Where Lie Algebra meets Probability?!

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One of the biggest challenges in statistical physics is to understand phenomena out of equilibrium. A common setting to model non-equilibrium dynamics is to consider stochastic processes of Markovian type with an open boundary acting on the system at different values, thus creating a flux. In these notes, we consider an interacting particle system known in the literature as the symmetric simple exclusion process (SSEP) which is connected to two reservoirs. We show how the algebraic construction of such Markov jump processes helps in analyzing microscopic quantities used to derive macroscopic universal laws. In particular, we will characterize through its moments the non-equilibrium stationary measure.

1 Introduction

Microscopic dynamics of random walks interacting on a discrete space under some stochastic rules are known as interacting particle systems (IPS) and were introduced in the mathematics community by Spitzer in 1970 (see [14]) but before were widely used by physicists, see [15]. The idea of introducing such systems is that, as it often happens in mathematics and physics, they can be used as toy models to describe complex stochastic phenomena involving a large number (typically of the order of the Avogadro's number) of interrelated components. Regardless their simple rules at the microscopic level, IPS are often remarkably suitable models capable of capturing the sort of phenomena one is interested at the macroscopic level. Mathematically speaking, they are treated as continuous time Markov processes with a finite or countable discrete state space. Typically, in the field of IPS one is interested in deriving the macroscopic laws of some thermodynamical quantities by means of a scaling limit procedure. The setting can be described as follows. One considers a continuous space, which is called the macroscopic space. This space is then discretized by a scaling parameter n and time is speeded up by a function of n. On the discrete space one considers a microscopic dynamics consisting of the infinitesimal evolution of particles according to some stochastic law. The dynamics conserves one (or more) thermodynamical quantity and its (their) space/time evolution is the object of our interest. The mathematical rigorous derivation of the macroscopic laws for such quantity, which can be a PDE or a stochastic PDE, depending on whether one is at the level of the Law of Large Numbers or at the level of the Central Limit theorem, is a central problem in the field of IPS. This derivation gives not only validity to the equations obtained but also some physical motivation for their study. In these notes we present, as toy model, the most classical IPS and our aim is, first, explain how to rewrite the Markovian generator of the process in terms of the generators of a Lie algebra, this is a known procedure in the literature. This technique allows to derive a dual process for our model, whose dynamics is simpler. It can be used to give relevant information about our original model; second, explain how to extract from our random dynamics a solution to a PDE, describing the space-time evolution of the density of our model.

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2 The model

To make our presentation simple, we consider as macroscopic space the interval $[0,1]$. Let n be a scaling parameter, we split that continuous space into intervals of size $1/n$.

To an interval of the form $[x/n,(x+1)/n)$ we associate the microscopic position x and then we have a discrete space which is the microscopic space. Now we define the random dynamics. Among the simplest and most widely studied IPS there is the SSEP, see e.g. [12], whose dynamics can be described as follows. After a certain random time, a particle decides to jump to a position of the microscopic space. In SSEP, particles jump among sites under the *exclusion* rule, namely each site can accommodate at most one particle, therefore, if a particle wants to jump to an occupied site, that jump is forbidden. Interactions are to nearest-neighbors (and this is why the process coins the name *simple*) and the jump rates to left and right are identical (*symmetric*). Our toy model is the SSEP with an open boundary, namely we attach two reservoirs that can inject or remove particles from their neighbor positions. The time between jumps is exponentially distributed, which guarantees that this process is Markovian, therefore its evolution can be entirely described via its Markov generator. In the next subsection we define it rigorously.

2.1 Probabilistic description

Consider the microscopic space $\Sigma_n := \{1,\dots,n-1\}$, called bulk, which corresponds to the macroscopic interval $[0,1]$. The construction of the SSEP evolving on Σ_n is done in the following way. To properly define the exchange dynamics, for each $x \in \Sigma_n$, we call $\eta(x)$ the occupation variable at site x: if $\eta(x)=0$ (resp. $\eta(x)=1$) it means that site x is empty (resp. occupied). With this restriction, the state space of our Markov process is $\Omega_n := \{0,1\}^{\Sigma_n}$. We denote by $\eta \in \Omega_n$ a configuration of particles. To each bond of the form $\{x,x+1\}$ with $x=1,\dots,n-2$, we associate a Poisson process of parameter 1, that we denote by $N_{x,x+1}(t)$. Now we describe the boundary dynamics. We artificially add the sites $x=0$ and $x=n$ that stand for the left and right reservoirs, respectively. We associate two independent Poisson Processes to each bond $\{0,1\}$ and $\{n-1,n\}$ in the following way: $N_{0,1}(t)$ (resp. $N_{n,n-1}(t)$) with parameter $\alpha n^{-\theta}$ (resp. $\delta n^{-\theta}$) and $N_{1,0}(t)$ (resp. $N_{n-1,n}(t)$) with parameter $\gamma n^{-\theta}$ (resp. $\beta n^{-\theta}$). All the Poisson processes described above are independent, so that the probability that two of them take the same value is equal to zero. This means that only one jump occurs whenever there is a possible transition. Before we proceed, we note that the role of the parameters $\alpha,\gamma,\beta,\delta \geq 0$ is to fix the reservoirs' density/current, while the role of $\theta \in \mathbb{R}$ is to tune the strength of the reservoirs' according to the scale parameter n. Taking, for example, θ negative the reservoirs are strong and for θ positive, they are weak and interactions between the boundary and the bulk is weaker as the value of θ increases.

We observe that given the initial configuration of the system plus the realization of all the Poisson processes, it is straightforward to obtain the whole evolution of the system. The role of the Poisson processes is to fix the random time between jumps. We show an example in figure 1 where we consider $n=5$: an initial condition is given namely $\eta_0=\delta_2$, ie the configuration with just a particle at site 2, together with all the realizations of the Poisson processes.

In figure 2 we exhibit all the configurations that we obtained from the initial configuration $\eta_0=\delta_2$ and all the realizations of the Poisson processes given in figure 1.

We warn the reader that below we indexed the configurations in terms of the marks of the Poisson processes and not in time, since our Markov chain evolves in continuous time.

We denote by $\eta^{x,x+1}$ the configuration obtained from η by swapping the values $\eta(x)$ and $\eta(x+1)$, that is $\eta^{x,x+1}(z) = \mathbf{1}_{\Sigma_n\backslash\{x,x+1\}}(z)\eta(z) + \mathbf{1}_{\{x\}}(z)\eta(x+1) + \mathbf{1}_{\{x+1\}}(z)\eta(x)$. On the other hand, when we see a mark of a Poisson process from the boundary, for example, $N_{0,1}(t)$ (resp. $N_{1,0}$), this means that we inject (resp. remove) a particle at the position $x=1$, if this site is empty (resp. occupied), otherwise nothing happens. More precisely, η^1 is the configuration obtained from η by flipping the occupation variable at 1, that is $\eta^1(z) = \mathbf{1}_{\Sigma_n\backslash\{1\}}(z)\eta(z) + \mathbf{1}_{\{1\}}(z)(1-\eta(1))$.

The exchange dynamics is described by the generator L_{ex}, which acts on functions $f:\Omega_n \to \mathbb{R}$ as $L_{ex}f(\eta) = \sum_{x=1}^{n-2} L_{x,x+1}f(\eta)$ where

$$L_{x,x+1}f(\eta) = c_{x,x+1}(\eta)\left[f(\eta^{x,x+1}) - f(\eta)\right] \qquad (2)$$

and the rates are

$$c_{x,x+1}(\eta) = \{\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))\}.$$

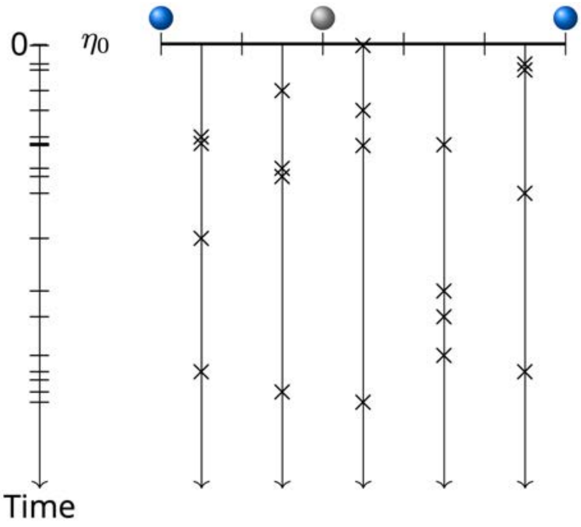

Figure 1.— An initial configuration and marks of the Poisson clocks between each bond.

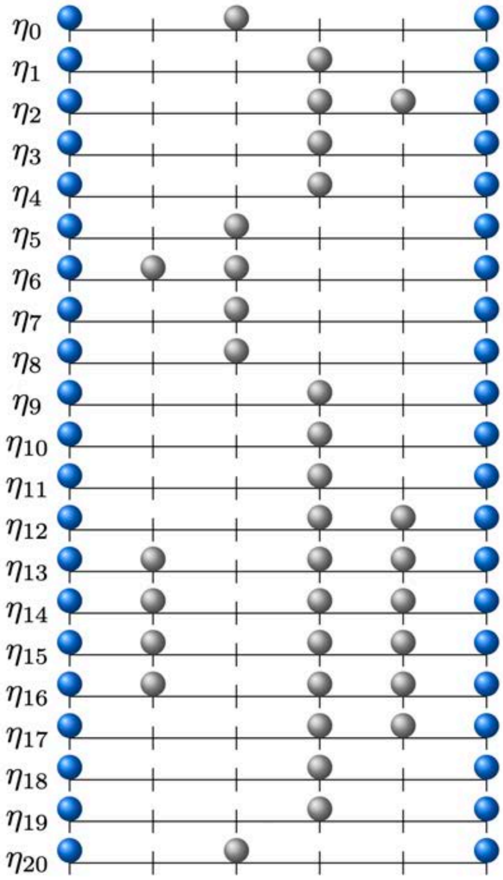

Figure 2.—Configurations evolving according to the marks of the Poisson processes.

The left reservoir generator acts on functions $f : \Omega_n \to \mathbb{R}$ as

$$L_\ell f(\eta) = \frac{1}{n^\theta}\{\alpha(1-\eta(1)) + \gamma\eta(1)\}\left[f(\eta^1) - f(\eta)\right]$$

and the right reservoir L_r is defined analogously, with 1 replaced by $n-1$, α by δ and γ by β. Finally, the open SSEP dynamics is described by a superposition of the two dynamics described above, the exchange and the flip dynamics, so that its full generator is given by

$$L_{\text{SSEP}} = L_\ell + L_{ex} + L_r. \tag{3}$$

Observe that the left and right reservoirs at different densities (respectively $\rho_a = \alpha/(\gamma+\alpha)$ and $\rho_b = \delta/(\beta+\delta)$) impose a flux of particles throughout the system. See a picture below for an illustration of the dynamics just defined.

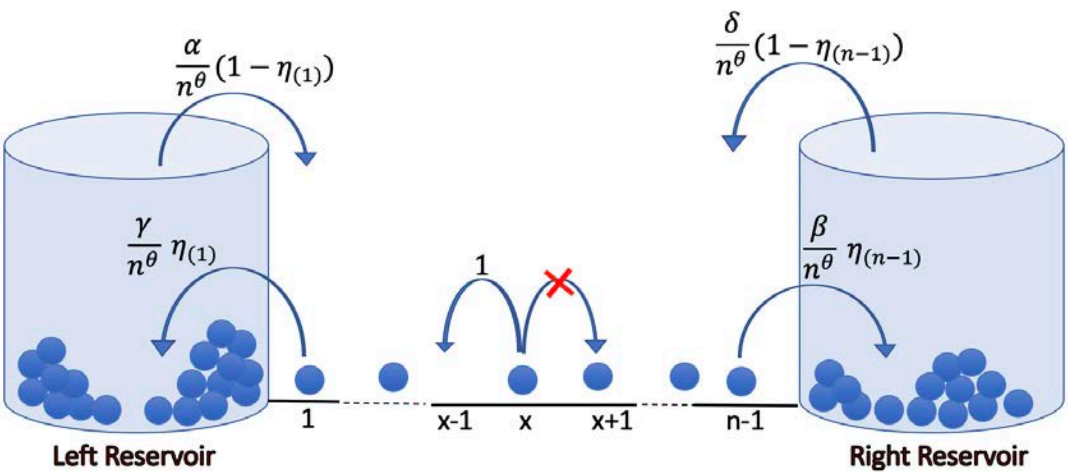

Figure 3.—Schematic description of dynamics of open SSEP.

2.2 Algebraic description

An interesting feature of some IPS is that their generators can be entirely described by the generators of a suitable algebra, for our toy model this will be the Lie algebra $\mathfrak{su}(2)$. More of such constructions were introduced in [10] and further developed in [4]. The Lie algebra $\mathfrak{su}(2)$ is a 3-dimensional vector space of traceless matrices together with the bilinear map $[\cdot,\cdot] : \mathfrak{su}(2)\times\mathfrak{su}(2) \to \mathfrak{su}(2)$ called Lie bracket, which is anti-symmetric, i.e. $[x,y] = -[y,x]$ and satisfies the Jacobi identity, i.e. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for all $x,y,z \in \mathfrak{su}(2)$. We equip $\mathfrak{su}(2)$ with the adjoint map $* : \mathfrak{su}(2) \to \mathfrak{su}(2)$, i.e. $x \to x^*$ such that $\left(x^*\right)^* = x$ and $\left[x^*, y^*\right] = [y,x]^*$. Usually a basis for $\mathfrak{su}(2)$ is given by the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are hermitian and unitary. Such matrices satisfy the following commutator and adjoint relations $[\sigma_j, \sigma_{j+1}] = 2i\sigma_{j+2}$ and $\sigma_j^* = \sigma_j$ for $j \in \mathbb{N} \pmod 3$. We also introduce the quadratic element called Casimir (which does not belong to $\mathfrak{su}(2)$) as $C = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$; it is central, i.e. it commutes with all the elements of $\mathfrak{su}(2)$ and it is self-adjoint. For our purpose, it will be more convenient to introduce the basis of real operators J^0, J^+ and J^- which we call generators of the Lie algebra $\mathfrak{su}(2)$. They are given by

$$J^- := \frac{\sigma_1 - i\sigma_2}{2}, \quad J^+ := \frac{\sigma_1 + i\sigma_2}{2}, \quad J^0 := \frac{\sigma_3}{2},$$

and they satisfy the following commutation and adjoint relations $[J^0, J^\pm] = \pm J^\pm, [J^+, J^-] = 2J^0$ and $(J^0)^* = J^0, (J^+)^* = J^-$. The Casimir element in this setting is $\mathscr{C} = 2(J^0)^2 + J^+J^- + J^-J^+$. Besides the matrices representation, an equivalent representation is given by the action on functions $f : \{0,1\} \to \mathbb{R}$ as

$$\begin{cases} (J^- f)(\eta_x) = (1-\eta_x) f(\eta_x + 1) \\ (J^+ f)(\eta_x) = \eta_x f(\eta_x - 1) \\ (J^0 f)(\eta_x) = (\eta_x - 1/2) f(\eta_x) \end{cases}$$

where we made the convention $f(-1) = f(2) = 0$. The operators above are also known as angular momentum operators. We now show how to write the open SSEP dynamics in this context. In particular, it is verified that the exchange generator defined in (3) can be written as the tensor product of the Casimir element. For sites $x, x+1$ we get

$$L_{x,x+1} = J_x^+ J_{x+1}^- + J_x^- J_{x+1}^+ + 2J_x^0 J_{x+1}^0 - 1/2\,, \quad (4)$$

and then summing over Σ_n we obtain L_{ex}.

A similar description holds true for the generators of the boundary reservoirs,

$$L_\ell = \frac{1}{n^\theta}\left\{\alpha\left[J_1^- + J_1^0 - \frac{1}{2}\right] + \gamma\left[J_1^+ - J_1^0 - \frac{1}{2}\right]\right\}$$

and similarly L_r is obtained replacing the algebra generators acting on site $n-1$. Note that above the notation J_x^a for $a \in \{0,+,-\}$ means that the generator is acting on the occupation variable at site $x \in \Sigma_n$. Why is this algebraic description useful? In the next section we see that, whenever it is possible to describe a Markov generator using an algebra representation then a useful property, duality, can be derived.

3 Duality for Markov generators

The advantage of dealing with a stochastic evolution lies in the possibility to use probabilistic techniques which considerably simplify the analysis of the system. A powerful tool to deal with Markov processes is *duality theory*, see [13]. This theory allows several simplifications: in a nutshell, one can infer information on a given process by using a simpler one, its dual. For our toy model, we will see how to relate the open SSEP with a simpler system where the open boundary is turned into an absorbing boundary. Indeed, duality in the context of IPS allows "replacing" boundary reservoirs, modeling birth and death processes, with absorbing reservoirs which, as time goes to infinity, will eventually absorb all the particles in the system. It is due to this simplification that one can study properties such as the k-point correlation function of a non-equilibrium system using properties of a dual system consisting of only k dual particles. The link between these two processes, the original, denoted by η_t and with state space Ω, and its dual, denoted by $\hat\eta_t$ and with state space $\hat\Omega$, is provided by a set of so-called *duality functions* $D : \Omega\times\hat\Omega \to \mathbb{R}$, i.e. a set of observables that are functions of both processes and whose expectations, with respect to the two randomness, satisfy the following relationship for all $t \geq 0$

$$\mathbb{E}_\eta[D(\eta_t, \hat\eta)] = \hat{\mathbb{E}}_{\hat\eta}[D(\eta, \hat\eta_t)]\,. \quad (5)$$

Above $\mathbb{E}_\eta$ (resp. $\hat{\mathbb{E}}_{\hat\eta}$) is the expectation with respect to the law of the η_t process initialized at η (resp. the $\hat\eta_t$ process initialized at $\hat\eta$). If the generators of the processes are explicit, denoting by $\mathscr{L}$ the generator of $\hat\eta_t$ and by $\hat{\mathscr{L}}$ the generator of its dual, $\hat\eta_t$, then a duality relation with duality function D translates in saying that

$$\Big(\mathscr{L} D(\cdot,\hat\eta)\Big)(\eta) = \Big(\hat{\mathscr{L}} D(\eta,\cdot)\Big)(\hat\eta). \tag{6}$$

In other words, the action of $\mathscr{L}$ on the first variable of D is equivalent to the action of $\hat{\mathscr{L}}$ on the second variable of D. This is when the algebra comes in. Proving the above relation knowing just the definition of the original process η_t by its generator would be very complicated, however with the algebraic description of η_t one can have a feeling of what to look for. The idea is that we can decompose the Markov generator using the algebra generators as building blocks. This simplifies the analysis because instead of looking for a duality function, one looks for an intertwiner function between two representations of the Lie algebra $\mathfrak{su}(2)$. Such intertwiner function between the J^0, J^-, J^+ representations yields the duality function. Given the special features of our toy model, the duality function will turn out to be product of indicator functions, for which the direct computation is not hard. Nevertheless, for more general dynamics which, for example, allows more than a particle per site, the duality function has a more complicated form and the aforementioned decomposition brings advantages in the proof of the duality relationship.

3.1 Duality for open SSEP

In the following result we give all the ingredients to find a duality relation for our model: it states that, the open SSEP is dual, via a moment duality function D, to a Markov process with the same exclusion dynamics in Σ_n but with only absorbing reservoirs at the boundary.

Theorem 1 (Duality for open SSEP).— For the open SSEP with generator given in 3, the duality relation in (6) is verified for $D : \Omega_n \times \hat\Omega_n \to \mathbb{R}$ given by

$$D(\eta,\hat\eta) = \left(\rho_a\right)^{\hat\eta_0} \prod_{x\in\Sigma_n} \mathbf{1}_{\{\eta_x \geq \hat\eta_x\}} \left(\rho_b\right)^{\hat\eta_n} \tag{7}$$

where $\hat\Omega_n := \mathbb{N}_0 \times \Omega_n \times \mathbb{N}_0$ and the dual generator $\hat L = \hat L_\ell + \hat L_0 + \hat L_r$ acts on functions $f : \hat\Omega_n \to \mathbb{R}$ as $\hat L_{ex} f(\hat\eta) = \sum_{x=1}^{n-2} L_{x,x+1} f(\hat\eta)$ in the bulk, where $L_{x,x+1}$ is as in (2); and

$$\hat L_\ell f(\hat\eta) = \frac{1}{n^\theta}(\alpha+\gamma)\hat\eta(1)\left[f(\hat\eta^1) - f(\hat\eta)\right]$$

and analogously for the right reservoir with $\alpha+\gamma$ replaced by $\beta+\delta$.

For the interested reader, a rigorous proof of this theorem (for $\theta = 0$) can be found in [5] and it is not hard to generalize it for any value of the parameter θ.

Note that the action in the bulk of the dual generator has the same dynamics of the original one, while the boundary generators only absorb particles from sites 1 and $n-1$. The plan is to describe the dual process with another $\mathfrak{su}(2)$ and find the duality function as intertwiner function of the algebra generators. This is given by the following representation which act on the same functions $f : \{0,1\} \to \mathbb{R}$ as

$$\begin{cases} (\hat J^- f)(\hat\eta_x) = \hat\eta_x f(\hat\eta_x - 1) \\ (\hat J^+ f)(\hat\eta_x) = (1-\hat\eta_x) f(\hat\eta_x+1) + (2\hat\eta_x - 1) f(\hat\eta_x) \\ \qquad\qquad -\hat\eta_x f(\hat\eta_x - 1) \\ (\hat J^0 f)(\hat\eta_x) = (\hat\eta_x - 1/2) f(\hat\eta_x) - \hat\eta_x f(\hat\eta_x - 1) \end{cases}$$

where, again, $f(-1) = f(2) = 0$. One can check that the above generators are a representation for the Lie algebra $\mathfrak{su}(2)$. Moreover, since the Casimir is irreducible in any representation we can describe the dual process in the same way as in equation (4). At this point one can see that

$$g(\eta_x,\hat\eta_x) = \frac{\eta_x!}{(\eta_x - \hat\eta_x)!}\Gamma(2-\hat\eta_x)\mathbf{1}_{\{\eta_x \geq \hat\eta_x\}}$$

satisfy

$$J^a g(\cdot,\hat\eta_x)(\eta_x) = \hat J^a g(\eta_x,\cdot)(\hat\eta_x)$$

for $a \in \{+,-,0\}$, i.e. $g(\eta_x,\hat\eta_x)$ intertwines two representations of the $\mathfrak{su}(2)$ algebra. Note that since both $\eta_x, \hat\eta_x \in \{0,1\}$ the above function correspond to the duality function in Ω_n. For the left reservoir generator (the right is analogous) one has to check that

$$\left(L_\ell D(\cdot,\hat\eta)\right)(\eta) = \left(\hat L_\ell D(\eta,\cdot)\right)(\hat\eta),$$

namely that

$$\begin{aligned} &\frac{\alpha(1-\eta_1)}{n^\theta}\rho_a^{\hat\eta_0}\left[\mathbf{1}_{\{\eta_1+1\geq\hat\eta_1\}} - \mathbf{1}_{\{\eta_1\geq\hat\eta_1\}}\right] \\ &-\frac{\gamma\eta_1}{n^\theta}\rho_a^{\hat\eta_0}\left[\mathbf{1}_{\{\eta_1-1\geq\hat\eta_1\}} - \mathbf{1}_{\{\eta_1\geq\hat\eta_1\}}\right] \\ &= \frac{(\alpha+\gamma)\hat\eta_1}{n^\theta}\left[\rho_a^{\hat\eta_0+1}\mathbf{1}_{\{\eta_1\geq\hat\eta_1-1\}} - \rho_a^{\hat\eta_0}\mathbf{1}_{\{\eta_1\geq\hat\eta_1\}}\right], \end{aligned}$$

which is verified since both η_1 and $\hat\eta_1$ are either 0 or 1.

4 Stationary probability measure and correlations via duality

The open SSEP is an irreducible continuous time Markov process with finite state space, therefore, by a classical theorem we know that there exists a unique stationary measure, that we denote by μ_{ss}. When $\rho = \rho_a = \rho_b$ the stationary measure of our process is an homogeneous product Bernoulli measure with parameter ρ. Moreover, this measure is also reversible. Nevertheless, when the equality $\rho = \rho_a = \rho_b$

fails, the invariant measure is no longer of product form. Heuristically speaking, the density/current at the reservoirs has a different intensity with respect to left/right reservoirs intensity and, therefore, there is an induction of a current flow of particles in the system. Take, for example, $\alpha = \beta = 0$ and $\gamma = \delta = 1$, so that particles are injected in the system from the right reservoir and only exit through the left one. Below we explain briefly how to get some information regarding this measure. Without loss of generality, in order to get information about the stationary measure it will be easier to consider the special case where the reservoirs' rates satisfy $\gamma = 1-\alpha$ and $\beta = 1-\delta$. From here we assume that this is the case. Under these conditions the density of the reservoirs coincide with their injection rate.

4.1 Application of duality

The peculiarity of having a dual process where the boundary becomes only absorbing relies on the fact that, even if two extra sites are considered, the total mass of the dual process can only decrease during the time evolution. As time increases, the bulk will become empty and all the dual particles will eventually stay either on the left or the right reservoirs. In particular, we now show how duality connects the moments of the initial process η with the absorption probabilities of the dual process $\hat{\eta}$. This is done via the following formula

$$\mathbb{E}_{\mu_{ss}}[D(\eta,\hat{\eta})] = \sum_{m=0}^{k} \rho_a^m \rho_b^{k-m} \mathbb{P}_{\hat{\eta}}(m) \,, \qquad (7)$$

where $k = \sum_{x=0}^{n} \hat{\eta}_x$ is the total number of dual particles and $\mathbb{P}_{\hat{\eta}}(m)$ is the probability that m particles are absorbed at the left reservoir (and the remaining $k-m$ go to the right reservoir) starting from the configuration $\hat{\eta}$. The proof relies on the fact that, as $t \to \infty$, all the dual particles will be at sites 0 or n. More details can be found in [5] and [6]. Indeed,

$$\begin{aligned}
&\mathbb{E}_{\mu_{ss}}[D(\eta,\hat{\eta})] = \lim_{t\to\infty} \mathbb{E}_{\eta}[D(\eta_t,\hat{\eta})] = \\
&\lim_{t\to\infty} \mathbb{E}_{\hat{\eta}}[D(\eta,\hat{\eta}_t)] = \mathbb{E}_{\hat{\eta}} \rho_a^{\hat{\eta}_\infty(0)} \rho_b^{\hat{\eta}_\infty(n)} = \\
&\sum_{m=0}^{k} \rho_a^m \rho_b^{k-m} \mathbb{P}_{\hat{\eta}}(\hat{\eta}_\infty(0) = m,\ \hat{\eta}_\infty(n) = k-m) \,.
\end{aligned}$$

Suppose we start with a dual configuration $\hat{\eta} = \delta_{x_1} + \delta_{x_2} + \dots \delta_{x_k}$, namely we choose to put a dual particle in each site $x_1, x_2, \dots, x_k$. In this case equation (7) reads as

$$D(\eta,\hat{\eta}) = \prod_{i=1}^{k} \eta_{x_i},$$

which is exactly the function of our interest for the initial process η_t. We now show how to find the 2-point correlation function via the absorption probabilities of two dual exclusion particles. Note that equation (7) specialized for $k = 2$ and $\hat{\eta} = \delta_x + \delta_y$ reads

$$\mathbb{E}_{\mu_{ss}}[\eta_x \eta_y] = \rho_b^2 \mathbb{P}_{x,y}(0) + \rho_a \rho_b \mathbb{P}_{x,y}(1) + \rho_a^2 \mathbb{P}_{x,y}(2) \quad (8)$$

where $\mathbb{P}_{x,y}(m)$ for $m = 0,1,2$ is the probability that m particles are absorbed on the left reservoir starting with a particle in x and a particle in y. Before going to the two particles' problem we show how to solve the absorption probabilities for just one particle in the same setting.

4.2 Absorption probability for one dual walker: drunkard's walk

This is a common exercise in probability, known as the drunkard's walk. Recall our dual process, imagine that site 0 is the drunk man's house and site n is a dangerous cliff. The man is at site $x \in \Sigma_n$ and he takes random steps to the left and to the right with the same probability: what is his chance of escaping the cliff? The house and the cliff are absorbing sites in the sense that once he reaches one of them, he will stay there forever. The jump rates are described by the dual generator $\hat{L}_{\text{SSEP}}$. Let us call $p_x := \mathbb{P}_x(1)$ the probability that he reaches home starting at x. Then obviously $p_0 = 1$ and $p_n = 0$. For $x \in \{2, \dots, n-2\}$ the probability of jumping right or left is the same, 1/2, while if he is in 1 (resp. $n-1$), goes to 0 (resp. n) with probability $1/(n^\theta + 1)$ and to 2 (resp. $n-3$) with the complement probability, $n^\theta/(n^\theta + 1)$. Mathematically we have to solve the following system, which is found by conditioning on the first possible jump of the drunk man:

$$\begin{cases}
p_1 = \frac{1}{n^\theta+1} + \frac{n^\theta}{n^\theta+1} p_2 \\
p_x = \frac{p_{x-1}+p_{x+1}}{2} & \text{for } x \in \{2, \dots, n-2\} \\
p_{n-1} = \frac{n^\theta}{n^\theta+1} p_{n-2} \,.
\end{cases}$$

A simple computation shows that last identities can be rewritten in such a way that $p_x = p(x)$ is the solution of $(\mathscr{B}_n^\theta p)(x) = 0$, where the operator $\mathscr{B}_n^\theta$ acts on functions $f : \{0, \cdots, n\} \to \mathbb{R}$ as

$$\begin{aligned}
(\mathscr{B}_n^\theta f)(x) &= \tfrac{1}{2}\Delta_n f(x), \quad \text{for } x \in \{2, \cdots, n-2\}, \\
(\mathscr{B}_n^\theta f)(1) &= n^2(f(2) - f(1)) + \frac{n^2}{n^\theta}(f(0) - f(1)), \\
(\mathscr{B}_n^\theta f)(n{-}1) &= n^2(f(n{-}2) {-} f(n{-}1)) + \frac{n^2}{n^\theta}(f(n) {-} f(n{-}1)) .
\end{aligned}$$

From this we know that, for $x \in \{2, \cdots, n-2\}$, we are looking for an harmonic function of the one dimensional discrete laplacian. Therefore p_x is a polynomial in x, i.e. $p_x = Ax + B$, for $A, B \in \mathbb{R}$ for $x \in \{2, \cdots, n-2\}$.

By using the boundary conditions, we find $A = -1/(n-2+2n^\theta)$ and $B = (n-1+n^\theta)/(n-2+2n^\theta)$, so that, for $x \in \{2, \cdots, n-2\}$ we have

$$p_x = -\frac{x}{n-2+2n^\theta} + \frac{n-1+n^\theta}{n-2+2n^\theta} \tag{9}$$

We now turn to the absorption probabilities of two exclusion processes.

4.3 Absorption probabilities for two dual walkers

The idea is the same as before, we condition on the first possible jump and we obtain a difference equation which is close to a two dimensional laplacian with some boundary conditions. Recall that $\mathbb{P}_{x,y}(m)$, for $m = 0, 1, 2$ is the probability that m particles are absorbed on the left boundary starting from the configuration with one particle in x and one particle in y. To simplify notation we use $p_{x,y} := \mathbb{P}_{x,y}(m)$ and we neglect the dependence on θ, n and m. Conditioning on the first jump we get the following identities

$$\begin{cases}
p_{x,y} = \frac{1}{4}[p_{x-1,y} + p_{x+1,y} + p_{x,y-1} + p_{x,y+1}] \\
\qquad\qquad \text{for } 1 \neq x \nsim y \neq n-1 \\
p_{x,x+1} = \frac{1}{2}[p_{x-1,x+1} + p_{x,x+2}] \\
\qquad\qquad \text{for } x \neq 1, n-2 \\
p_{1,y} = \frac{n^\theta}{1+3n^\theta}[p_{2,y} + p_{1,y-1} + p_{1,y+1}] \\
\qquad + \frac{1}{1+3n^\theta} p_{0,y} \quad \text{for } 2 < y < n-1 \\
p_{x,n-1} = \frac{n^\theta}{1+3n^\theta}[p_{x,n-2} + p_{x-1,n-1} + p_{x+1,n-1}] \\
\qquad + \frac{1}{1+3n^\theta} p_{x,n} \quad \text{for } 2 < y < n-1 \\
p_{1,n-1} = \frac{1}{2+2n^\theta}[p_{0,n-1} + p_{1,n}]\frac{n^\theta}{2+2n^\theta}[p_{2,n-1} + p_{1,n-2}] \\
p_{1,2} = \frac{1}{1+n^\theta} p_{0,2} + \frac{n^\theta}{1+n^\theta} p_{1,3} \\
p_{n-2,n-1} = \frac{1}{1+n^\theta} p_{n-2,n} + \frac{n^\theta}{1+n^\theta} p_{n-3,n-1}.
\end{cases} \tag{10}$$

We observe that the above identities do not depend on the choice of m, nevertheless, as we will see below, the boundary conditions satisfied by $p_{x,y}$ do depend on m.

As above, by introducing the operator $\mathscr{O}_n^\theta$, that we define below, we can write the above system in a concise form. The operator acts on functions $f : \{0, \cdots, n\} \times \{0, \cdots, n\} \to \mathbb{R}$ in the following way: for $x \nsim y \in \{1, \cdots, n-1\}$ we have

$$\begin{aligned}
(\mathscr{O}_n^\theta f)(x, y) &= a_{x-1} f(x-1, y) + f(x+1, y) \\
&\quad + a_{y+1} f(x, y+1) + f(x, y-1) \\
&\quad - (a_{x-1} + 2 - a_{y+1}) f(x, y)
\end{aligned}$$

and for $x \in \{1, \cdots, n-2\}$ we have

$$\begin{aligned}
(\mathscr{O}_n^\theta f)(x, x+1) &= a_{x-1} f(x-1, x+1) \\
&\quad + a_{x+2} f(x-1, x+2) \\
&\quad - (a_{x-1} + a_{x+2}) f(x, x+1).
\end{aligned}$$

The coefficients satisfy $a_0 = a_n = \frac{1}{n^\theta}$, otherwise $a_x = 1$. We know that, conditioning on the fist jump, $p_{x,y} = p(x, y)$ satisfies $(\mathscr{O}_n^\theta p)(x, y) = 0$. The above equation tells us that we are looking for the harmonic function of a two dimensional laplacian which is reflected if $x \sim y$ and deformed by a factor that depends on θ if we are close to the boundary. A general solution for $m = 0, 1, 2$ is of the form $p_{x,y} = p_{x,y}(m) = A_m x + B_m y + C_m xy + D_m$. The twelve unknown constants are found by using the boundary conditions, the law of total probability, some geometric symmetries because the walk gives symmetric jumps and also the previous result regarding the drunkard's walk. For $m = 2$, it is easy to deduce that $p_{x,y}(2)$ satisfies

$$\begin{cases}
p_{0,y}(2) := p_y(1) = \frac{n-1-y+n^\theta}{n-2+2n^\theta} \\
p_{0,0}(2) := 1 \\
p_{x,n}(2) := 0
\end{cases}$$

For $m = 0$, it is easy to deduce that $p_{x,y}(0)$ satisfies

$$\begin{cases}
p_{x,n}(0) := p_x(0) = \frac{x-1+n^\theta}{n-2+2n^\theta} \\
p_{n,n}(0) := 1 \\
p_{0,y}(0) := 0
\end{cases}$$

For $m = 1$, it is easy to deduce that $p_{x,y}(1)$ satisfies

$$\begin{cases}
p_{0,y}(1) := p_y(0) = \frac{y-1+n^\theta}{n-2+2n^\theta} \\
p_{x,n}(1) := p_x(1) = \frac{n-1-x+n^\theta}{n-2+2n^\theta} \\
p_{0,n}(1) := 1 \\
p_{0,0}(1) = p_{n,n}(1) := 0
\end{cases}$$

Where the $p_x(1)$ is the probability found in the previous section and $p_x(0) = 1 - p_x(1)$, its complement. Using the last three equations of (10) together with the boundary conditions of each absorbed probability we can write for all $m = 0, 1, 2$, the probability $p_{x,y}(m)$ in terms of the C_m's only. Now we consider the following identity given by symmetry arguments

$$p_{x,y}(2) = p_{n-y,n-x}(0)$$

which allows immediately to identify $C := C_2 = C_0$. Using the law of total probability:

$$p_{x,y}(2) + p_{x,y}(1) + p_{x,y}(0) = 1$$

we get $C_1 = -2C$. And, finally, using the condition for $p_{x,y}(1)$:

$$2p_{x,x+1}(1) = p_{x-1,x+1}(1) + p_{x,x+2}(1),$$

we find that $C = \frac{1}{(n-2+2n^\theta)(n-3+2n^\theta)}$. For sake of completeness we write below the explicit values of the three absorption probabilities.

$$\begin{aligned} p_{x,y}(2) &= \frac{(n-x-2+n^\theta)(n-1-y+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)} \\ p_{x,y}(0) &= \frac{(x-1+n^\theta)(y-2+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)} \\ p_{x,y}(1) &= \frac{x(n+1)+y(n-1)-2xy}{(n-2+2n^\theta)(n-3+2n^\theta)} \\ &\qquad + \frac{2(n^\theta-1)(1+n+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)}. \end{aligned}$$

At this point one can easily find the stationary correlation by plugging the above result into equation (8). All the computations presented above are in agreement with those obtained by another method, called the matrix ansatz product [7], where the stationary correlations are also found, for details we refer the reader to [8]. An intuitive representation of the dynamics of two dual exclusion particles on $\{0, 1, \dots, n\}$ is given in the picture below. This dynamics can always be represented by the dynamics of a single particle which is performing a symmetric random walk but now evolving inside the two dimensional simplex. The red points are the traps where the random walk is absorbed forever. They represent the three possible ways that two dual exclusion particles can be absorbed in the boundary of the lattice $\{0, 1, \dots, n\}$. If the random walk reaches the vertical cathetus it means the leftmost exclusion particle has been absorbed in 0, while if it reaches the horizontal cathetus, the rightmost exclusion particle has been absorbed in n. Note that one of these two events has to happen in order that the random walk hits one of the three traps. Once the random walk reaches one of the two cathetus it cannot leave that cathetus, since in the dynamics of the two exclusion one particle is already absorbed. On the cathethus the dynamics of the two dimensional random walk is exactly the same as the one of the one dimensional random walk with absorbing boundary, whose absorption probability is given by the drunkard's walk. Note that since two exclusion particles cannot be on the same site, we removed the diagonal $y = x$, while the upper diagonal $y = x + 1$ represents the sites where the two exclusion particles are neighbors.

We observe that these arguments can be extended to higher point correlations functions like $\mathbb{E}_{\mu_{ss}}[\eta_{x_1} \cdots \eta_{x_k}]$ and also to higher dimensions, but for the purposes of this article we decide to present only the one-dimensional case and the two-point correlation function.

5 The evolution of density

The dynamics described above in different ways, if not in the presence of stochastic reservoirs, would conserve one quantity: *the number of particles*. More precisely, starting from a configuration η_0 with $k \le n-1$ particles, at any time t we would see exactly the same number of particles on η_t. Adding the stochastic reservoirs, this conservation law is destroyed and the goal is to see the effect at the macroscopic level of adding reservoirs to the system. We define then a random measure π^n that gives weight $1/n$ to each particle as

$$\pi^n(\eta, du) = \frac{1}{n}\sum_{x=1}^{n-1} \eta(x)\delta_{x/n}(du)$$

which is a positive measure with total mass bounded by 1. We assume that we start our process η_t from a measure μ_n for which the following result is true: $\pi^n(\eta, du)$ converges, as $n \to +\infty$, to the measure $\pi(du) = g(u)du$ where $g : [0,1] \to \mathbb{R}$ is a measurable function. Observe that π^n is a random measure while π is deterministic. The above convergence is in the weak sense and, by the randomness of π^n, it is also in probability with respect to μ_n, more precisely, μ_n is such that, for any $\delta > 0$ and any function $f \in C([0,1])$ it holds

$$\lim_n \mu_n\Big(\eta : \Big|\pi^n(\eta, du)(f) - \langle f, g\rangle\Big| > \delta\Big) = 0. \quad (11)$$

Above $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2[0,1]$ and $\pi^n(\eta, du)(f)$ denotes the integral of f with respect to the measure $\pi^n(\eta, du)$. The goal is then to show that the same result holds true at any later time t, but the limit measure is given by $\pi_t(du) = \rho(t,u)du$, where the density $\rho(t,u)$ is the solution of a PDE. This result is known in the literature as hydrodynamic limit and the PDE is the hydrodynamic equation. In the case of the open SEEP we have the following result.

Theorem 2 (Hydrodynamics for SSEP).— Starting from μ_n as described above i.e. satisfying (11) for a certain measurable function $g[0,1] \to \mathbb{R}$; the trajectory of random measures $\pi_t^n(\eta_{tn^2}, du)$ converges, as $n \to +\infty$, to the trajectory of deterministic measures given by $\pi_t(du) = \rho(t,u)du$, where $\rho(t,u)$ is the unique weak solution of the heat equation $\partial_t \rho(t,u) = \partial_u^2 \rho(t,u)$ starting from g and with:

- Dirichlet boundary conditions $\rho(t,0) = \rho_a$ and $\rho(t,1) = \rho_b$, for any $t > 0$, when $\theta < 1$;
- Robin boundary conditions $\partial_u \rho(t,0) = (\alpha + \gamma)(\rho(t,0) - \rho_a)$ and $\partial_u \rho(t,1) = (\beta+\delta)(\rho_b - \rho(t,1))$, for any $t > 0$, when $\theta = 1$;
- Neumann boundary conditions $\partial_u \rho(t,0) = \partial_u \rho(t,1) = 0$, for any $t > 0$, when $\theta > 1$.

The proof of the previous theorem, by using the entropy method developed in [11], can be seen in [1] for the regime $\theta \geq 0$ and in [2] for the regime $\theta < 0$. We observe that above the time scale has been re-scaled to tn^2, which is the time scale for which the evolution of the density is non- trivial, known as diffusive time scale. What if one takes shorter time scales of the form n^s with $s < 2$? Then we do not see any space/time evolution of $\rho(t,u)$. As a consequence of the previous result we see that on a strong action regime of the reservoir dynamics, the density profile is fixed at the boundary; while on the weak action regime, the space derivative (current) of the profile becomes fixed.

6 Hydrostatics and correlation functions

The reader now might ask about the stationary measure. Can we obtain the previous result starting from the measure μ_{ss}? This result is know in the literature as *hydrostatic limit* and to recover it from last theorem one just has to derive (11) for a certain function g. The candidate is exactly the stationary solution of the corresponding PDE, which in the cases above is of the form $\bar{\rho}(u) = au + b$, where a and b are fixed by the boundary conditions. To prove the result we need two things:

1. Define for $x \in \Sigma_n$ the discrete profile $\rho_t^n(x) = \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)]$ and extend it to the boundary by setting $\rho_t^n(0) = \rho_a$, $\rho_t^n(n) = \rho_b$. Taking $\mu_n = \mu_{ss}$, we need to know that the stationary discrete profile $\rho^n(x)$ is close to $\bar{\rho}(\frac{x}{n})$. One way to do it is from Kolmogorov's equation, in which one finds that it solves the equation
$$\partial_t \rho_t^n(x) = (\mathscr{B}_n^\theta \rho_t^n)(x)\,, \quad x \in \Sigma_n\,, \quad t \geq 0$$
where the operator $\mathscr{B}_n^\theta$ was defined in previously.

   Observe that the above equation is closed in terms of $\rho_t^n(\cdot)$, this is a consequence of the fact that the generator of the dynamics does not increase the degree of functions. A simple computation allows to derive the stationary solution of the previous equation and to show that it is close to $\bar{\rho}(\cdot)$. Alternatively, we could use the results we obtained by duality which give
$$\mathbb{E}_{\mu_{ss}}[\eta(x)] = \rho_b \mathbb{P}_x(0) + \rho_a \mathbb{P}_x(1). \tag{12}$$
   From (9) we conclude that
$$\rho_{ss}^n(x) = \frac{\beta-\alpha}{2n^\theta+n-2}x + \frac{\beta-\alpha}{2n^\theta+n-2}(n^\theta - 1) + \alpha, \tag{13}$$
   from where we can easily check that
$$\lim_{n\to+\infty} \max_{x\in\Sigma_n} |\rho_{ss}^n(x) - \bar{\rho}(\tfrac{x}{n})| = 0.$$

2. We need to study the behavior of the two-point correlation function defined generally by
$$\varphi_t^n(x,y) = \mathbb{E}_{\mu_n}[\bar{\eta}_{tn^2}(x)\bar{\eta}_{tn^2}(y)],$$
where $\bar{\eta}_{tn^2}(x) = \eta_{tn^2}(x) - \rho_t^n(x)$ and show that, for $\mu_n = \mu_{ss}$, it vanishes as $n \to +\infty$. As for the discrete profile, we can also apply Kolmogorov's equation and derive a discrete equation for the evolution of this function and then obtain its stationary solution. Alternatively, we could use the results we obtained by duality from where we can get the explicit expression for the stationary correlations. A simple, but long, computation shows that
$$\varphi_{ss}^n(x,y) = \mathbb{E}_{\mu_{ss}}^n[\bar{\eta}(x)\bar{\eta}(y)] = -\frac{(\beta-\alpha)^2(x+n^\theta-1)(n-y+n^\theta-1)}{(2n^\theta+n-2)^2(2n^\theta+n-3)}$$
from where we conclude that
$$\max_{x,y} |\varphi_{ss}^n(x,y)| \to_{n\to+\infty} 0.$$

Even if for hydrostatics it is enough to know the order of decay of $\varphi_{ss}^n(x,y)$, thanks to the approach shown above we were able to write the actual form of the two point correlation function. We note that from the previous identity we can obtain the following relationship

$$\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{2n^\theta+n-3}p_x(0)p_y(1)$$

where $p_x(1)$ is given in (9). We also note that for $\theta=0$ the above identity becomes

$$\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{n-1}G^{\rm Dir}\left(\frac{x}{n},\frac{y}{n}\right)$$

where $G^{\rm Dir}(u,v)=u(1-v)$ is the Green function of the 2-dimensional laplacian on $\{(u,v):0\le u\le v\le 1\}$, reflected on the line $u=v$ and with homogeneous Dirichlet boundary conditions, that is $G^{\rm Dir}(u,v)$ is the solution of

$$\Delta^R G^{\rm Dir}(u,v) = -\delta_{u=v}$$

where for $u\neq v$,

$$\Delta^R G^{\rm Dir}(u,v)=\partial_u^2 G^{\rm Dir}(u,v)+\partial_v^2 G^{\rm Dir}(u,v)$$

and for $u=v$,

$$\Delta^R G^{\rm Dir}(u,v)=\partial_v G^{\rm Dir}(u,v)-\partial_u G^{\rm Dir}(u,v)$$

and $G^{\rm Dir}(0,v)=G^{\rm Dir}(u,1)=0$. We get the scaling form

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y) = -(\beta-\alpha)^2 G^{\rm Dir}(u,v)$$

for the continuous correspondents $\frac{x}{n}\to u$ and $\frac{y}{n}\to v$. We now see what happens for the cases when $\theta\neq 0$. For $0<\theta<1$, a simple computation shows that the limit above also holds. For $\theta=1$ we get

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{9}G^{\rm Rob}(u,v)$$

where $G^{\rm Rob}(u,v)=\frac{1}{3}(u+1)(2-v)$ and corresponds to the Green function of the 2-dimensional laplacian defined above, but with homogeneous Robin boundary conditions given by $\partial_u G^{\rm Rob}(0,v)=G^{\rm Rob}(0,v)$ and $\partial_v G^{\rm Rob}(u,1)=-G^{\rm Rob}(u,1)$. Finally, for $\theta>1$, if we use the same scaling as above, we see that

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y)=0.$$

Nevertheless, a simple computation shows that

$$\lim_{n\to+\infty} n^\theta\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{8}(u+1)(1-v)$$

for the continuous correspondents $x/n^\theta\to u$ and $y/n^\theta\to v$; and this is the correct order to see a non-trivial limit in the case of very slow boundary. For higher point correlation functions, we can use exactly the same argument as above in order to obtain the exact rates of convergence of the corresponding stationary correlations. Moreover, we conjecture that we can write the stationary k-th point correlation function $\varphi^n_{ss}(x_1,\dots,x_k)$ as a product between a scaling factor and the absorption probabilities of k independent one-dimensional random walks. We also believe that this argument could be extended to other models for which duality is known but all this is left for a future work.

Recently, it has been developed a method in [9] to derive the hydrodynamic and the hydrostatic limits in presence of duality for a similar model called the symmetric inclusion process, where many particles can occupy the same site and show a preference of laying together. The macroscopic behavior for this process is the same as the one described above, but the proof now boils down to the sole use of duality.

We conclude by saying that there are many other models for which one has to explore the notion of duality, specially for asymmetric models where the equations for correlation functions are no longer closed. There is a long and standing work to develop around these problems and here we just collected some nice and simple results for a toy model where Lie algebra and, consequently, duality allows getting a lot of relevant information about our model.

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