Symbolic Powers

by **Eloísa Grifo***

Symbolic powers arise naturally in commutative algebra from the theory of primary decomposition, but they also contain geometric information, thanks to a classical result of Zariski and Nagata. Computing primary decompositions is a difficult computational problem, and as a result, many natural questions about symbolic powers remain wide open. We will briefly introduce symbolic powers and describe some of the main open problems on the subject, and point to some recent research advances.

1 Introduction

Given a finite set of points $P_1, \dots, P_s$ in projective space $\mathbb{P}^d_{\mathbb{C}}$, what is the lowest degree of a hypersurface passing through $P_1, \dots, P_s$? How about a hypersurface passing through each given point P_i with the same multiplicity n? More generally, given an affine or projective variety V, which polynomials vanish to order n at every point in V? These classical geometric questions can be studied with commutative algebra tools once we reframe them within the language of symbolic powers.

Let us formalize what we mean by vanishing to order n. Given a point a in either affine space $\mathbb{A}^{d+1}_{\mathbb{C}}$ or projective space $\mathbb{P}^d_{\mathbb{C}}$, a polynomial $f \in R := \mathbb{C}[x_0, \dots, x_d]$, which we assume to be homogeneous in the projective case, vanishes to order n at a if

$$\frac{\partial^{c_0+\cdots+c_d} f}{\partial x_0^{c_0} \cdots \partial x_d^{c_d}}(a) = 0 \quad \text{for all } c_0 + \cdots + c_d < n.$$

Notice that with this definition, f vanishes to order 1 at a if and only if $f(a) = 0$. More generally, given an algebraic set V — the solution set to some system of polynomial equations in $d+1$ variables, which are homogeneous in the projective case — consider the ideal I of all the polynomials in R that vanish at every $a \in V$. A (homogeneous, in the projective case) polynomial f **vanishes to order n along V** if

$$\frac{\partial^{c_0+\cdots+c_d}}{\partial x_0^{c_0} \cdots \partial x_d^{c_d}}(f) \in I \quad \text{for all } c_0 + \cdots + c_d < n.$$

Let us give examples of polynomials vanishing to order n on a given algebraic set. The **nth power of I** is the ideal generated by all the n-fold products of elements in I, which we write as

$$I^n := (f_1 \cdots f_n \mid f_i \in I).$$

Here the notation $I = (g_1, \dots, g_m)$ stands for the ideal generated by $g_1, \dots, g_m$, so the elements in I^n are all the R-linear combinations of n-fold products of polynomials that vanish at V. It is elementary to show that every element in I^n must vanish to order n along V. However, we may have other more interesting polynomials vanishing to order n along V.

Example 1.— Let V be the union of the 3 coordinate lines in affine 3-space, which corresponds to the ideal $I = (xy, xz, yz)$. The polynomial $f = xyz$ vanishes to order 2 along V, since $\partial f/\partial x = yz \in I$, $\partial f/\partial y = xz \in I$, and $\partial f/\partial z = xy \in I$. On the other hand, all the nonzero polynomials in I^2 have degree 4 or higher, so $f \notin I^2$.

In particular, we may have polynomials vanishing to order n along V that live in an unexpected degree — meaning, a degree d such that I^n has no polynomials of degree d. Completely describing which polynomials vanish to order n along a given algebraic set V,

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determining whether those are exactly the polynomials in I^n, or giving (lower) bounds for the degrees of polynomials vanishing to order n along V are all very delicate questions.

We can attack these questions using purely algebraic tools, thanks to a classical result of Zariski and Nagata [Zar49, Nag62] which says that the polynomials that vanish to order n along V are exactly the polynomials in the nth symbolic power of I, which we will introduce in the next section. Despite being a classical topic that has been around for a century, many natural questions about symbolic powers remain unanswered, in part because it is computationally difficult to calculate symbolic powers and test conjectures. We will first introduce symbolic powers in Section 2, and then quickly survey some of the current active research questions related to symbolic powers in the remaining sections. For a more detailed survey of symbolic powers, see [DDSG⁺18]. Throughout, let R be a commutative Noetherian ring; a good working example is the case when R is a polynomial ring in finitely many variables over a field k.

2 Symbolic powers: definition and basic properties

Symbolic powers arise naturally in commutative algebra from the theory of primary decomposition. Roughly speaking, primary decomposition is an ideal-theoretic version of the Fundamental Theorem of Arithmetic — the theorem which says that every nonzero integer can be written as a product of prime integers that is unique up to sign and the order of the factors. Once we replace the integers with other commutative rings, there are many examples of rings where we *cannot* write every element as a product of irreducibles that is unique up to multiplication by units or the order of the factors; for example, in $\mathbb{Z}[\sqrt{-5}]$, $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ are two distinct factorizations into irreducibles. One way to avoid this failure of the Fundamental Theorem of Arithmetic is to focus on ideals rather than elements: every ideal in a Noetherian ring can be written as a finite intersection of primary ideals [Las05, Noe21], and while this *primary decomposition* is not necessarily unique, there are certain aspects of it that are in fact unique.

Let us start with prime ideals. An ideal P is **prime** if $ab \in P$ implies that $a \in P$ or $b \in P$. When $R = \mathbb{C}[x_0, \ldots, x_d]$, prime ideals are precisely the ideals that correspond to varieties: a variety is an *irreducible* algebraic set, meaning it cannot be decomposed as a finite union of two or more proper algebraic subsets.

Definition 1.— Let P be a prime ideal. The **nth symbolic power** of P is the ideal

$$P^{(n)} := \{ f \in R \mid sf \in P^n \text{ for some } s \notin P \}.$$

Note that $P^n \subseteq P^{(n)}$, since every $f \in P^n$ satisfies $1 \cdot f \in P^n$ for $1 \notin P$. In general, $P^n \neq P^{(n)}$.

Example 2.— Let $R = k[x, y, z]/(xy - z^2)$, where k is an arbitrary field, and consider the prime ideal $P = (x, z)$ in R. Since $xy = z^2 \in P^2$ and $y \notin P$, we have $x \in P^{(2)}$, while $x \notin P^2$.

While we will not define primary decomposition, it turns out that when writing a primary decomposition for P^n, one of the components — the P-primary component — will be precisely $P^{(n)}$. Historically, this is the context where symbolic powers first arose.

More generally, let us consider a radical ideal I, which means that I is a finite intersection of prime ideals. Geometrically, Hilbert's Nullstellensatz gives us a bijection between algebraic sets and radical ideals, so for our purposes these are the only ideals we care about.

Definition 2.— Let $P_1, \ldots, P_k$ be prime ideals, and let $I = P_1 \cap \cdots \cap P_k$. The **$n$th symbolic power** of I is the ideal

$$\begin{aligned} I^{(n)} &:= P_1^{(n)} \cap \cdots \cap P_k^{(n)} \\ &= \{ f \in R \mid sf \in I^n \text{ for some } s \notin \textstyle\bigcup_{i=1}^k P_i \}. \end{aligned}$$

The following properties can be shown via elementary commutative algebra methods.

Theorem 3.— Let I be a radical ideal in a Noetherian ring R.

1. $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.
2. $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.
3. $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all $a, b \geqslant 1$.

The last property allows us to construct the *symbolic Rees algebra of I*, which packages together all the symbolic powers of I into one graded object. The symbolic Rees algebra of I is the graded R-algebra with $I^{(n)}$ in degree n, $\mathcal{R}_s(I) = \bigoplus I^{(n)} t^n \subseteq R[t]$, where the t keeps track of degrees. It turns out that this algebra can fail to be finitely generated over R — or equivalently, it can fail to be a Noetherian ring — which means that for arbitrarily high values of n, there are elements in $I^{(n)}$ that do not live in the product of symbolic powers of I of lower order. While we will not have a chance to discuss symbolic Rees algebras in detail, we point the reader to [GS20] for a survey on symbolic Rees algebras and the fascinating problem of when they are finitely generated.

In the next section we will discuss some of the geometric motivations to study $I^{(n)}$. Note that there are also many algebraic reasons to study symbolic powers, including the fact that they can be used as effective tools to answer questions that are a priori unrelated to symbolic powers, and that symbolic powers are used in the proofs of important results in commutative algebra, such as Krull's Height Theorem and the Hartshorne-Lichtenbaum Vanishing Theorem in local cohomology, even though these results are not about symbolic powers.

3 Higher order vanishing

A classical result of Zariski and Nagata [Zar49, Nag62] and its modern generalization by Eisenbud and Hochster [EH79] give us the connection with our opening questions.

Theorem 4 (Zariski-Nagata, 1949 and 1962).— Let I be a radical ideal in $R=\mathbb{C}[x_0,\dots,x_d]$. Then

$$I^{(n)} = \bigcap\{\mathfrak{m}^n \mid \mathfrak{m} \supseteq I, \mathfrak{m} \text{ maximal ideal}\} = \\ = \left\{ f \in R \;\middle|\; \frac{\partial^{c_0+\cdots+c_d}}{\partial x_0^{c_0}\cdots \partial x_d^{c_d}}(f) \in I, \text{ for all } c_0+\cdots+c_d < n \right\}.$$

(See [Zar49, Nag62, EH79, DDSG+18].)

This is the classical result we alluded to in the introduction: that $I^{(n)}$ is precisely the set of polynomials that vanish to order n along the algebraic set corresponding to I. The maximal ideals $\mathfrak{m}$ that contain the radical ideal I correspond to each point in the affine algebraic set that I defines, and $\mathfrak{m}^n$ is the set of polynomials vanishing to order n at the particular point corresponding to $\mathfrak{m}$. From this perspective, our opening questions can be answered by studying the elements in $I^{(n)}$ and their degrees.

This result can be stated in a lot more generality, via differential operators.

Definition 5 (Grothendieck).— Given an A-algebra R, the **A-linear differential operators on R of order up to n**, $D^n_{R|A}$, are defined inductively as follows:

- $D^0_{R|A} = \operatorname{Hom}_R(R,R) \subseteq \operatorname{Hom}_A(R,R)$ where $\operatorname{Hom}_A(R,R)$ consists of the A-module homomorphisms $f: R \to R$.
- $\delta \in D^n_{R|A}$ if and only if $\delta \in \operatorname{Hom}_A(R,R)$ and $\delta f - f\delta \in D^{n-1}_{R|A}$ for every $f \in D^0_{R|A}$.

(See section 16.8 of [Gro67].)

When $R=\mathbb{C}[x_1,\dots,x_d]$, the $\mathbb{C}$-linear differential operators on R of order up to n are

$$D^n_{R|\mathbb{C}} = \bigoplus_{a_1+\cdots+a_d \leqslant n} \mathbb{C}\cdot \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1}\cdots \partial x_d^{a_d}}.$$

The following result is the *differential version* of Zariski-Nagata, see Proposition 2.4 in [DDSG+18].

Theorem 6.— Let k be a perfect field and consider any radical ideal I in $R=k[x_1,\dots,x_d]$. Then

$$I^{(n)} = \left\{ f \in R \mid \partial(f) \in I \text{ for every } \partial \in D^{n-1}_{R|k} \right\}.$$

If we replace k by $\mathbb{Z}$ or some other ring of mixed characteristic, this description no longer holds; roughly speaking, the differential operators cannot see what happens in the arithmetic direction.

Example 3.— In $R=\mathbb{Z}[x]$, the symbolic powers of the maximal ideal $\mathfrak{m}=(2,x)$ coincide with its powers, so $2 \notin \mathfrak{m}^n$ for any $n>1$. However, any differential operator $\partial \in D^n_{R|\mathbb{Z}}$ of any order is $\mathbb{Z}$-linear, so $\partial(2) = 2\cdot\partial(1) \in \mathfrak{m}$.

To describe symbolic powers in mixed characteristic, we need to consider differential operators together with p-derivations, a tool from arithmetic geometry introduced independently in [Joy85] and [Bui95]; for a thorough development of the theory of p-derivations, see [Bui05].

Definition 7 (p-derivation).— Fix a prime $p \in \mathbb{Z}$, and let R be a ring on which p is a nonzerodivisor. A set-theoretic map $\delta: R \to R$ is a **p-derivation** if $\phi_p(x) := x^p + p\delta(x)$ is a ring homomorphism. Equivalently, δ is a p-derivation if $\delta(1)=0$ and δ satisfies the following identities for all $x, y \in R$:

(1) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$,

(2) $\delta(x+y) = \delta(x)+\delta(y)+\mathscr{C}_p(x,y)$

where $\mathscr{C}_p(X,Y) = \frac{X^p+Y^p-(X+Y)^p}{p} \in \mathbb{Z}[X,Y]$. If δ is a p-derivation, we set δ^a to be the a-fold self-composition of δ; in particular, δ^0 is the identity.

Roughly speaking, a p-derivation and its powers play the role of differential operators in the arithmetic direction.

Theorem 8 (De Stefani-Grifo-Jeffries, 2020).— Let $p \in \mathbb{Z}$ be a prime. Let $A = \mathbb{Z}$ or a DVR with uniformizer p. Let R be an essentially smooth A-algebra that has a p-derivation δ. Let Q be a prime ideal of R that contains p, and assume that A/pA is perfect, or more generally that the field extension $A/pA \hookrightarrow R_Q/QR_Q$ is separable. Then

$$Q^{(n)} = \{ f \in S \mid (\delta^s \circ \partial)(f) \in I \text{ for all } \partial \in D^t_{R|A} \text{ with } s+t \leqslant n-1 \}.$$

(See [DSGJ20].)

For prime ideals that do not contain p, the usual description using only differential operators, as in Theorem 6, still holds [DSGJ20, Theorem 3.9].

Example 4.— The maximal ideal $\mathfrak{m} = (2, x)$ in $R = \mathbb{Z}[x]$ contains the prime 2, so to describe its symbolic powers we need to consider a 2-derivation. The map $\delta_2 : R \to R$

$$\delta_2(f(x)) = \frac{f(x^2) - f(x)^2}{2}$$

is a 2-derivation on R. By Theorem 8, the symbolic powers of $\mathfrak{m} = (2, x)$ are given by

$$\mathfrak{m}^{(n)} = \left\{ f \in \mathbb{Z}[x] \;\middle|\; \delta_2^a \left(\frac{\partial^b f}{\partial x^b} \right) \in (2, x), \text{ for } a + b \leqslant n-1 \right\}.$$

In particular, we can now see that $2 \notin \mathfrak{m}^{(2)}$, since

$$\delta_2(2) = \frac{2 - 2^2}{2} = -1 \notin \mathfrak{m},$$

while as we saw in Example 3 there are no $\mathbb{Z}$-linear differential operators ∂ of order up to 1 (or any order!) satisfying $\partial(2) \notin \mathfrak{m}$.

4 Some open Problems

There are many interesting open problems related to symbolic powers. We collect a quick survey of some of those problems, but must necessarily leave a lot of the story to be told elsewhere. For a survey of symbolic powers and other related problems, see [DDSG+18].

4.1 Equality

While the symbolic powers $I^{(n)}$ of I can be computationally difficult to compute, its ordinary powers I^n are very easy to describe. It is thus desirable to understand when $I^{(n)} = I^n$ for some or all n. We do have $I^{(n)} = I^n$ for all n whenever I defines a complete intersection — meaning I is generated by a regular sequence — though this condition is far from being necessary [Hoc73, LS]. A necessary and sufficient condition can be found in [Hoc73], though this condition is not suitable to test in practice outside of special cases. When we restrict to squarefree monomial ideals a polynomial ring $k[x_1, \dots, x_d]$ over a field k, it is conjectured that the condition $I^{(n)} = I^n$ for all n is equivalent to a combinatorial condition. A monomial ideal is an ideal generated by monomials, and it is **squarefree** if it is generated by products of distinct variables; (xy, xz, yz) is a squarefree monomial ideal, (x^2y, z) is a monomial ideal but not squarefree, and $(x+y, z)$ is not a monomial ideal.

Definition 9 (König ideal).— Let I be a squarefree monomial ideal of height c in a polynomial ring over a field. We say that I is **könig** if I contains c monomials with no common variables. A squarefree monomial ideal of height c is said to be **packed** if every ideal obtained from I by setting any number of variables equal to 0 or 1 is könig.

The following is a restatement by Gitler, Valencia, and Villarreal [GVV05] in the setting of symbolic powers of a conjecture of Conforti and Cornuéjols about max-cut min-flow properties.

Conjecture 1 (Packing Problem).— Let I be a squarefree monomial ideal in a polynomial ring over a field k. We have $I^{(n)} = I^n$ for all $n \geqslant 1$ if and only if I is packed.

The Packing Problem has been solved for edge ideals of simple graphs, in which case $I^n = I^{(n)}$ for all n if and only if I is the edge ideal of a bipartite graph [GVV05], but it remains open in the general setting.

One may also wonder if it is sufficient to check the equality $I^n = I^{(n)}$ for finitely many n; this has recently been shown to hold in the case when I is generated by monomials.

Theorem 10 (Montaño-Núnez Betancourt, 2019).— Let I be a monomial ideal in $k[x_1, \dots, x_d]$, where k is a field, and suppose that I is generated by μ monomials. If $I^n = I^{(n)}$ for all $n \leqslant \frac{\mu}{2}$, then $I^n = I^{(n)}$ for all n. (See [MnNnB19].)

It is an open question whether such a theorem holds for a general ideal, and if it does, what values of n we need to test to guarantee $I^n = I^{(n)}$ for all n.

4.2 Degree bounds

Given a nonzero homogeneous ideal I in $k[x_1, \dots, x_d]$, we write $\alpha(I)$ for the minimum degree of a nonzero homogeneous $f \in I$. The questions that we opened the paper with asked about $\alpha(I)$ and $\alpha(I^{(n)})$; giving lower bounds for these quantities can be quite challenging.

Conjecture 2 (Chudnovsky, 1981 [Chu81]).— If I defines a finite set of points in $\mathbb{P}^N$, then for all $m \geqslant 1$ we have

$$\frac{\alpha(I^{(m)})}{m} \geqslant \frac{\alpha(I) + N - 1}{N}.$$

Chudnovsky's conjecture holds for any set of points in $\mathbb{P}^2$ [Chu81, HH13], a general set of points in $\mathbb{P}^3$ [Dum15], a set of at most $N + 1$ points in generic position in $\mathbb{P}^N$ [Dum15], a set of a binomial coefficient number of points forming a star configuration [BH10, GHM13], a set of points in $\mathbb{P}^N$ lying on a quadric [FMX18], a very general set of points in $\mathbb{P}^N$ [DTG17, FMX18], and sets of $s \geqslant 4^N$ general points in $\mathbb{P}^N$ [BGHN]. The case of an arbitrary set of points remains open.

4.3 The Containment Problem

When is $I^{(a)} \subseteq I^b$? Necessary and sufficient conditions for this question to make sense — so that given I and b, we can always find such an a — were studied by Schenzel in the 1980s [Sch85]. For each I and each b, we want to find the smallest possible a with $I^{(a)} \subseteq I^b$. If $I^{(b)} \subseteq I^b$, then $I^{(b)} = I^b$, so this question contains the equality problem as a subproblem. When equality does not hold, we may think of the Containment Problem as a way of comparing the ordinary and symbolic powers of I. Notice also that if $I^{(a)} \subseteq I^b$, then $\alpha(I^{(a)}) \geqslant b\alpha(I)$, so answering the Containment Problem for I will in particular provide lower bounds for the degrees of elements in the symbolic powers of I.

Over $R = k[x_1, \dots, x_d]$, or more generally any regular ring, the answer depends on the **big height** of I, the largest codimension of an irreducible component of the algebraic set corresponding to I, which in algebraic terms is the same as the largest height of a minimal prime of I.

This answer is a beautiful theorem of Ein—Lazersfeld—Smith, Hochster—Huneke, and Ma—Schwede.

Theorem 11.— Let R be a regular ring and I a radical ideal in R. If h is the big height of I, then

$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geqslant 1.$$

(See [ELS01, HH02, MS17, M].)

In particular, when k is a field and $R = k[x_1, \dots, x_d]$, the theorem says that $I^{(dn)} \subseteq I^n$ for every I. This type of uniform behavior — in this case, independent of the ideal I we choose — appears in many shapes and forms throughout commutative algebra. For example, for a prime ideal P of height 2, the theorem says that $P^{(4)} \subseteq P^2$; in 2000, Huneke asked if this could be improved to $P^{(3)} \subseteq P^2$ under some technical hypothesis, which inspired the following conjecture.

Conjecture 3 (Harbourne, 2006).— Let I be a radical homogeneous ideal in $k[x_1, \dots, x_d]$, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Hochster and Huneke's proof of Theorem 11 uses prime characteristic techniques and reduction to characteristic p to do the case when the ring contains a field, and their proof in the prime characteristic p case for $n = p^e$ turns out to be a beautiful application of the Pigeonhole Principle. A more careful application of the Pigeonhole Principle gives Harbourne's Conjecture for powers of p: $I^{(hq-h+1)} \subseteq I^q$ for all $q = p^e$. In an amazing twist, however, 3 is not true as stated: there is a set of 12 points in $\mathbb{P}^{12}$ [DSTG13] with $h = 2$ that fails $I^{(3)} \subseteq I^2$, among other families of counterexamples [HS15, Sec15, DS21].

Despite these counterexamples, Conjecture 3 does hold for some large classes of ideals, such as monomial ideals [BDRH+09, Example 8.4.5], generic sets of points in $\mathbb{P}^2$ [BH10] or $\mathbb{P}^3$ [Dum15], for matroid configurations [GHMN17], and for star configurations [HH13]. The conjecture also holds if R/I has nice singularities: if R/I is F-pure in prime characteristic or of dense F-pure type in equicharacteristic 0 [GH]. This class of rings contains Veronese rings, generic determinantal rings, and more generally rings of invariants of linearly reductive groups.

Moreover, every counterexample to 3 known to date actually satisfies the following open conjecture:

Conjecture 4 (Stable Harbourne [Gri20]).— If I is a radical ideal of big height h in a regular ring, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

We are asking if Harbourne's Conjecure holds for n large — where large enough should depend on I. The philosophy is that when one asks for the smallest a_n such that $I^{(a_n)} \subseteq I^n$, things get better as n grows. Not only do we have no counterexamples to this conjecture, the evidence supporting it keeps growing [Gri20, BGHN, GHM20a, GHM20b]. In fact, every counterexample known to date to the original conjecture, Conjecture 3, satisfies the stable conjecture.

If studying $I^{(n)}$ is hard, the computational problems only get harder as n grows. As such, testing conjectures such as this one can be quite challenging. Many of the results in this direction rely on proving that certain particular containments are sufficient to obtain an eventual containment statement for large n, a technique which has also found applications [BGHN, BGHN22] in the degree problem we mentioned in Section 4.2.

The problems we discussed here have been open for decades, but have paved the way for many new research avenues in recent years. For more on recent advances in the topic of symbolic powers, and their connections to other topics, see [DDSG+18].

Acknowledgements

The author was partially supported by NSF grant DMS-2140355. The author thanks the anonymous referee for helpful comments and corrections.

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The CIM Bulletin is published once a year. Material intended for publication should be sent to one of the editors. The Bulletin is available at www.cim.pt. The CIM acknowledges the financial support of FCT–Fundação para a Ciência e Tecnologia.