An introduction to Analysis on Graphs: what are Quantum Graphs and (why) are they Interesting?

by **James B. Kennedy***

We give a gentle introduction to analysis on graphs. We focus on the construction of prototypical difference operators on discrete graphs, differential operators on metric graphs, and the parallels between the two. The latter lead naturally to quantum graphs, metric graphs on which a Schrödinger-type differential operator acts, for which we finish by discussing a number of recent applications and ongoing areas of investigation. These are drawn mostly, but not exclusively, from mathematical physics.

1 Introduction: two complementary types of graphs

Probably everyone has at least an intuitive notion of what a graph is: a collection of vertices, or nodes, joined by edges. Most mathematicians, perhaps even some non-mathematicians, probably have some idea of the role that graphs play in modelling phenomena as diverse as fine structures such as crystals and carbon nanostructures, social networks, the PageRank algorithm, data processing and machine learning, …, but may not be so familiar with the details.

Generally speaking, at a mathematical level, we are interested in some process taking place on the graph, such as described by a difference or differential equation. The mathematics behind such equations combines ideas from graph theory (obviously), linear algebra, functional analysis and the theory of differential equations, operator theory, and mathematical physics; yet many of the details seem to be largely unknown to the wider mathematical community. As a test: do you know what quantum graphs are?

Our goal here is to give somewhat uneven introduction to analysis on graphs: we first describe, in hopefully accessible terms, what this is: how to define functions and difference and differential operators on graphs, and study them – and in particular what are quantum graphs. Our starting point is that there are (at least) two natural, somewhat parallel, notions of graphs: discrete and metric graphs; the former give rise to difference operators, the latter to differential operators. We will first discuss the construction of these graphs, and then introduce prototypical difference and differential operators, principally realisations of the Laplacian, on each.

But our second goal is to highlight some of the parallels between the two kinds of graphs: indeed, one speaks of Laplacians in both the discrete and the metric case, nomenclature which is justified for various reasons, as we shall see. Finally, we will turn to quantum graphs, which in simple terms are metric graphs equipped with differential operators. We will describe a number of areas of current interest, especially within (parts of) the mathematical physics community. The list of topics we have selected is somewhat idiosyncratic; we include a brief mention of, and references to the literature for, a variety of others. The reader interested in discovering more is referred to the book [BK13], considered a standard reference in the area, the recent survey paper [BK20], the elementary introduction [Ber17], and the somewhat older volume [EKKST08], which contains a large number of still useful review articles.

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1.1 Discrete graphs

In the case of *discrete graphs* we are more interested in the vertices and consider the edges as relations between the vertices, without necessarily having any direct physical meaning. More formally, a discrete graph G is a pair (V, E), where the vertex set V is any countable (in practice usually finite) set and each edge e in the edge set E may be regarded as a pair of vertices, that is, E may be identified with a subset of $\mathsf{V}\times\mathsf{V}$. Already here we have a further decision to make: whether to treat the edges $\mathsf{e} = (\mathsf{v}, \mathsf{w})$, $\mathsf{v}, \mathsf{w} \in \mathsf{V}$ as *ordered* or *unordered* pairs; we speak of *directed edges* (also called *bonds* in some circles) and *undirected edges*, respectively. In the case of directed edges $\mathsf{e} = (\mathsf{v}, \mathsf{w})$, we may distinguish between the *initial vertex* v and the *terminal vertex* w.

Many social networks may be modelled in this framework; for example, Facebook is a network in which each person (or entity) represents a vertex, and being (Facebook) friends corresponds to an undirected edge between the two vertices. Twitter, on the other hand, is directed, if one considers the edge (v, w) to mean v *is a follower of* w – as is the internet itself with links being edges between the pages represented by vertices. More generally, any model of a network in which there is no natural *distance* between vertices, nor physical bond linking them, is likely to fit into the framework of discrete graphs. This is of course a considerable simplification; for example, one may assign a weight function to the edges of a discrete graph to give a notion of the proximity of the respective vertices.

To do any sort of analysis, of course we need to define functions on our graph. In the case of discrete graphs, this is easy: if functions live on the vertices, then the space of all functions may be identified with $\mathbb{R}^{|\mathsf{V}|}$ or $\mathbb{C}^{|\mathsf{V}|}$. Some care must be taken if the vertex set V is infinite; it becomes natural to work with ℓ^p-spaces.

1.2 Metric graphs

Metric graphs, on the other hand, focus attention on the edges, and are thus more suited to modelling actual physical networks, or fine ramified structures such as nanostructures. We will write $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ for a metric graph, where now each edge $e \in \mathscr{E}$ is identified with a closed interval which may be finite, of some given length $\ell(e) > 0$, i.e. $e \simeq [0, \ell(e)] \subset \mathbb{R}$, or a half-line, $[0, +\infty)$. Care must obviously be taken with the latter, so here we will restrict ourselves to compact intervals.

In order to encode the topological structure of the graph, or equivalently to create a metric, one identifies all interval endpoints which correspond to a given vertex. While intuitively this is very simple, formally it is somewhat fiddly and may be done in a number of ways: for example:

- identify equivalence classes of endpoints, or
- define the underlying metric directly by declaring that the distance between two different interval endpoints corresponding to the same vertex is zero, thus allowing the construction paths between any two points on different edges, or alternatively
- work directly at the level of continuous functions.

For more details we refer to [BK13, Section 1.3], [Mug19] and [KKLM20, Section 2].

At any rate, this gives rise naturally to a metric space; the distance between two given points is the (Euclidean) distance of the shortest path between them. Technically the metric is a pseudometric, as it may take the value $+\infty$ if there is no path between a given pair of points, but it becomes a metric if and only if the graph is connected.

When it comes to defining spaces of functions, metric graphs are, unsurprisingly, more interesting than their discrete counterparts, albeit not yet at the level of L^p-spaces: we may simply define, for a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with a finite edge set $\mathscr{E}$,

$$L^p(\mathscr{G}) = \bigoplus_{e\in\mathscr{E}} L^p(e) \simeq \bigoplus_{e\in\mathscr{E}} L^p(0, \ell(e))$$

(each edge being prototypically equipped with Lebesgue measure on the interval $[0, \ell(e)]$); indeed, L^p-functions will never see the vertices as the latter form a set of measure zero. Correspondingly, to integrate a function over the graph we integrate over each edge and sum the result. The structure of the graph is only encoded at the level of continuous functions: $C(\mathscr{G})$ will consist of those functions which are continuous on every edge, such that their values at all endpoints meeting at a vertex should agree. These are of course exactly the functions which are continuous with respect to the metric.

To define differentiable functions becomes more challenging because of the issue of defining the derivative across the vertices; instead, it becomes more nat-

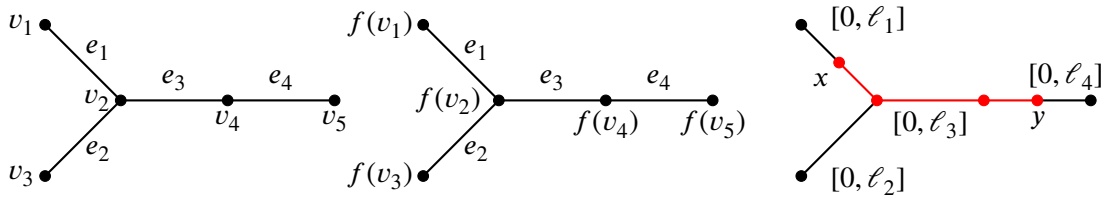

Figure 1. A simple graph on 5 vertices and 4 edges (left); to define a function f on a discrete realisation of the graph we specify the values at the vertices — this also gives rise naturally to difference operators (centre); one may instead identify each edge e_i with an interval $[0,\ell_i] \subset \mathbb{R}$ (or a half-line $[0,+\infty)$), and *glue together* the intervals at their endpoints in the right way, to form a metric graph (right). Here a path between two points x and y is marked in red.

ural to speak of vertex conditions which the functions should satisfy (such as continuity at the vertices, as imposed in $C(\mathcal{G})$). Two of the most natural such conditions to satisfy are the Dirichlet, or zero, condition, and the Kirchhoff condition, where the sum of inward-pointing derivatives at a vertex equals zero.[1]

In practice, one usually works with Sobolev spaces of weakly differentiable functions; for example, $H^1(\mathcal{G})$ is defined as those functions which are edgewise L^2-integrable with edgewise L^2-integrable weak derivative, and which are continuous across the vertices. This makes sense since by standard Sobolev embedding theorems one-dimensional H^1-functions are continuous and thus, up to choosing the correct representative, defined pointwise.

Three final observations are in order: firstly, thus defined, our graphs are not considered to be embedded in Euclidean space; there is no curvature of the edges or angle between them. Secondly, by labelling one vertex of an edge $e \simeq [0, \ell(e)]$ as 0 and the other as $\ell(e) > 0$, we are implicitly (or explicitly) imposing an orientation. However, for many practical purposes this orientation is irrelevant; the differential operators we shall define in the sequel are independent of this choice up to unitary equivalence. Thirdly, in the case of metric graphs it is easy to allow multiple edges between vertices, as well as loops (edges which begin and end at the same vertex); in the case of discrete graphs this is a bit more complicated and we will tacitly assume that our graphs are free of such features, even though most of what we discuss will remain true even with multiple parallel edges and loops.

2 Difference and differential operators

We see immediately that on discrete graphs, since the functions are identifiable with vectors, difference operators (or more generally matrices) will arise; while on metric graphs we may define (ordinary) differential expressions on the edges. In the latter case the point of interest becomes specifying the vertex conditions, or equivalently the domain of definition of the differential operator; a metric graph is essentially a smooth one-dimensional manifold with isolated singularities (the vertices). In both cases we will illustrate this via a prototypical operator, the Laplacian; note that here, in both cases, our edges will be undirected.

Let us start with metric graphs, as here we are closer to the *traditional* Laplacian from the theory of PDEs. In fact, we start with the differential expression $-f''$ on each edge. It is natural to impose continuity at all vertices, as this is essentially the minimal requirement for the functions to *see* the graph. Additionally imposing the Kirchhoff condition, which we may write as

$$\sum_{e \text{ adjacent to } v} \frac{\partial f}{\partial v}(v) = 0,$$

that is, the sum of the derivatives of f at the endpoint of each edge e directed into the vertex v, gives rise to the Laplacian with vertex conditions variously known as standard, natural, continuity-Kirchhoff, and even Neumann-Kirchhoff (if the vertex has *degree one*, i.e.,

[1] This condition is sometimes loosely called a *flow in equals flow out* condition, although this expression must obviously be interpreted with care, depending on the kind of *flow* one is imagining.

only one edge attached, then this reduces to the Neumann condition. Roughly speaking, in many ways the Laplacian with standard conditions behaves somewhat like the Neumann Laplacian on domains, or the Laplace-Beltrami operator on manifolds without boundary). For the Dirichlet condition at a vertex v, instead of the Kirchhoff condition we require that $f(v) = 0$.

If the graph $\mathcal{G}$ has finite total length, then such operators are self-adjoint, semi-bounded from below, and have compact resolvent; thus they behave exactly like Laplacians or Schrödinger operators on bounded domains and manifolds. Generalisations, such as adding a potential to each edge, are easy to incorporate in this framework.

All this is perhaps more naturally seen at the level of forms/weak solutions: the associated positive, symmetric sesquilinear form reads

$$a(f, g) = \int_{\mathcal{G}} f' \cdot \overline{g}' \, \mathrm{d}x$$

with form domain exactly $H^1(\mathcal{G})$ in the case of the standard Laplacian; if Dirichlet conditions are imposed at one or more vertices then the functions should additionally take on the value 0 there. (All this is a short exercise in integration by parts.) The eigenvalues and eigenfunctions of the Laplacian admit the usual min-max variational characterisation; for example, the smallest eigenvalue can be obtained by minimising $a(f, f)$ among all $f \in H^1(\mathcal{G})$ whose L^2-norm is 1.

On discrete graphs, the (discrete or combinatorial) Laplacian is defined purely in terms of the graph structure. We suppose $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ to be a discrete graph with finite vertex set $\mathsf{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ and finite edge set $\mathsf{E} = \{\mathsf{e}_1, \dots, \mathsf{e}_m\}$. We take as a starting point the following matrices:

- the *adjacency matrix*, the symmetric matrix whose (i, j)-entry is 1 if v_i and v_j share an edge, or 0 otherwise (in the case of directed edges this matrix can still be defined but will no longer be symmetric);

- the *degree matrix*, the diagonal matrix whose (i, i)-entry is the degree of v_i, i.e., the number of edges emanating from v_i.

The *(discrete)* Laplacian is the difference operator corresponding to the symmetric, positive semidefinite matrix $L := D - A$. For example, for the graph depicted in Figure 1, with the order of vertices as specified there, the Laplacian would be

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The fact that this is a plausible discrete version of the Laplacian may be recognised in (at least) two ways:

- vectors x satisfying $Lx = 0$ have the *mean value property*, as can be checked with the above example: since the sum of each row of L is zero, the value of x at a vertex is equal to the sum of the values at the surrounding vertices – just as harmonic functions in $\mathbb{R}^d$, solutions of $\Delta f = 0$, satisfy the (continuous) mean value property;

- at the level of forms: L is associated with the positive, symmetric sesquilinear form

$$a(x, y) = \sum_{\mathsf{e} \in \mathsf{E}} (\mathcal{J}^T x)(\mathsf{e}) \overline{(\mathcal{J}^T y)(\mathsf{e})},$$

  where $\mathcal{J} \in \mathbb{R}^{n \times m}$ is the so-called (signed) *incidence matrix* encoding which vertices are the terminal and initial endpoints of which edges; in fact L may also be represented as $L = \mathcal{J}\mathcal{J}^T$, and we may intuitively think of $\mathcal{J}$ as a discrete counterpart of the divergence operator.

A word of caution: there is a common, *normalised* variant, namely

$$L_{\mathrm{norm}} := \mathrm{Id} - D^{-1/2} A D^{-1/2}$$

(that is, we normalise the operator by the degree of the vertices); its $|\mathsf{V}|$ eigenvalues (counting multiplicities) always lie in the interval $[0, 2]$. The standard reference on the topic is [Chu97]; see also [Mug14, Chapter 2] for the construction of all these matrices as well as their directed counterparts.

A strong mathematical parallel between the standard Laplacian on a metric graph and the normalised Laplacian on the corresponding discrete graph was established by von Below in 1985 [Bel85]. Namely, if all the edges of the metric graph have length 1, and we denote by λ_k the ordered eigenvalues of the standard Laplacian, then, up to certain special cases (corresponding to the normalised Laplacian eigenvalues 0 and 2) the eigenvalues of L_{norm} are given by $1 - \cos(\sqrt{\lambda_k})$; the values of the eigenvectors correspond to the values of the eigenfunctions at the vertices. Thus, at least for this special class of *equilateral* metric graphs, the standard Laplacian as a differential operator is *essentially determined* by the corresponding

discrete (normalised) Laplacian. Such connections between discrete and continuous versions have naturally been explored further over the last 30-odd years; see [LP16] and the references therein.

2.1 So what are quantum graphs?

We can now finally answer the first question posed in the title. By a quantum graph we understand a metric graph on which acts a differential operator, most often (but not necessarily) some kind of Schrödinger operator or *Hamiltonian* [Ber17, BK13]; of course this includes various realisations of the Laplacian.

While differential operators on metric graphs have been studied for a long time – they were actively studied in the 1980s, often under the name c^2*-networks* (e.g., [Bel85, Nic87]), and there are applications going back much further [RS53] – the name *quantum graph* is generally considered [Ber17] to trace back to the article *Quantum Chaos on Graphs* [KS97] from 1997, possibly as a contraction of the title.

3 A cornucopia of applications

We finish with a discussion of some current topics of interest in the community, as an answer to the second question posed in the title: we wish to give some idea of the variety of problems and applications which arise in the context of quantum graphs. One may broadly and imperfectly group the applications into models where it is intrinsically sensible to consider ramified structures (atomic or crystalline structures, honeycombs, ramified traps,…) and those where the graph represents a toy model used to study mathematical or quantum physical phenomena: graphs are simple one-dimensional objects which often display complex behaviour typical of higher-dimensional problems.

In the following list no claim is made to completeness, either in the list of topics or in the references given, which largely reflect the author's personal taste and prejudices. Where possible we have tried to provide some of the most recent references available to act as a starting point for a further literature search.

In keeping with these prejudices, as well as the general focus of the quantum graph community, we will mostly be interested in differential operators such as the Laplacian and Schrödinger operators, and their spectra. This is natural since by the spectral theorem the spectrum completely determines such self-adjoint operators.

3.1 Approximation of, or by, higher-dimensional objects

There are two senses in which graphs, be they discrete or metric, can be related to higher-dimensional domains or manifolds: one can consider a (metric) graph as the limit of a sequence of thin branching domains (*shrinking tubes*, or *fattened graphs*), or one can try and approximate a domain or manifold as the limit of as sequence of graphs. In the latter case one usually takes discrete graphs as the approximating objects, as a kind of discretisation of the domain or manifold.

Needless to say, there is an extensive literature on both. The latter is sometimes used to extend results from discrete graphs to manifolds (as in [LLPO15], see also Section 3.3). The former provides a justification for using quantum graphs to study phenomena like waveguides, be they acoustic, quantum or electromagnetic, thin super-conducting structures and so on; here we will follow, and refer to, [BK13, Section 7.5]. Another standard reference for shrinking tubes is the review paper [Gri08] contained in the volume [EKKST08]. Typical questions include whether the solutions of differential equations in the thin domains converge to the solution of some differential equation on the graph, and if so, what vertex conditions the problem in the limit satisfies. (More technically, we are interested in convergence of the resolvents of the operators in various norms, as well as of the operator eigenvalues and eigenfunctions.)

The precise results depend very much on the nature of the approximation, but in perhaps the simplest and most important case of Schrödinger operators in *Neumann tubes* (thin perfectly insulated tubes) shrinking uniformly, one does at least have convergence of the eigenvalues to the eigenvalues, in the correct order, of the Schrödinger operator on the graph with standard vertex conditions, and where the electric potential is, roughly speaking, the restriction of the potential on the thin domain to the graph it contains. Work is still ongoing to establish other kinds of convergence, in particular under different kinds of domain convergence.

In this case the limit object, the quantum graph with its Schrödinger operator, *forgets* many geometric features of the domains, such as angles between branches, curvature of edges and so on. If one allows the Neumann tubes to shrink in a *wilder*, non-

uniform fashion, then one may obtain more interesting limit quantum graphs, including where the operators satisfy other vertex conditions than the standard/Kirchhoff ones.

3.2 Spectral geometry

We have seen that discrete Laplacians may be defined directly in terms of the structure (topology) of a discrete graph, and that at least at the spectral level this can be transferred to equilateral metric graphs via von Below's formula. Can we say something about (non-equilateral) metric graphs, where we have to contend with both the topology and the edge length?

In the case of domains and manifolds a group of questions revolves around understanding how the eigenvalues and eigenfunctions depend on the geometry of the underlying domain or manifold. The classical example is the theorem of Faber-Krahn from the 1920s, based on an earlier conjecture of Lord Rayleigh, that among all domains of given volume the ball is the one whose first Dirichlet Laplacian eigenvalue is smallest. This is an analytic translation of the geometric isoperimetric inequality, that the ball minimises surface area for given volume; the first (nonzero) eigenvalue is of particular interest because it controls the rate of heat loss in the heat equation, the lowest frequency of the object, and so on. We refer to [Pay67] for a (classical) introduction and [Heno6, Hen17] for more modern surveys of the area of *shape optimisation and spectral theory*. The corresponding inverse problem, determining the domain/manifold based on the spectrum of a differential operator on it, corresponds to the question made famous by Mark Kac, "can one hear the shape of a drum?"; see [LR15].

On metric graphs the equivalent of the theorem of Faber-Krahn states that the smallest nonzero eigenvalue of the Laplacian with standard vertex conditions is minimised when the graph is an interval of the same length; this theorem first appeared around 30 years ago [Nic87]. It turns out that graphs are far more amenable to this kind of analysis than domains; see [BL17, BKKM19, KKMM16]. A surprisingly subtle question is which (geometric or topological) properties of a graph are sufficient to bound its eigenvalues and which are not. For example, fixing the diameter D (length of the longest path within the graph) alone places no control on the smallest nonzero standard Laplacian eigenvalue: it may be arbitrarily large or small [KKMM16]; however, if we restrict to *trees*, graphs without cycles, then it cannot exceed π^2/D^2, the corresponding eigenvalue of an interval of length D.

Work has also been done on *isospectral* graphs, quantum graphs which are different but have the same Laplacian spectra. On graphs the problem can be given a new twist since one has more chance of describing the corresponding eigenfunctions: one considers the so-called *nodal count*, the number of *nodal domains*, which are by definition the connected components of the set where the eigenfunction is nonzero. See [BK13, Section 7.1].

3.3 Clustering and partitions

A major preoccupation in applied graph theory is to detect the presence of clusters in a (usually discrete) graph. One might ask whether a given social network such as Facebook tends to be divided into groups of hig hly interconnected individuals with few links between the groups, thus creating the infamous *echo chambers*. Alternatively, one might wish to identify, say, weaknesses in a road network or an electricity grid: if the power lines here go down, does half the country lose power?

There are various ways to measure this. One natural way is the notion of *Cheeger constants* and *Cheeger cuts* borrowed from geometric analysis, originally introduced for manifolds. Say we wish to cut the graph G into two pieces S and $S^c = \mathsf{G} \setminus S$, which we do by cutting through edges. Then for each possible cut we look at the ratio

$$\frac{|\partial S|}{\min\{|S|, |S^c|\}}$$

of edges cut $|\partial S|$ to the smaller of the two sets S e S^c, as measured by the number of vertices in the set.

The infimum of this quotient over all possible cuts is the *Cheeger constant*; the smaller the constant, the easier it is to cut the graph into two (the traditional image for this is the dumbbell manifold, cut through its thin handle).

Figure 2 gives an example on graphs: on this graph of 20 vertices, there is a way to make just two cuts to separate the graph into two groups of 10 vertices each, labelled as blue and red; this is in fact the optimal cut. One might imagine a social network where the vertices represent users; the blue users tend only to have friends with other blue users, while the red users likewise stay amongst themselves. In this case the Cheeger constant will be $2/\min\{10, 10\} = 1/5$, which may be considered small (the number has no absolute meaning but should be viewed in conjunc-

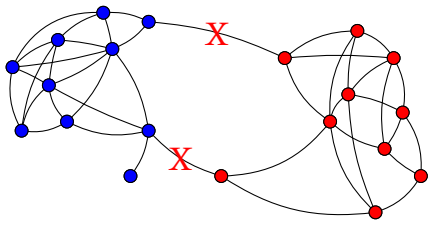

Figure 2. The Cheeger cut of a graph whith 20 vertices.

tion with the total number of vertices and edges).

One can also consider higher order Cheeger constants, which partition the graph into more than two pieces, and in fact estimating these constants, in particular in terms of Laplacian eigenvalues, is one instance where results were first proved on discrete graphs and then transferred to manifolds [LLPO15]. Cheeger constants have also been introduced on metric graphs, with the numerator remaining the same and the denominator becoming the total length (sum of edge lengths) of each piece [KM16, Nic87].

The Cheeger constant corresponds to the first eigenvalue of the so-called 1-Laplacian, i.e., the p-Laplacian when $p = 1$; the eigenvector/eigenfunction has two nodal domains which correspond to the Cheeger cut. This operator is singular, its eigenvalues lack the easy L^p-variational characterisation of the p-Laplacian eigenvalues, and actually calculating the constant of a large graph becomes a computationally hard problem.

Thus one can use the (2-)Laplacian (say, with standard vertex conditions) as a natural proxy, as its variational structure makes determining the eigenvalues and eigenfunctions much easier, both analytically and computationally. Ideally one would use the nodal domains of the k-th eigenvalue as an "optimal" partition into k pieces, but in general there is no simple relationship between the number of nodal domains of an eigenfunction and its number in the sequence. A natural alternative is to consider *spectral minimal partitions*, whereby one looks to minimise a functional of the eigenvalues over all partitions; prototypically this problem might take the form

$$\inf_{\mathscr{P}} \max_{i=1,\dots,k} \lambda_1(\Omega_i),$$

where the infimum is taken over all partitions $\mathscr{P}$ of the object (domain, graph, …) into k pieces $\Omega_1, \dots, \Omega_k$, and $\lambda_1(\Omega_i)$ is the first nontrivial eigenvalue of a suitable Laplacian on Ω_i; one could equally take a p-norm of the eigenvalues in place of the ∞-norm. Such problems were originally considered, and have been studied intensively, on domains and some manifolds; see [Hen17, Chapter 10] for a survey.

On metric graphs this topic is new: the first systematic study of spectral minimal partitions was undertaken in [KKLM20]. As is the case for spectral geometry, and actually for many problems considered here, one can say far more on metric graphs than on domains. Here, far more functionals can be meaningfully defined on the former than the latter, including more exotic combinations of eigenvalues (such as max-min rather than min-max problems). Understanding how these optimal partitions differ and what they reveal about the structure of the graph will be a topic of interest in the next few years.

3.4 Nonlinear Schrödinger equations

Until now we have always considered linear differential operators, as has historically usually been the case on metric graphs. There is, however, a notable family of exceptions, first considered just a few years ago [AST15a]. This principally involves studying existence, or nonexistence, of certain solutions of stationary nonlinear Schrödinger equations on metric graphs (NLSE for short). A stationary NLSE typically takes the form

$$-\Delta u + f(u) = \lambda u, \tag{1}$$

where in place of the usual potential term Vu a nonlinearity $f(u)$ is introduced; here, as in the literature, we will consider the prototypical power nonlinearity $f(u) = |u|^{p-1}u$. These equations, a bedrock of the Calculus of Variations literature, are most commonly studied in d-dimensional space, see [Caz03] for an introduction, but a number of applications, such as Bose-Einstein condensates in traps or optical fibres [AST15a], make it reasonable to consider NLSE in ramified structures, that is, on metric graphs, most commonly and naturally with standard vertex condi-

tions. Of most interest are the ground states, minimisers of the energy functional for which 1 is the Euler-Lagrange equation:

$$E(u) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

where $\|\cdot\|_2$ and $\|\cdot\|_p$ are, respectively, the L^2- and L^p-norms, here on some graph $\mathcal{G}$. Here one usually considers *unbounded* graphs, with a finite number of edges but where some of them are half-lines ($\mathcal{G} = \mathbb{R}$ itself is a prototype, being two half-lines glued together at the origin), as well as the *subcritical case* $2 < p < 6$, which guarantees the Sobolev embedding $H^1 \hookrightarrow L^p$ in dimension 1.

It turns out that the existence or non-existence of ground states on such graphs depends heavily on the topology of the graph, as shown in a series of landmark papers [AST15a, AST15b, AST16, AST17]. Further research, including into stability of solutions and standing waves, other types of metric graphs, other restrictions on the parameters, and other equations is ongoing; see, for example, [DST20, Hof19, NP20] and the references therein.

3.5 Final remarks

The above list excludes a huge and growing number of topics from various areas of mathematics. We could mention quantum chaos (the presumable source of the name *quantum graph*, as discussed in Section 2.1; see also [BK13, Chapter 6]), as well as various other applications in mathematical physics such as scattering and inverse scattering, the Bethe-Sommerfeld property on the gap structure of the spectrum of periodic objects [ET17], Anderson localisation [DFS, DS19], the spectra of graphene and carbon nanotubes, Bose-Einstein condensates, and the quantum Hall effect. Differential equations on metric graphs also feature in other areas of mathematics as diverse as neural networks and models of population dynamics [DLPZ20, SCA14]. Surveys of many of these and further applications in mathematical physics may be found in [BK20], [BK13, Chapter 7] and the collection [EKKST08].

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