : \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(R)/\mathcal{S}$$

that sends all objects of $\mathcal{S}$ to the zero object and that, moreover, is universal with respect to this property. It is then to be expected that by studying the abelian category $\mathsf{Mod}(R)/\mathcal{S}$ together with the associated Serre subcategory $\mathcal{S}$, one might be able to glue data to the larger category $\mathsf{Mod}(R)$. While this may, in general, still be quite difficult, the task becomes easier if we require that $\mathcal{S}$ is also closed under coproducts, i.e. if $\mathcal{S}$ is localising. Indeed, given a localising subcategory $\mathcal{S}$ of $\mathsf{Mod}(R)$, both the inclusion functor of $\mathcal{S}$ into $\mathsf{Mod}(R)$ and the quotient functor $q_{\mathcal{S}}$ admit right adjoints ([4]). These adjoints can then be used to better relate the structures of these three categories.

Example 3.— Consider again the category $\mathsf{Mod}(\mathbb{Z})$ and the localising subcategory Tors from Example 2. It turns out that the categorical quotient $\mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ is equivalent to $\mathsf{Mod}(\mathbb{Q})$, i.e. the category of $\mathbb{Q}$-vector spaces. Moreover, the quotient functor $q_{\mathsf{Tors}}\colon \mathsf{Mod}(\mathbb{Z}) \longrightarrow \mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ can be shown to be naturally equivalent to the tensor product $-\otimes_{\mathbb{Z}}\mathbb{Q}$, and its right adjoint identifies $\mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ with the torsionfree and divisible abelian groups.

2.2 Ring epimorphisms

Epimorphisms in the category of (unital) rings are not just, as one naively could expect, surjective ring homomorphisms. In fact, it is an easy exercise to check that the embeeding of $\mathbb{Z}$ into $\mathbb{Q}$ is a ring epimorphism or, in other words, that any ring homomorphism from $\mathbb{Q}$ to a ring C is uniquely determined by the image of the integers. As it turns out, any ring of fractions of a ring R (where one formally inverts a suitably selected subset of R) yields a ring epimorphism from R.

Ring epimorphisms from a ring R are relevant in the representation theory of R because they correspond bijectively (up to a suitable notion of equivalence) to bireflective subcategories of $\mathsf{Mod}(R)$ ([5]). Note that any ring homomorphism $f\colon R \longrightarrow S$ induces an R-module structure on any (right) S-module M: just set the action of $r \in R$ on $m \in M$ by $m \cdot r := mf(r)$. This defines a faithful functor

$$f_*\colon \mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(R)$$

which is called *restriction of scalars*. Moreover, it turns out that f is an epimorphism if and only if every R-linear map between S-modules is also S-linear (or, in other words, f_* is full). The assignment can now be defined by associating to a ring epimorphism $f\colon R \longrightarrow S$ the subcategory of $\mathsf{Mod}(R)$ obtained as the essential image of the functor f_* (which is naturally equivalent to $\mathsf{Mod}(S)$).

Classifying bireflective subcategories of $\mathsf{Mod}(R)$ then amounts to classifying families of modules that share the property of being modules over some *epimorphic image* of R (in a compatible way). In this note we restrict our aim to classifying those bireflective subcategories which are also extension-closed. This is because the associated ring epimorphisms exhibit a better homological behaviour (see [6] for details).

Example 4.— Consider yet again the category $\mathsf{Mod}(\mathbb{Z})$ and the bireflective subcategories $\mathscr{U}_n$ ($n > 1$) from Example 2. The ring epimorphism associated to $\mathscr{U}_n$ turns out to be $f_n\colon \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$. This is because the abelian groups admitting a natural $\mathbb{Z}/n\mathbb{Z}$-module structure are precisely those annihilated by n. These bireflective subcategories are not, however, extension-closed (see Example 2).

Consider now the ring epimorphism $f\colon \mathbb{Z} \longrightarrow \mathbb{Q}$. The restriction functor $f_*\colon \mathsf{Mod}(\mathbb{Q}) \longrightarrow \mathsf{Mod}(\mathbb{Z})$ is naturally equivalent to the right adjoint of q_{Tors} (see Example 3). One can then conclude that indeed the essential image of f_* (or, in other words, the bireflective subcategory associated to f) consists of the torsionfree divisible abelian groups and it is, therefore, extension-closed.

3 The prime spectrum

We turn our focus to the study of commutative noetherian rings. The structure of these rings is rather well-understood, and a key part of that understanding comes from their prime ideals. It is therefore not surprising that prime ideals also play an important role in the classification results we aim to survey.

Let R be a commutative ring. Recall that an ideal $\mathfrak{p}$ of R is said to be **prime** if $\mathfrak{p} \neq R$ and whenever ab lies in $\mathfrak{p}$ for two elements a and b in R, then at least one of them must lie in $\mathfrak{p}$. The set of prime ideals of R is called the **spectrum of** R and is denoted by $\mathsf{Spec}(R)$. The spectrum of R admits a natural partial order induced by inclusion of ideals and, moreover, it can also be endowed with an interesting topology, the **Zariski topology**, by declaring the closed subsets to be the ones of the form

$$V(I) := \{\mathfrak{p} \in \mathsf{Spec}(R)\colon\ I \subseteq \mathfrak{p}\}$$

for some ideal I of R. It is an easy exercise to check that this indeed yields a topology. Note that the closed points of $\mathsf{Spec}(R)$ are then precisely the maximal ideals of R. There is a full characterisation of the topological spaces arising as Zariski spectra of commutative rings. These are precisely the compact spaces which are T_0[1] and for which the compact open subsets form a basis for the topology and are closed under finite intersections ([7]). These are the so-called **spectral spaces**.

Example 5.— Let us go back to $R = \mathbb{Z}$. The prime ideals are precisely those generated by a prime natural number and, in addition, the zero ideal (because $\mathbb{Z}$ is an integral domain!). Every non-zero prime ideal is maximal, and thus, we have countably many closed points, and a point (the zero ideal) whose closure is the whole spectrum. The poset of prime ideals can be depicted as in

[1] For every pair of distinct points there is an open subset containing one and not the other

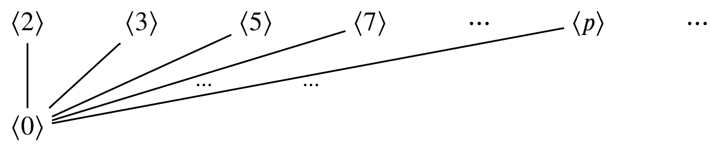

Figure 1: $\mathsf{Spec}(\mathbb{Z})$.

Figure 1 where the convention is that an edge represents an inclusion from the lower row to the upper row.

Given a commutative ring R and a prime ideal $\mathfrak{p}$ of R, we define the **height** of $\mathfrak{p}$ (denoted by $\mathsf{ht}(\mathfrak{p})$) as the largest integer n for which there is a chain of prime ideals of R of the form

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

The **Krull dimension** of R (denoted by $\mathsf{Kdim}(R)$) is then defined to be the supremum of the heights of its prime ideals, i.e. $\mathsf{Kdim}(R) := \sup\{\mathsf{ht}(\mathfrak{p}) : \mathfrak{p} \in \mathsf{Spec}(R)\}$.

We will also restrict ourselves to the class of commutative noetherian rings. Recall that a commutative ring is said to be **noetherian** if there are no infinite strictly ascending chains of ideals. Examples of such rings are $\mathbb{Z}$ or quotients of polynomial rings by any ideal: $\mathbb{K}[x_1, ..., x_n]/I$ ($\mathbb{K}$ is a field). It is an easy exercise to check that the spectrum of a commutative noetherian ring R is a noetherian space (any ascending chain of open subsets stabilises). It can also be shown that, for every ideal I of such a ring R, there is a finite set of prime ideals $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ such that $V(I) = \cup_{i=1}^{k} V(\mathfrak{p}_i)$. This implies that the Zariski topology on $\mathsf{Spec}(R)$ can be easily recovered from the poset of prime ideals. Moreover, we also have that the poset of prime ideals over a commutative noetherian ring satisfies the following theorem.

Theorem 2.— [8, Theorem 144] *Let R be a commutative noetherian ring. Whenever there are prime ideals $\mathfrak{p}, \mathfrak{t}$ and $\mathfrak{q}$ such that $\mathfrak{p} \subsetneq \mathfrak{t} \subsetneq \mathfrak{q}$, then there are also infinitely many prime ideals $\mathfrak{s}$ such that $\mathfrak{p} \subsetneq \mathfrak{s} \subsetneq \mathfrak{q}$.*

In particular, any commutative noetherian ring R which contains a prime ideal which is neither a maximal nor a minimal prime ideal has the property that $\mathsf{Spec}(R)$ is infinite. As a consequence, if a commutative noetherian ring R has only finitely many prime ideals, then R must have Krull dimension at most 1.

By definition, the complement of a prime ideal is closed under multiplication (and contains the unit of the ring). As such, it is a good candidate for a set of denominators in a ring of fractions. Given a prime ideal $\mathfrak{p}$ in a commutative ring R, we may consider the ring of fractions obtained by adding to R formal inverses to the elements in the complement $S = R \setminus \mathfrak{p}$, and we denote it by $R_\mathfrak{p}$. Moreover, there is a natural ring homomorphism $\psi_S : R \longrightarrow R_\mathfrak{p}$ sending an element r to the fraction $\frac{r}{1}$. This ring homomorphism is, in fact, a ring epimorphism and its associated bireflective subcategory is extension-closed. The following proposition recalls how $\mathsf{Spec}(R)$, $\mathsf{Spec}(R/\mathfrak{p})$ and $\mathsf{Spec}(R_\mathfrak{p})$ are related.

Proposition 3.— *Let R be a commutative noetherian ring and $\mathfrak{p}$ a prime ideal of R. Then we have that*

1. *$\mathsf{Spec}(R/\mathfrak{p})$ is homeomorphic to the subspace $V(\mathfrak{p})$ of $\mathsf{Spec}(R)$;*
2. *$\mathsf{Spec}(R_\mathfrak{p})$ is homeomorphic to the subspace $\Lambda(\mathfrak{p}) := \{\mathfrak{q} \in \mathsf{Spec}(R) : \mathfrak{q} \subseteq \mathfrak{p}\}$ of $\mathsf{Spec}(R)$;*
3. *$\mathsf{Kdim}(R_\mathfrak{p}) = \mathsf{ht}(\mathfrak{p})$ and $\mathsf{Kdim}(R/\mathfrak{p}) \leq \mathsf{Kdim}(R) = \mathsf{ht}(\mathfrak{p})$.*

Example 6.— Let R be again the ring of integers $\mathbb{Z}$, and let $\mathfrak{p}$ be the ideal generated by the prime 2. Note that $\mathbb{Z}$ is a ring of Krull dimension 1 and $\langle 2 \rangle$ is a prime ideal of height 1. Then $R/\mathfrak{p}$ is the field $\mathbb{Z}/2\mathbb{Z}$ which has a unique prime ideal: the zero ideal. This fits with the fact that $\mathsf{Spec}(\mathbb{Z}/2\mathbb{Z})$ ought to be homeomorphic to the subspace of $\mathsf{Spec}(\mathbb{Z})$ given by $V(\langle 2 \rangle) = \{\langle 2 \rangle\}$. A similar check can be done for the localisation of $\mathbb{Z}$ at $\langle 2 \rangle$. Indeed, the ring of fractions $\mathbb{Z}_{\langle 2 \rangle}$ can be described as a subring of $\mathbb{Q}$ as

$$\mathbb{Z}_{\langle 2 \rangle} = \{\frac{a}{b} \in \mathbb{Q} : \mathsf{gcd}(b, 2) = 1\}$$

where gcd denotes de greatest common divisor. The spectrum of $\mathbb{Z}_{\langle 2 \rangle}$ is indeed homeomorphic to $\Lambda(\langle 2 \rangle)$ and consists of two prime ideals: the zero ideal and the maximal ideal $2\mathbb{Z}_{\langle 2 \rangle}$.

4 Classification results

We want to illustrate the idea that for a commutative noetherian ring R, the structure of $\mathsf{Spec}(R)$ controls much of the representation theory of R, i.e. the structure of $\mathsf{Mod}(R)$. For this purpose we introduce the following notion of support. Given an R-module M, consider the set of primes

$$\mathsf{Supp}(M) = \{\mathfrak{p} \in \mathsf{Spec}(R) : \exists i \geq 0 : \mathsf{Tor}_i^R(M, k(\mathfrak{p})) \neq 0\}$$

where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of R at $\mathfrak{p}$ and $\mathrm{Tor}_i^R(-, k(\mathfrak{p}))$ is the i-th derived functor of $- \otimes_R k(\mathfrak{p})$. We refer to [2] for further details.

If M is a finitely generated R-module, $\mathrm{Supp}(M)$ is easier to calculate: it coincides with $\{\mathfrak{p} \in \mathrm{Spec}(R) : M \otimes_R R_\mathfrak{p} \neq 0\}$ ([2, Lemma 2.2]). From a geometric standpoint this is the support of the associated coherent sheaf $\widetilde{M}$ over $\mathrm{Spec}(R)$, i.e. the set of points p in $\mathrm{Spec}(R)$ where the stalk $\widetilde{M}_p$ does not vanish.

We can also consider support of subcategories of R-modules: we define $\mathrm{Supp}(\mathcal{U})$, for $\mathcal{U}$ a subcategory of $\mathrm{Mod}(R)$, to be the union of $\mathrm{Supp}(U)$ where U runs over all R-modules in $\mathcal{U}$. This provides us with a way to assign a subset of $\mathrm{Spec}(R)$ to any subcategory of $\mathrm{Mod}(R)$, and this is the key tool to the classification results we will discuss next.

4.1 Localising subcategories

The first theorem we want to present is a well-known classification by support of localising subcategories of modules over a commutative noetherian ring. Moreover, it turns out that the subsets of the spectrum that arise as support of localising subcategories are arbitrary unions of closed sets. Such subsets are called **specialisation-closed subsets**. Equivalently, a subset V of the spectrum of a commutative noetherian ring R is specialisation-closed if for any $\mathfrak{p}$ in V, any prime $\mathfrak{q}$ containing $\mathfrak{p}$ must also lie in V.

It can be shown that a specialisation-closed subset V has minimal elements (for the partial order induced by inclusion) and that these completely determine V. Indeed, if V is specialisation-closed and $(\mathfrak{p}_i)_{i\in I}$ is the collection of the minimal elements of V, then $V = \cup_{i\in I} V(\mathfrak{p}_i)$.

Theorem 4.— [9, Ch.VI,§5, §6] For a commutative noetherian ring R, the assignment of support yields a bijection between

1. localising subcategories of $\mathrm{Mod}(R)$;
2. specialisation-closed subsets of $\mathrm{Spec}(R)$.

Note that the inverse assignment sends a specialisation-closed subset V of $\mathrm{Spec}(R)$ to the subcategory of all modules whose support is contained in V.

Remark 1.— It can be shown that for a commutative noetherian ring R, there is a topology on $\mathrm{Spec}(R)$ for which the open sets are the (Zariski) specialisation-closed subsets above described. This is called the Hochster dual topology. For more details on this duality, we refer to [7].

Recall that an R-module M is said to be **flat** if $-\otimes_R M$ is an exact functor or, equivalently, if the derived functors $\mathrm{Tor}_i^{\mathbb{R}}(-, \mathbb{M})$ are identically zero.

Example 7.— Consider again the localising subcategory Tors of $\mathrm{Mod}(\mathbb{Z})$ (see Example 2). We show that $\mathrm{Supp}(\mathrm{Tors}) = \mathrm{Spec}(\mathbb{Z}) \setminus \{\langle 0\rangle\}$ (this is specialisation-closed: it is the set of all maximal ideals of $\mathbb{Z}$).

First observe that $\mathbb{Z}_{\langle 0\rangle} = \mathbb{Q}$ and, therefore, $k(\langle 0\rangle) = \mathbb{Q}$. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, we have that $\mathrm{Tor}_i^{\mathbb{Z}}(-, \mathbb{Q}) = 0$ for all $i \geq 1$. Hence $\langle 0\rangle$ lies in $\mathrm{Supp}(M)$ (for some abelian group M) if and only if $M \otimes_\mathbb{Z} \mathbb{Q} \neq 0$. Observe, however, that if M is an abelian group where every element has finite order, $M \otimes_\mathbb{Z} \mathbb{Q} = 0$. Indeed, if m is an element of M of order $n \geq 1$, then for any $q \in \mathbb{Q}$, we have that $m \otimes q = mn \otimes \frac{q}{n} = 0$. This shows that $\mathrm{Supp}(\mathrm{Tors}) \subseteq \mathrm{Spec}(\mathbb{Z}) \setminus \{\langle 0\rangle\}$.

To prove the converse we show that $\mathrm{Supp}(\mathbb{Z}/p\mathbb{Z}) = \{\langle p\rangle\}$ for any prime p in $\mathbb{Z}$. Since $\mathbb{Z}/p\mathbb{Z}$ is finitely generated, then it is enough to compute $\mathbb{Z}/p\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_{\langle q\rangle}$ for any prime q. Since p is invertible in $\mathbb{Z}_{\langle q\rangle}$ whenever $p \neq q$, a similar argument to the one above shows that $\mathbb{Z}/p\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_{\langle q\rangle} = 0$ for all $q \neq p$. Finally, it can be shown that $\mathbb{Z}/p\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_{\langle p\rangle} \cong k(\mathfrak{p}) \neq 0$. This proves the claim.

4.2 Extension-closed bireflective subcategories

We now turn our attention to extension-closed bireflective subcategories of $\mathrm{Mod}(R)$, where R is commutative and noetherian as before. As mentioned earlier, the condition of extension-closure imposes some *nice* homological behaviour on the associated ring epimorphism. It turns out that, in the context of commutative noetherian rings, the extension-closure requirement gives us an *extremely nice* homological behaviour: flatness.

Theorem 5.— [1] Let R be a commutative noetherian ring and let $f : R \longrightarrow S$ be a ring epimorphism. Then the bireflective subcategory associated to f is extension-closed if and only if S is flat as an R-module.

Ring epimorphisms as those in the theorem above are called **flat ring epimorphisms**. The theorem is far from being true outside the commutative noetherian setting.

A classification of extension-closed bireflective subcategories therefore amounts to a classification of flat ring epimorphisms (up to equivalence). Observe moreover that, given a flat ring epimorphism $f : R \longrightarrow S$, the R-modules M for which $M \otimes_R S = 0$ form a localising subcategory of $\mathrm{Mod}(R)$ which is supported, by Theorem 4, on a specialisation-closed subset V. This gives us a way of assigning a specialisation-closed subset in $\mathrm{Spec}(R)$ to any given equivalence class of flat ring epimorphisms of R. What are the properties of this assignment?

THEOREM 6.— [1] Let R be a commutative noetherian ring. Let Ψ be the assignment sending the equivalence class of a flat ring epimorphism $f\colon R \longrightarrow S$ to the support of the subcategory of R-modules M such that $M \otimes_R S = 0$. Then:

1. Ψ is an injective assignment;
2. The image of Ψ is contained in the set of specialisation-closed subsets of $\mathrm{Spec}(R)$ whose minimal elements have height at most 1;
3. If $\mathrm{Kdim}(R) \leq 1$ or if R is regular, then Ψ induces a bijection between flat ring epimorphisms up to equivalence and specialisation-closed subsets of $\mathrm{Spec}(R)$ whose minimal elements have height at most 1.

Recall that the commutative noetherian regular rings are precisely those of finite global dimension. Geometrically, regularity can be interpreted as the smoothness of the corresponding affine scheme. It is always possible to describe the image of Ψ, and the assumptions in point (3) of the theorem can be significantly relaxed - all of this at the expense of requiring some more technical tools ([1]). The following corollary characterises completely the support of extension-closed bireflective subcategories under the same assumptions as in (3) above. This follows from the fact that this support will be the complement of the specialisation-closed subset arising through Ψ.

COROLLARY 7.— [1] Let R be a commutative noetherian ring and suppose that R either has Krull dimension at most one or is a regular ring. Then the assignment of support establishes a bijection between

1. extension-closed bireflective subcategories of $\mathrm{Mod}(R)$;
2. subsets of $\mathrm{Spec}(R)$ whose complement is specialisation-closed and the minimal primes of the complement have height at most one.

REMARK 2.— Note that for commutative noetherian rings of Krull dimension at most one, the assumption on the height of minimal primes in the specialisation-closed subsets is automatically fulfilled. Therefore, if R has Krull dimension at most one, there is a bijection between flat ring epimorphisms up to equivalence (or extension-closed bireflective subcategories) and specialisation-closed subsets of $\mathrm{Spec}(R)$.

EXAMPLE 8.— We give a complete list of flat ring epimorphisms (up to equivalence) starting in $\mathbb{Z}$. Such ring epimorphisms are classified by specialisation-closed subsets of $\mathrm{Spec}(\mathbb{Z})$, and these are: $\varnothing$, elements of $\mathscr{P}(\mathrm{Max}(\mathbb{Z}))$ (the power set of maximal ideals of $\mathbb{Z}$) and $\mathrm{Spec}(\mathbb{Z})$.

1. Let us consider the first specialisation-closed subset $V = \varnothing$. The corresponding extension-closed bireflective subcategory $\mathscr{B}$ must be supported on $\mathrm{Spec}(\mathbb{Z}) \setminus V = \mathrm{Spec}(\mathbb{Z})$. Therefore, we have $\mathscr{B} = \mathrm{Mod}(\mathbb{Z})$ and the associated flat ring epimorphism (up to equivalence) is the identity on $\mathbb{Z}$.
2. Let V be the set of prime ideals determined by a subset of prime natural numbers P. The associated flat ring epimorphism can be checked to be (up to equivalence) the map
$$f_P\colon \mathbb{Z} \longrightarrow \mathbb{Z}[P^{-1}]$$
where $\mathbb{Z}[P^{-1}]$ can be identified with the subring of $\mathbb{Q}$ consisting of the fractions a/b such that b has no prime factors which are not in the set P and $f_P(r) = r/1$.
3. Finally, let V be $\mathrm{Spec}(\mathbb{Z})$. Since its complement is empty, the flat ring epimorphism (up to equivalence) that we are looking for is the trivial one: $f\colon \mathbb{Z} \longrightarrow 0$.

The situation described in the example above is extremely nice and not at all typical. In this case we were able to describe all flat ring epimorphisms as rings of fractions, but in general we may need more robust localisation techniques. This is explored in detail in [1].

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