What's in a Circle Action?

by **Leonor Godinho***

Given a compact connected Lie group G and a closed manifold M it is natural to ask if M admits a nontrivial action of G and, if yes, how many different actions it can have. The existence of even the simplest case of a circle action already imposes strong restrictions on the topology of the manifold. We will explore some of these restrictions, illustrating how the simple existence of a circle symmetry already provides much information on the underlying manifold.

1 Introduction

Given a compact connected Lie group G and a closed manifold M it is natural to ask if M admits a nontrivial action of G and, if yes, how many different actions it can have. The existence of even the simplest case of a circle action already imposes strong restrictions on the topology of the manifold. For instance, in the 1970's, Petrie proved that if M is homotopy equivalent to a complex projective space and admits a circle action with isolated fixed points, then its Pontrjagin classes are determined by the representations at the fixed points [27]. Based on this, he formulated what is known as the *Petrie conjecture*: if M is homotopy equivalent to a complex projective space and admits a circle action with isolated fixed points then its Pontrjagin classes are the same as those of the projective space. This was proved in many situations [7, 13, 19, 24, 25, 33, 34] but it is still open in general.

If we consider an almost complex manifold, the existence of a circle action again restricts many topological (or almost complex) invariants of the manifold. A simple example is the Euler characteristic. Indeed, if M^{2n} admits a circle action with isolated fixed points that preserves the almost complex structure, then the number of fixed points coincides with the Euler characteristic of M [11, Section 3]. If, in addition, M is symplectic[1] and the action is Hamiltonian[2] then the fixed points are the critical points of the Hamiltonian function (a perfect Morse function) and so, as the Morse inequalities become equalities, the number of fixed points is equal to the sum of the even Betti numbers of M (all critical points have an even index). Since the classes $[\omega^k] \in H^{2k}(M;\mathbb{R})$ are non zero for $k = 0, \dots, (\dim M)/2$, the number of fixed points, and consequently the Euler characteristic of M, is at least $n+1$.

Another topological invariant of an almost complex manifold that is determined by the circle action is the Chern number $c_1c_{n-1}[M]$, where $c_j \in H^{2j}(M;\mathbb{Z})$ is the degree-$2j$ Chern class of TM, for $j = 0, \dots, n$. Salamon [30] showed that this Chern number can be obtained from the Hirzebruch genus[3] of M as

$$c_1c_{n-1}[M] = 6 \left.\frac{d^2\chi_y(M)}{dy^2}\right|_{y=-1} + \frac{n(5-3n)}{2}\,\chi_{-1}(M). \tag{1}$$

[1] A symplectic manifold is a pair (M,ω) where M is a smooth manifold and ω is a closed non-degenerate 2-form on M called a *symplectic form*.

[2] A symplectic circle action on (M,ω) is said to be Hamiltonian if there exists an S^1-invariant function $\psi : M \to \mathbb{R}$ (called the *moment map* or *Hamltonian function*) such that $d\psi = -\iota(\xi^\sharp)\omega$, where $\xi^\sharp$ is the vector field generated by the circle action.

[3] The Hirzebruch genus $\chi_y(M)$ is the genus corresponding to the power series $Q_y(x) = (x(1+ye^{-x(1+y)}))/(1-e^{-x(1+y)})$ (cf. [16]).

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Since the Hirzebruch genus is rigid for almost complex manifolds admitting a circle action, this Chern number is determined by a set of integers N_i defined by the action as follows:

Theorem 1.— [11, Theorem 1.2] Let (M^{2n}, J) be a closed almost complex manifold with an S^1-action that preserves the almost complex structure J and has isolated fixed points. For every $i = 0, \dots, n$, let N_i be the number of fixed points with exactly i negative weights[4]. Then

$$c_1 c_{n-1}[M] = \sum_{i=0}^{n} N_i \left[6i(i-1) + \frac{5n - 3n^2}{2} \right]. \quad (2)$$

In the following sections we will see some interesting applications of this result. The goal is to illustrate how the simple existence of a circle symmetry can provide so much information on the underlying manifold.

2 Possible weights for a circle action

The collection of possible weights for a circle action on an almost complex manifold must satisfy many strong conditions imposed, for instance, by the Localization Theorem in equivariant cohomology [2, 5]. Using (2) we can obtain additional linear relations through an algorithm constructed in [11].

These relations are very powerful. In particular, when M is symplectic of dimension smaller than 10 and the action is Hamiltonian with a **minimal** number of fixed points, it is possible to determine all families of weights, proving a symplectic generalization of the Petrie conjecture proposed by Tolman [32]:

Conjecture 1 (Symplectic Petrie Conjecture).— If a symplectic manifold (M, ω) satisfying $H^{2i}(M; \mathbb{R}) = H^{2i}(\mathbb{CP}^n; \mathbb{R})$ for all i admits a Hamiltonian circle action, then $H^j(M; \mathbb{Z}) = H^j(\mathbb{CP}^n; \mathbb{Z})$ for all j. Moreover, the total Chern class $c(TM)$ is completely determined by the cohomology ring $H^*(M; \mathbb{Z})$.

In dimension 4, the weights obtained by the algorithm agree with those of the standard circle action on the complex projective plane, and so do the (equivariant) cohomology ring and Chern classes of the manifold.

In dimension 6 we recover previous results of Ahara [1] and Tolman [32].

Theorem 2.— [32, Theorem 1] Let (M^6, ω) be a closed symplectic manifold with a Hamiltonian circle action with 4 fixed points. Then one of the following holds:

1. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^4)$,
   and $c(TM) = 1 + 4x + 6x^2 + 4x^3$;
2. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 2y, y^2)$,
   and $c(TM) = 1 + 3x + 8y + 4xy$;
3. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 5y, y^2)$,
   and $c(TM) = 1 + 2x + 12y + 4xy$;
4. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 22y, y^2)$,
   and $c(TM) = 1 + x + 24y + 4xy$;

(where, in all cases, x has degree 2 and y has degree 4).

In (1) the weights agree with those of the standard circle action on $\mathbb{CP}^3$. In (2) they agree with those of a circle action on the Grassmannian of oriented 2-planes[5] $Gr_2^+(\mathbb{R}^5)$ as a subgroup of $SO(5)$. In (3) and (4) they are the same as those of circle actions on the Fano manifolds V_5 and V_{22} [26].

In dimension 8 the algorithm yields the following result [11, 21].

Theorem 3.— Let (M^8, ω) be a closed symplectic manifold with a Hamiltonian S^1-action with 5 fixed points. Then the weights agree with those of the standard circle action on $\mathbb{CP}^4$ as well as the cohomology ring and Chern classes, i.e.

$$H^*(M; \mathbb{Z}) = \mathbb{Z}[y]/(y^5) \quad \text{and} \quad c(TM) = (1 + y)^5,$$

where y has degree 2.

3 Lower bounds for the number of fixed points

Theorem 1 imposes several restrictions on the possible number of fixed points of a circle action. If M

[4] Given a fixed point $p_i \in M$, the S^1-representation on $T_{p_i}M$ is determined by a multiset of integers $\{w_{i1}, \dots, w_{in}\}$ called the weights of the action at p_i and we can equivariantly identify $T_{p_i}M$ with $\mathbb{C}^n$ with a circle action given by $\lambda \cdot (z_1, \dots, z_n) = (\lambda^{w_{i1}} z_1, \dots, \lambda^{w_{in}} z_n)$, for $\lambda \in S^1$.

[5] An $SO(5)$ coadjoint orbit.

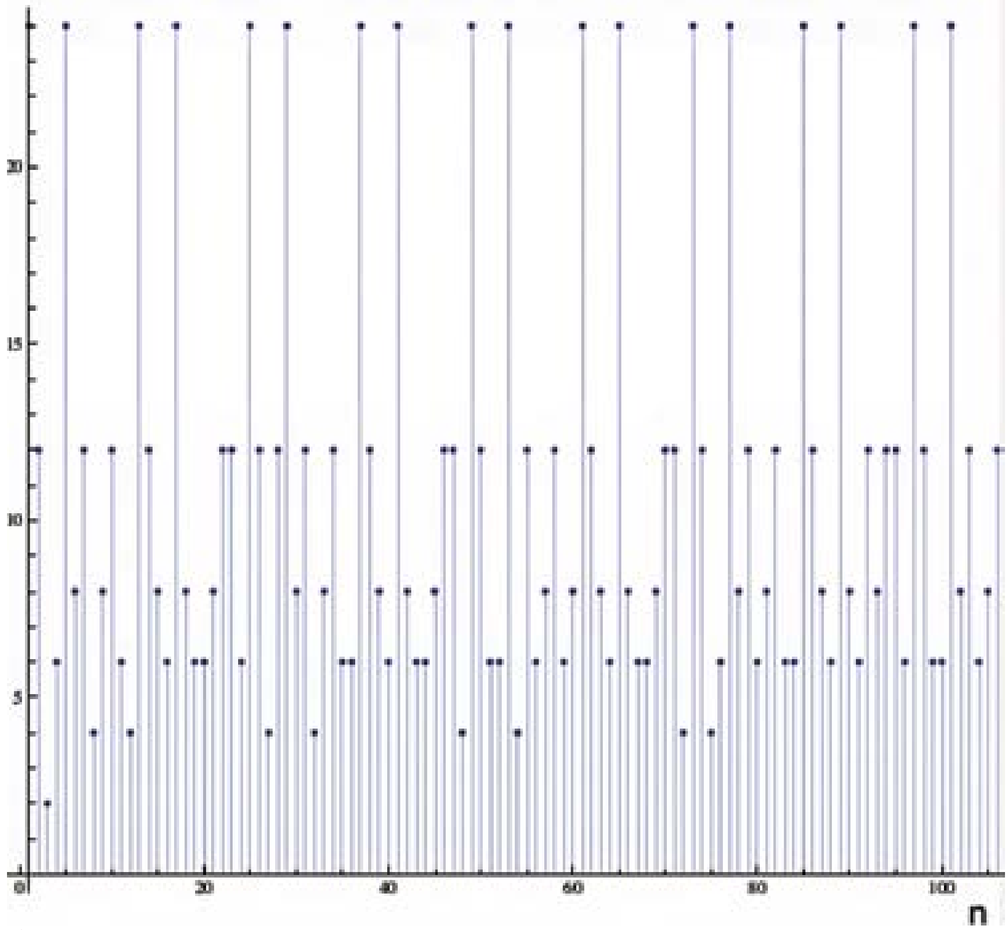

Figure 1.—Lower bounds for the number of fixed points, $n \leq 100$.

is symplectic and the action is Hamiltonian we have seen that there are at least $n+1$ fixed points. Moreover, for general unitary S^1-manifolds[6] there exists an open conjecture stated by Kosniowsky [22].

Conjecture 2 (Kosniowski).— There exists a linear function f such that, for every $2n$-dimensional compact unitary S^1-manifold M with isolated fixed points which is not equivariantly unitary cobordant with the empty set, the number of fixed points is greater than $f(n)$. In particular, $f(x) = x/2$ should satisfy this condition, implying that the number of fixed points is expected to be at least $\lfloor n/2 \rfloor + 1$.

Using information from a non-vanishing Chern number, several lower bounds have been obtained (see for example [14, 29, 23, 6, 20]). If, on the other hand, we have $c_1 c_{n-1}[M] = 0$, which is satisfied, for example by all symplectic Calabi-Yau manifolds, then Theorem 1 provides additional lower bounds (see Figure 1) in the case of an almost complex manifold and, in particular, for symplectic circle actions [12, Theorem B]. This requires the use of classical number theory results involving polygonal numbers originally stated by Fermat and proved later by Legendre, Lagrange, Euler, Gauss and Ewell [8, 9]. In some cases the lower bounds obtained are stronger than those conjectured by Kosniowski. However, our bounds are at most 24 in all dimensions, and so they do not provide evidence of a lower bound that depends linearly on the dimension, as proposed by Kosniowski.

Still when $c_1 c_{n-1}[M] = 0$, Theorem 1 provides strong divisibility conditions that must be satisfied by the number of fixed points. These improve the existing divisibility results for the Euler characteristic of almost complex manifolds satisfying $c_1 c_{n-1}[M] = 0$ obtained by Hirzebruch in [15], adding that the Euler characteristic must be divisible by 3, whenever the dimension of the manifold is not a multiple of 6.

Theorem 4.— [12, Theorem A] Let (M^{2n}, J) be a closed connected almost complex manifold equipped with a J-preserving circle action with nonempty, discrete fixed point set M^{S^1} and such that $c_1 c_{n-1}[M] = 0$. Let m be such that $n = 2m$ ($m \geq 1$) when n is even, and $n = 2m+3$ ($m \geq 1$) when n is odd. If

[6] Unitary S^1-manifolds are smooth manifolds with a fixed S^1-invariant complex structure on the stable tangent bundle.

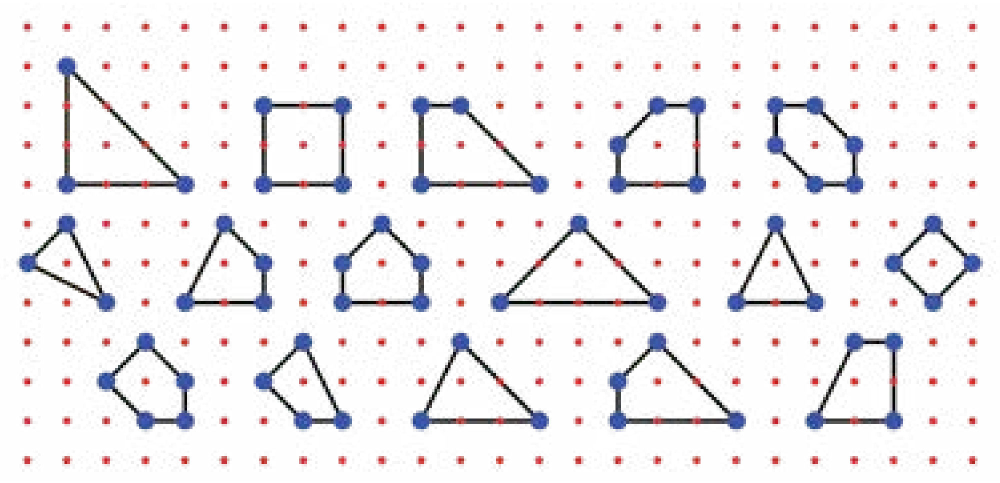

Figure 2.—Reflexive Polygons.

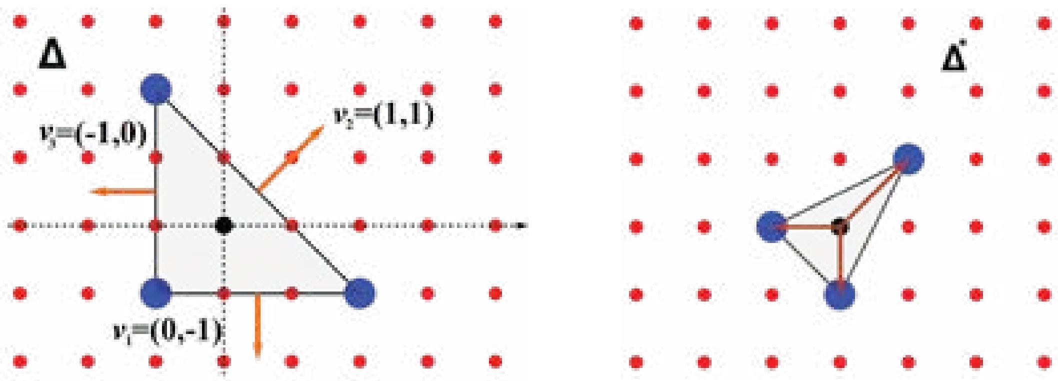

Figure 3.—A reflexive triangle Δ and its polar dual Δ^*.

$r = \gcd(m, 12)$, then

$$|M^{S^1}| \equiv 0 \mod \frac{12}{r} \quad \text{if } n \text{ is even}$$

and

$$|M^{S^1}| \equiv 0 \mod \frac{24}{r} \quad \text{if } n \text{ is odd.}$$

If we restrict to Hamiltonian actions, keeping the hypothesis that $c_1 c_{n-1}[M] = 0$, we can improve the existing lower bound of $n + 1$.

Theorem 5.— [12, Theorem 2.8] Let M be a $2n$-dimensional closed connected symplectic manifold with $c_1 c_{n-1}[M] = 0$. Then the number of fixed points of a Hamiltonian circle action on M is at least

- $(n+1)(n+2)$, if n is even;
- $n^2 + 6n + 17 + \dfrac{24}{\gcd(\frac{n-3}{2}, 12)}$, if $n > 3$ is odd.

4 Reflexive polytopes

Another interesting application of Theorem 1 concerns reflexive polytopes. A polytope Δ is called *reflexive* if it is integral, contains the origin in the interior and can be written as

$$\Delta = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \leq 1\},$$

where $v_i \in \mathbb{Z}^n$ are the primitive outward normal vectors to the hyperplanes supporting the facets of Δ (see Figure 2).

They were first defined by Batyrev [3], play an important role in mirror symmetry and satisfy many special combinatorial properties. For example, they have only one interior lattice point (the origin) and their polar duals are also reflexive. Moreover,

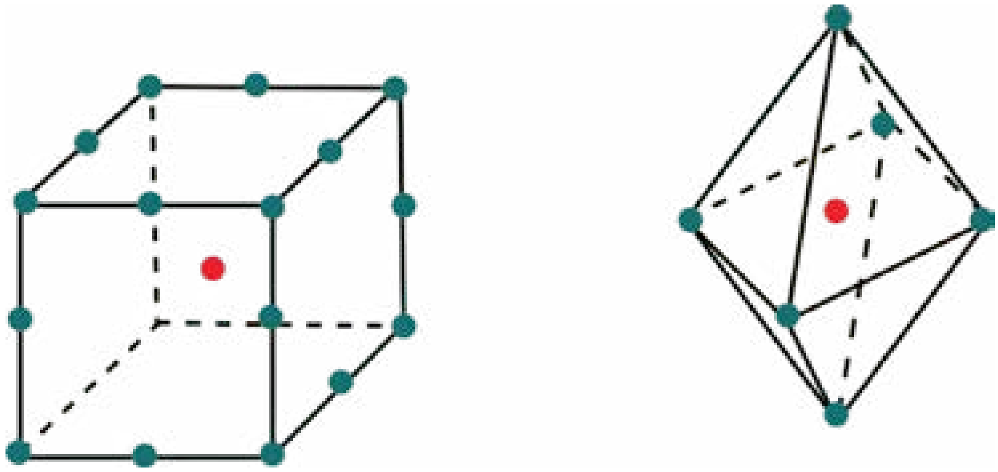

Figure 4.—The reflexive cube and its dual.

they satisfy the following property in dimensions 2 and 3, involving the relative (lattice) length of their edges and of their polar duals.

Theorem 6.— (12 and 24 -Theorem) Let Δ be a reflexive polytope of dimension n with edge set E.

- If $n = 2$ then
$$\sum_{e\in E} l(e) + \sum_{f\in E^*} l(f) = 12\,;$$
- If $n = 3$ then
$$\sum_{e\in E} l(e)l(e^*) = 24\,,$$

where E^* denotes the edge set of the dual polytope Δ^*, the edge $e^* \in E^*$ is dual to the edge $e \in E$ and $l(e)$ is the relative length of e.

One can prove this theorem in many ways. For example, since there exists only a finite number of reflexive polytopes in each dimension (up to a lattice isomorphism) one can prove it by exhaustion. In dimension two, there are other proofs [28, 17], involving modular forms, toric geometry and certain relations in $SL(2,\mathbb{Z})$. In dimension three, this result was obtained by Dais and Batyrev [4, Corollary 7.10] using toric geometry. A combinatorial proof is presented in [18, Section 5.1.2].

Surprisingly, we can use Theorem 1 to generalize Theorem 6 to all Delzant[7] reflexive polytopes, i.e. those arising as moment map images of closed symplectic toric manifolds (M,ω) with $c_1 = [\omega]$. In particular, we have the following result.

Theorem 7.— [10, Theorem 1.2] Let Δ be a Delzant reflexive polytope of dimension n with edge set E. Then
$$\sum_{e\in E} l(e) = 12 f_2 + (5-3n) f_1, \tag{3}$$
where f_k is the number of faces of Δ of dimension k.

For $n = 2$, the Delzant reflexive polygons are depicted in the first line of Figure 2. Moreover, in this dimension, Theorem 6 is equivalent to the property that the sum of the relative lengths of the edges of Δ and the number of vertices of Δ is always equal to 12 (see Fgure 2). Indeed, all the edges of the dual polygon Δ^* have length equal to 1 (as Δ is Delzant) and the number of edges of Δ^* is equal to the number of vertices of Δ. On the other hand, the relation in (3) tell us that the sum of the relative lengths of the edges of Δ is equal to $12 - f_1$ or, equivalently, to $12 - f_0$ (as the number of edges of a polygon is equal to the number of vertices) and so the two theorems agree.

When $n = 3$, the relation in (3) tells us that the sum of the integer lengths of the edges of a Delzant reflexive polytope is equal to $12f_2 - 4f_1$. Using the Euler relation $f_0 - f_1 + f_2 = 2$ and the fact that $3f_0 = 2f_1$ (as the polytope is simple), we obtain that this sum is always 24. This agrees with Theorem 6 since the length of every edge of the dual of a Delzant reflexive polytope is always 1 (see, for example, Figure 4 for the reflexive cube and its dual). Note that, in this dimension, the sum of the relative lengths of the edges of Δ has a nice geometric interpretation: it is the Euler characteristic of a Calabi Yau surface (a

[7] A polytope of dmension n is said to be *Delzant* if it is simple (each vertex is the intersection of exactly n edges), rational (the n edges that intersect at a vertex v are contained in affine lines of the form $v + \langle u_i \rangle$ with $u_i \in \mathbb{Z}^n$) and smooth (for each vertex, the edge vectors $u_i \in \mathbb{Z}^n$ can be chosen so that $\langle u_1, \dots, u_n \rangle_{\mathbb{Z}} = \mathbb{Z}^n$).

$K3$ surface) obtained from Δ through a construction described, for example, in [3]. To prove Theorem 7 using Theorem 1, we consider the symplectic toric manifold (M_Δ, ω, ψ) corresponding to the Delzant polytope $\Delta = \psi(M_\Delta)$ (where ψ is the toric moment map) and the preimage $\mathcal{S} := \psi^{-1}(E)$ of the edge set. Then $\mathcal{S}$ is a union of smoothly embedded 2-spheres $\mathcal{S} = \cup_{e\in E} S_e^2$ and is Poincaré dual to c_{n-1}. Hence, we have

$$c_1 c_{n-1}[M_\Delta] = \sum_{S_e^2\in\mathcal{S}} c_1[S_e^2] = \sum_{S_e^2\in\mathcal{S}} [\omega]([S_e^2]) = \\ = \sum_{S_e^2\in\mathcal{S}} \mathrm{Vol}_\omega(S_e^2) = \sum_{e\in E} l(e),$$

and so this Chern number is the sum of the relative lengths of the edges of Δ. Taking a generic subcircle of the torus acting on M_Δ and the corresponding N_i (the number of fixed points of this circle action with exactly i negative weights which, in turn, is equal to the Betti number $b_{2i}(M_\Delta)$), and expressing the Betti numbers of M_Δ in terms of the face numbers of Δ [31], we obtain the relation in (3).

This result can also be proved without any symplectic or toric geometry, using only the combinatorial properties of Delzant reflexive polytopes (see [10] for details).

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