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Contents

Editorial

The COVID pandemic is still conditioning CIM's activities and, in particular, some events partially supported by CIM occurred online. This was the case of the Workshop CIM & Enterprises, Dynamic Control and Optimization International Conference and SPM's National Meeting.

In this issue, we include three articles on topical research subjects. One of them illustrates how to use commutative algebra tools to study classical geometric problems once these are reframed within the language of symbolic powers. Another paper shows how to use topological and variational tools to analyse semilinear elliptic problems exemplified with the Lane-Emden equation. The last article focuses on the intimate relations between the existence of a circle action on a closed manifold and special properties of its geometry and topology.

We feature an interview with Adélia Sequeira, where we took a look back at her career, and, within the cycle of historical articles, we feature an article about Hugo Baptista Ribeiro, a distinguished mathematician from the *Geração de 40*. We also include a report with an overview of the Portuguese Mathematical Typography, which was the motif of an exhibition that took place in the sexcentenary Moinho de Papel in Leiria.

We recall that the bulletin continues to welcome the submission of review, feature, outreach and research articles in Mathematics and its applications.

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Symbolic Powers

by **Eloísa Grifo***

Symbolic powers arise naturally in commutative algebra from the theory of primary decomposition, but they also contain geometric information, thanks to a classical result of Zariski and Nagata. Computing primary decompositions is a difficult computational problem, and as a result, many natural questions about symbolic powers remain wide open. We will briefly introduce symbolic powers and describe some of the main open problems on the subject, and point to some recent research advances.

1 Introduction

Given a finite set of points $P_1, \dots, P_s$ in projective space $\mathbb{P}^d_{\mathbb{C}}$, what is the lowest degree of a hypersurface passing through $P_1, \dots, P_s$? How about a hypersurface passing through each given point P_i with the same multiplicity n? More generally, given an affine or projective variety V, which polynomials vanish to order n at every point in V? These classical geometric questions can be studied with commutative algebra tools once we reframe them within the language of symbolic powers.

Let us formalize what we mean by vanishing to order n. Given a point a in either affine space $\mathbb{A}^{d+1}_{\mathbb{C}}$ or projective space $\mathbb{P}^d_{\mathbb{C}}$, a polynomial $f \in R := \mathbb{C}[x_0, \dots, x_d]$, which we assume to be homogeneous in the projective case, vanishes to order n at a if

$$\frac{\partial^{c_0+\cdots+c_d} f}{\partial x_0^{c_0} \cdots \partial x_d^{c_d}}(a) = 0 \quad \text{for all } c_0 + \cdots + c_d < n.$$

Notice that with this definition, f vanishes to order 1 at a if and only if $f(a) = 0$. More generally, given an algebraic set V — the solution set to some system of polynomial equations in $d+1$ variables, which are homogeneous in the projective case — consider the ideal I of all the polynomials in R that vanish at every $a \in V$. A (homogeneous, in the projective case) polynomial f **vanishes to order n along V** if

$$\frac{\partial^{c_0+\cdots+c_d}}{\partial x_0^{c_0} \cdots \partial x_d^{c_d}}(f) \in I \quad \text{for all } c_0 + \cdots + c_d < n.$$

Let us give examples of polynomials vanishing to order n on a given algebraic set. The **nth power of I** is the ideal generated by all the n-fold products of elements in I, which we write as

$$I^n := (f_1 \cdots f_n \mid f_i \in I).$$

Here the notation $I = (g_1, \dots, g_m)$ stands for the ideal generated by $g_1, \dots, g_m$, so the elements in I^n are all the R-linear combinations of n-fold products of polynomials that vanish at V. It is elementary to show that every element in I^n must vanish to order n along V. However, we may have other more interesting polynomials vanishing to order n along V.

Example 1.— Let V be the union of the 3 coordinate lines in affine 3-space, which corresponds to the ideal $I = (xy, xz, yz)$. The polynomial $f = xyz$ vanishes to order 2 along V, since $\partial f/\partial x = yz \in I$, $\partial f/\partial y = xz \in I$, and $\partial f/\partial z = xy \in I$. On the other hand, all the nonzero polynomials in I^2 have degree 4 or higher, so $f \notin I^2$.

In particular, we may have polynomials vanishing to order n along V that live in an unexpected degree — meaning, a degree d such that I^n has no polynomials of degree d. Completely describing which polynomials vanish to order n along a given algebraic set V,

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determining whether those are exactly the polynomials in I^n, or giving (lower) bounds for the degrees of polynomials vanishing to order n along V are all very delicate questions.

We can attack these questions using purely algebraic tools, thanks to a classical result of Zariski and Nagata [Zar49, Nag62] which says that the polynomials that vanish to order n along V are exactly the polynomials in the nth symbolic power of I, which we will introduce in the next section. Despite being a classical topic that has been around for a century, many natural questions about symbolic powers remain unanswered, in part because it is computationally difficult to calculate symbolic powers and test conjectures. We will first introduce symbolic powers in Section 2, and then quickly survey some of the current active research questions related to symbolic powers in the remaining sections. For a more detailed survey of symbolic powers, see [DDSG+18]. Throughout, let R be a commutative Noetherian ring; a good working example is the case when R is a polynomial ring in finitely many variables over a field k.

2 Symbolic powers: definition and basic properties

Symbolic powers arise naturally in commutative algebra from the theory of primary decomposition. Roughly speaking, primary decomposition is an ideal-theoretic version of the Fundamental Theorem of Arithmetic — the theorem which says that every nonzero integer can be written as a product of prime integers that is unique up to sign and the order of the factors. Once we replace the integers with other commutative rings, there are many examples of rings where we *cannot* write every element as a product of irreducibles that is unique up to multiplication by units or the order of the factors; for example, in $\mathbb{Z}[\sqrt{-5}]$, $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ are two distinct factorizations into irreducibles. One way to avoid this failure of the Fundamental Theorem of Arithmetic is to focus on ideals rather than elements: every ideal in a Noetherian ring can be written as a finite intersection of primary ideals [Las05, Noe21], and while this *primary decomposition* is not necessarily unique, there are certain aspects of it that are in fact unique.

Let us start with prime ideals. An ideal P is **prime** if $ab \in P$ implies that $a \in P$ or $b \in P$. When $R = \mathbb{C}[x_0, \dots, x_d]$, prime ideals are precisely the ideals that correspond to varieties: a variety is an *irreducible* algebraic set, meaning it cannot be decomposed as a finite union of two or more proper algebraic subsets.

Definition 1.— Let P be a prime ideal. The **nth symbolic power** of P is the ideal

$$P^{(n)} := \{ f \in R \mid sf \in P^n \text{ for some } s \notin P \}.$$

Note that $P^n \subseteq P^{(n)}$, since every $f \in P^n$ satisfies $1 \cdot f \in P^n$ for $1 \notin P$. In general, $P^n \neq P^{(n)}$.

Example 2.— Let $R = k[x,y,z]/(xy - z^2)$, where k is an arbitrary field, and consider the prime ideal $P = (x,z)$ in R. Since $xy = z^2 \in P^2$ and $y \notin P$, we have $x \in P^{(2)}$, while $x \notin P^2$.

While we will not define primary decomposition, it turns out that when writing a primary decomposition for P^n, one of the components — the P-primary component — will be precisely $P^{(n)}$. Historically, this is the context where symbolic powers first arose.

More generally, let us consider a radical ideal I, which means that I is a finite intersection of prime ideals. Geometrically, Hilbert's Nullstellensatz gives us a bijection between algebraic sets and radical ideals, so for our purposes these are the only ideals we care about.

Definition 2.— Let $P_1, \dots, P_k$ be prime ideals, and let $I = P_1 \cap \dots \cap P_k$. The **$n$th symbolic power** of I is the ideal

$$\begin{aligned} I^{(n)} &:= P_1^{(n)} \cap \dots \cap P_k^{(n)} \\ &= \{ f \in R \mid sf \in I^n \text{ for some } s \notin \textstyle\bigcup_{i=1}^k P_i \}. \end{aligned}$$

The following properties can be shown via elementary commutative algebra methods.

Theorem 3.— Let I be a radical ideal in a Noetherian ring R.

1. $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.
2. $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.
3. $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all $a, b \geqslant 1$.

The last property allows us to construct the *symbolic Rees algebra of I*, which packages together all the symbolic powers of I into one graded object. The symbolic Rees algebra of I is the graded R-algebra with $I^{(n)}$ in degree n, $\mathscr{R}_s(I) = \bigoplus I^{(n)} t^n \subseteq R[t]$, where the t keeps track of degrees. It turns out that this algebra can fail to be finitely generated over R — or equivalently, it can fail to be a Noetherian ring — which means that for arbitrarily high values of n, there are elements in $I^{(n)}$ that do not live in the product of symbolic powers of I of lower order. While we will not have a chance to discuss symbolic Rees algebras in detail, we point the reader to [GS20] for a survey on symbolic Rees algebras and the fascinating problem of when they are finitely generated.

In the next section we will discuss some of the geometric motivations to study $I^{(n)}$. Note that there are also many algebraic reasons to study symbolic powers, including the fact that they can be used as effective tools to answer questions that are a priori unrelated to symbolic powers, and that symbolic powers are used in the proofs of important results in commutative algebra, such as Krull's Height Theorem and the Hartshorne-Lichtenbaum Vanishing Theorem in local cohomology, even though these results are not about symbolic powers.

3 Higher order vanishing

A classical result of Zariski and Nagata [Zar49, Nag62] and its modern generalization by Eisenbud and Hochster [EH79] give us the connection with our opening questions.

Theorem 4 (Zariski-Nagata, 1949 and 1962).— Let I be a radical ideal in $R = \mathbb{C}[x_0, \ldots, x_d]$. Then

$$I^{(n)} = \bigcap \{\mathfrak{m}^n \mid \mathfrak{m} \supseteq I, \mathfrak{m} \text{ maximal ideal}\} =$$
$$= \left\{ f \in R \;\middle|\; \frac{\partial^{c_0+\cdots+c_d}}{\partial x_0^{c_0} \cdots \partial x_d^{c_d}}(f) \in I, \text{ for all } c_0 + \cdots + c_d < n \right\}.$$

(See [Zar49, Nag62, EH79, DDSG+18].)

This is the classical result we alluded to in the introduction: that $I^{(n)}$ is precisely the set of polynomials that vanish to order n along the algebraic set corresponding to I. The maximal ideals $\mathfrak{m}$ that contain the radical ideal I correspond to each point in the affine algebraic set that I defines, and $\mathfrak{m}^n$ is the set of polynomials vanishing to order n at the particular point corresponding to $\mathfrak{m}$. From this perspective, our opening questions can be answered by studying the elements in $I^{(n)}$ and their degrees.

This result can be stated in a lot more generality, via differential operators.

Definition 5 (Grothendieck).— Given an A-algebra R, the **A-linear differential operators on R of order up to n**, $D^n_{R|A}$, are defined inductively as follows:

- $D^0_{R|A} = \operatorname{Hom}_R(R, R) \subseteq \operatorname{Hom}_A(R, R)$ where $\operatorname{Hom}_A(R, R)$ consists of the A-module homomorphisms $f : R \to R$.
- $\delta \in D^n_{R|A}$ if and only if $\delta \in \operatorname{Hom}_A(R, R)$ and $\delta f - f\delta \in D^{n-1}_{R|A}$ for every $f \in D^0_{R|A}$.

(See section 16.8 of [Gro67].)

When $R = \mathbb{C}[x_1, \ldots, x_d]$, the $\mathbb{C}$-linear differential operators on R of order up to n are

$$D^n_{R|\mathbb{C}} = \bigoplus_{a_1+\cdots+a_d \leqslant n} \mathbb{C} \cdot \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}.$$

The following result is the *differential version* of Zariski-Nagata, see Proposition 2.4 in [DDSG+18].

Theorem 6.— Let k be a perfect field and consider any radical ideal I in $R = k[x_1, \ldots, x_d]$. Then

$$I^{(n)} = \left\{ f \in R \mid \partial(f) \in I \text{ for every } \partial \in D^{n-1}_{R|k} \right\}.$$

If we replace k by $\mathbb{Z}$ or some other ring of mixed characteristic, this description no longer holds; roughly speaking, the differential operators cannot see what happens in the arithmetic direction.

Example 3.— In $R = \mathbb{Z}[x]$, the symbolic powers of the maximal ideal $\mathfrak{m} = (2, x)$ coincide with its powers, so $2 \notin \mathfrak{m}^n$ for any $n > 1$. However, any differential operator $\partial \in D^n_{R|\mathbb{Z}}$ of any order is $\mathbb{Z}$-linear, so $\partial(2) = 2 \cdot \partial(1) \in \mathfrak{m}$.

To describe symbolic powers in mixed characteristic, we need to consider differential operators together with p-derivations, a tool from arithmetic geometry introduced independently in [Joy85] and [Bui95]; for a thorough development of the theory of p-derivations, see [Bui05].

Definition 7 (p-derivation).— Fix a prime $p \in \mathbb{Z}$, and let R be a ring on which p is a nonzerodivisor. A set-theoretic map $\delta : R \to R$ is a **p-derivation** if $\phi_p(x) := x^p + p\delta(x)$ is a ring homomorphism. Equivalently, δ is a p-derivation if $\delta(1) = 0$ and δ satisfies the following identities for all $x, y \in R$:

(1) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$,

(2) $\delta(x + y) = \delta(x) + \delta(y) + \mathscr{C}_p(x, y)$

where $\mathscr{C}_p(X, Y) = \frac{X^p+Y^p-(X+Y)^p}{p} \in \mathbb{Z}[X, Y]$. If δ is a p-derivation, we set δ^a to be the a-fold self-composition of δ; in particular, δ^0 is the identity.

Roughly speaking, a p-derivation and its powers play the role of differential operators in the arithmetic direction.

Theorem 8 (De Stefani-Grifo-Jeffries, 2020).— Let $p \in \mathbb{Z}$ be a prime. Let $A = \mathbb{Z}$ or a DVR with uniformizer p. Let R be an essentially smooth A-algebra that has a p-derivation δ. Let Q be a prime ideal of R that contains p, and assume that A/pA is perfect, or more generally that the field extension $A/pA \hookrightarrow R_Q/QR_Q$ is separable. Then

$$Q^{(n)} = \{ f \in S \mid (\delta^s \circ \partial)(f) \in I \text{ for all } \partial \in D^t_{R|A} \text{ with } s + t \leqslant n-1 \}.$$

(See [DSGJ20].)

For prime ideals that do not contain p, the usual description using only differential operators, as in Theorem 6, still holds [DSGJ20, Theorem 3.9].

Example 4.— The maximal ideal $\mathfrak{m} = (2, x)$ in $R = \mathbb{Z}[x]$ contains the prime 2, so to describe its symbolic powers we need to consider a 2-derivation. The map $\delta_2 : R \to R$

$$\delta_2(f(x)) = \frac{f(x^2) - f(x)^2}{2}$$

is a 2-derivation on R. By Theorem 8, the symbolic powers of $\mathfrak{m} = (2, x)$ are given by

$$\mathfrak{m}^{(n)} = \left\{ f \in \mathbb{Z}[x] \;\middle|\; \delta_2^a\left(\frac{\partial^b f}{\partial x^b}\right) \in (2, x), \text{ for } a + b \leqslant n-1 \right\}.$$

In particular, we can now see that $2 \notin \mathfrak{m}^{(2)}$, since

$$\delta_2(2) = \frac{2 - 2^2}{2} = -1 \notin \mathfrak{m},$$

while as we saw in Example 3 there are no $\mathbb{Z}$-linear differential operators ∂ of order up to 1 (or any order!) satisfying $\partial(2) \notin \mathfrak{m}$.

4 Some open Problems

There are many interesting open problems related to symbolic powers. We collect a quick survey of some of those problems, but must necessarily leave a lot of the story to be told elsewhere. For a survey of symbolic powers and other related problems, see [DDSG+18].

4.1 Equality

While the symbolic powers $I^{(n)}$ of I can be computationally difficult to compute, its ordinary powers I^n are very easy to describe. It is thus desirable to understand when $I^{(n)} = I^n$ for some or all n. We do have $I^{(n)} = I^n$ for all n whenever I defines a complete intersection — meaning I is generated by a regular sequence — though this condition is far from being necessary [Hoc73, LS]. A necessary and sufficient condition can be found in [Hoc73], though this condition is not suitable to test in practice outside of special cases. When we restrict to squarefree monomial ideals a polynomial ring $k[x_1, \ldots, x_d]$ over a field k, it is conjectured that the condition $I^{(n)} = I^n$ for all n is equivalent to a combinatorial condition. A monomial ideal is an ideal generated by monomials, and it is **squarefree** if it is generated by products of distinct variables; (xy, xz, yz) is a squarefree monomial ideal, (x^2y, z) is a monomial ideal but not squarefree, and $(x + y, z)$ is not a monomial ideal.

Definition 9 (König ideal).— Let I be a squarefree monomial ideal of height c in a polynomial ring over a field. We say that I is **könig** if I contains c monomials with no common variables. A squarefree monomial ideal of height c is said to be **packed** if every ideal obtained from I by setting any number of variables equal to 0 or 1 is könig.

The following is a restatement by Gitler, Valencia, and Villarreal [GVV05] in the setting of symbolic powers of a conjecture of Conforti and Cornuéjols about max-cut min-flow properties.

Conjecture 1 (Packing Problem).— Let I be a squarefree monomial ideal in a polynomial ring over a field k. We have $I^{(n)} = I^n$ for all $n \geqslant 1$ if and only if I is packed.

The Packing Problem has been solved for edge ideals of simple graphs, in which case $I^n = I^{(n)}$ for all n if and only if I is the edge ideal of a bipartite graph [GVV05], but it remains open in the general setting.

One may also wonder if it is sufficient to check the equality $I^n = I^{(n)}$ for finitely many n; this has recently been shown to hold in the case when I is generated by monomials.

Theorem 10 (Montaño-Núnez Betancourt, 2019).— Let I be a monomial ideal in $k[x_1, \ldots, x_d]$, where k is a field, and suppose that I is generated by μ monomials. If $I^n = I^{(n)}$ for all $n \leqslant \frac{\mu}{2}$, then $I^n = I^{(n)}$ for all n. (See [MnNnB19].)

It is an open question whether such a theorem holds for a general ideal, and if it does, what values of n we need to test to guarantee $I^n = I^{(n)}$ for all n.

4.2 Degree bounds

Given a nonzero homogeneous ideal I in $k[x_1, \dots, x_d]$, we write $\alpha(I)$ for the minimum degree of a nonzero homogeneous $f \in I$. The questions that we opened the paper with asked about $\alpha(I)$ and $\alpha(I^{(n)})$; giving lower bounds for these quantities can be quite challenging.

Conjecture 2 (Chudnovsky, 1981 [Chu81]).— If I defines a finite set of points in $\mathbb{P}^N$, then for all $m \geqslant 1$ we have

$$\frac{\alpha(I^{(m)})}{m} \geqslant \frac{\alpha(I) + N - 1}{N}.$$

Chudnovsky's conjecture holds for any set of points in $\mathbb{P}^2$ [Chu81, HH13], a general set of points in $\mathbb{P}^3$ [Dum15], a set of at most $N+1$ points in generic position in $\mathbb{P}^N$ [Dum15], a set of a binomial coefficient number of points forming a star configuration [BH10, GHM13], a set of points in $\mathbb{P}^N$ lying on a quadric [FMX18], a very general set of points in $\mathbb{P}^N$ [DTG17, FMX18], and sets of $s \geqslant 4^N$ general points in $\mathbb{P}^N$ [BGHN]. The case of an arbitrary set of points remains open.

4.3 The Containment Problem

When is $I^{(a)} \subseteq I^b$? Necessary and sufficient conditions for this question to make sense — so that given I and b, we can always find such an a — were studied by Schenzel in the 1980s [Sch85]. For each I and each b, we want to find the smallest possible a with $I^{(a)} \subseteq I^b$. If $I^{(b)} \subseteq I^b$, then $I^{(b)} = I^b$, so this question contains the equality problem as a subproblem. When equality does not hold, we may think of the Containment Problem as a way of comparing the ordinary and symbolic powers of I. Notice also that if $I^{(a)} \subseteq I^b$, then $\alpha(I^{(a)}) \geqslant b\alpha(I)$, so answering the Containment Problem for I will in particular provide lower bounds for the degrees of elements in the symbolic powers of I.

Over $R = k[x_1, \dots, x_d]$, or more generally any regular ring, the answer depends on the **big height** of I, the largest codimension of an irreducible component of the algebraic set corresponding to I, which in algebraic terms is the same as the largest height of a minimal prime of I.

This answer is a beautiful theorem of Ein—Lazersfeld—Smith, Hochster—Huneke, and Ma—Schwede.

Theorem 11.— Let R be a regular ring and I a radical ideal in R. If h is the big height of I, then

$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geqslant 1.$$

(See [ELS01, HH02, MS17, M].)

In particular, when k is a field and $R = k[x_1, \dots, x_d]$, the theorem says that $I^{(dn)} \subseteq I^n$ for every I. This type of uniform behavior — in this case, independent of the ideal I we choose — appears in many shapes and forms throughout commutative algebra. For example, for a prime ideal P of height 2, the theorem says that $P^{(4)} \subseteq P^2$; in 2000, Huneke asked if this could be improved to $P^{(3)} \subseteq P^2$ under some technical hypothesis, which inspired the following conjecture.

Conjecture 3 (Harbourne, 2006).— Let I be a radical homogeneous ideal in $k[x_1, \dots, x_d]$, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Hochster and Huneke's proof of Theorem 11 uses prime characteristic techniques and reduction to characteristic p to do the case when the ring contains a field, and their proof in the prime characteristic p case for $n = p^e$ turns out to be a beautiful application of the Pigeonhole Principle. A more careful application of the Pigeonhole Principle gives Harbourne's Conjecture for powers of p: $I^{(hq-h+1)} \subseteq I^q$ for all $q = p^e$. In an amazing twist, however, 3 is not true as stated: there is a set of 12 points in $\mathbb{P}^{12}$ [DSTG13] with $h = 2$ that fails $I^{(3)} \subseteq I^2$, among other families of counterexamples [HS15, Sec15, DS21].

Despite these counterexamples, Conjecture 3 does hold for some large classes of ideals, such as monomial ideals [BDRH+09, Example 8.4.5], generic sets of points in $\mathbb{P}^2$ [BH10] or $\mathbb{P}^3$ [Dum15], for matroid configurations [GHMN17], and for star configurations [HH13]. The conjecture also holds if R/I has nice singularities: if R/I is F-pure in prime characteristic or of dense F-pure type in equicharacteristic 0 [GH]. This class of rings contains Veronese rings, generic determinantal rings, and more generally rings of invariants of linearly reductive groups.

Moreover, every counterexample to 3 known to date actually satisfies the following open conjecture:

Conjecture 4 (Stable Harbourne [Gri20]).— If I is a radical ideal of big height h in a regular ring, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

We are asking if Harbourne's Conjecure holds for n large — where large enough should depend on I. The philosophy is that when one asks for the smallest a_n such that $I^{(a_n)} \subseteq I^n$, things get better as n grows. Not only do we have no counterexamples to this conjecture, the evidence supporting it keeps growing [Gri20, BGHN, GHM20a, GHM20b]. In fact, every counterexample known to date to the original conjecture, Conjecture 3, satisfies the stable conjecture.

If studying $I^{(n)}$ is hard, the computational problems only get harder as n grows. As such, testing conjectures such as this one can be quite challenging. Many of the results in this direction rely on proving that certain particular containments are sufficient to obtain an eventual containment statement for large n, a technique which has also found applications [BGHN, BGHN22] in the degree problem we mentioned in Section 4.2.

The problems we discussed here have been open for decades, but have paved the way for many new research avenues in recent years. For more on recent advances in the topic of symbolic powers, and their connections to other topics, see [DDSG+18].

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References

[BDRH+09] Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Michał Kapustka, Andreas Knutsen, Wioletta Syzdek, and Tomasz Szemberg. A primer on Seshadri constants. In *Interactions of classical and numerical algebraic geometry*, volume 496 of *Contemp. Math.*, pages 33–70. Amer. Math. Soc., Providence, RI, 2009.

[BGHN] Sankhaneel Bisui, Eloísa Grifo, Huy Tài Hà, and Thái Thành Nguyễn. Chudnovsky's conjecture and the stable Harbourne-Huneke containment. To appear in *Transactions of the American Mathematical Society*.

[BGHN22] Sankhaneel Bisui, Eloísa Grifo, Huy Tài Hà, and Thái Thành Nguyễn. Demailly's conjecture and the containment problem. *Journal of Pure and Applied Algebra*, 226 (2022), no. 4, Paper No. 106863, 21 pp.

[BH10] Cristiano Bocci and Brian Harbourne. Comparing powers and symbolic powers of ideals. *J. Algebraic Geom.*, 19(3):399–417, 2010.

[Bui95] Alexandru Buium. Differential characters of abelian varieties over p-adic fields. *Inventiones mathematicae*, 122(2):309–340, 1995.

[Bui05] Alexandru Buium. *Arithmetic differential equations*, volume 118 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[Chu81] Gregory V. Chudnovsky. Singular points on complex hypersurfaces and multidimensional schwarz lemma. In *Séminaire de Théorie des Nombres, Paris 1979-80, Séminaire Delange-Pisot-Poitou*, volume 12 of *Progress in Math.*, pages 29–69. Birkhäuser, Boston, Sasel, Stutgart, 1981.

[DDSG+18] Hailong Dao, Alessandro De Stefani, Eloísa Grifo, Craig Huneke, and Luis Núñez Betancourt. Symbolic powers of ideals. In *Singularities and foliations. geometry, topology and applications*, volume 222 of *Springer Proc. Math. Stat.*, pages 387–432. Springer, Cham, 2018.

[DS21] Benjamin Drabkin and Alexandra Seceleanu. Singular loci of reflection arrangements and the containment problem. *Mathematische Zeitschrift*, pages 1–29, 2021.

[DSGJ20] Alessandro De Stefani, Eloísa Grifo, and Jack Jeffries. A Zariski-Nagata theorem for smooth $\mathbb{Z}$-algebras. *J. Reine Angew. Math.*, 761:123–140, 2020.

[DSTG13] Marcin Dumnicki, Tomasz Szemberg, and Halszka Tutaj-Gasińska. Counterexamples to the $I^{(3)} \subseteq I^2$ containment. *J. Alg*, 393:24–29, 2013.

[DTG17] Marcin Dumnicki and Halszka Tutaj-Gasińska. A containment result in P^n and the Chudnovsky conjecture. *Proc. Amer. Math. Soc.*, 145(9):3689–3694, 2017.

[Dum15] Marcin Dumnicki. Containments of symbolic powers of ideals of generic points in $\mathbb{P}^3$. *Proc. Amer. Math. Soc.*, 143(2):513–530, 2015.

[EH79] David Eisenbud and Melvin Hochster. A Nullstellensatz with nilpotents and Zariski's main lemma on holomorphic functions. *J. Algebra*, 58(1):157–161, 1979.

[ELS01] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Invent. Math.*, 144(2):241–252, 2001.

[FMX18] Louiza Fouli, Paolo Mantero, and Yu Xie. Chudnovsky's conjecture for very general points in $\mathbb{P}_k^N$. *J. Algebra*, 498:211–227, 2018.

[GH] Eloísa Grifo and Craig Huneke. Symbolic powers of ideals defining F-pure and strongly F-regular rings. *International Mathematics Research Notices*, Volume 2019, Issue 10, May 2019, pages 2999–3014.

[GHM13] Anthony V. Geramita, Brian Harbourne, and Juan Migliore. Star configurations in $\mathbb{P}^n$. *J. Algebra*, 376:279–299, 2013.

[GHM20a] Eloísa Grifo, Craig Huneke, and Vivek Mukundan. Expected resurgences and symbolic powers of ideals. *J. Lond. Math. Soc. (2)*, 102(2):453–469, 2020.

[GHM20b] Eloísa Grifo, Craig Huneke, and Vivek Mukundan. Expected resurgence of ideals defining Gorenstein rings. To appear in the *Michigan Math Journal*.

[GHMN17] A. V. Geramita, B. Harbourne, J. Migliore, and U. Nagel. Matroid configurations and symbolic powers of their ideals. *Trans. Amer. Math. Soc.*, 369(10):7049–7066, 2017.

[Gri20] Eloísa Grifo. A stable version of Harbourne's Conjecture and the containment problem for space monomial curves. *J. Pure Appl. Algebra*, 224(12):106435, 2020.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[GS20] Eloísa Grifo and Alexandra Seceleanu. Symbolic Rees algebras. In Irena Peeva, editor, *Commutative Algebra: Expository papers dedicated to David Eisenbud on the occasion of his 75th birthday*. Springer, to appear.

[GVV05] Isidoro Gitler, Carlos Valencia, and Rafael H. Villarreal. A note on the Rees algebra of a bipartite graph. *J. Pure Appl. Algebra*, 201(1-3):17–24, 2005.

[HH02] Melvin Hochster and Craig Huneke. Comparison of symbolic and ordinary powers of ideals. *Invent. Math.*, 147(2):349–369, 2002.

[HH13] Brian Harbourne and Craig Huneke. Are symbolic powers highly evolved? *J. Ramanujan Math. Soc.*, 28A:247–266, 2013.

[Hoc73] Melvin Hochster. Criteria for equality of ordinary and symbolic powers of primes. *Mathematische Zeitschrift*, 133:53–66, 1973.

[HS15] Brian Harbourne and Alexandra Seceleanu. Containment counterexamples for ideals of various configurations of points in $\mathbf{P}^N$. *J. Pure Appl. Algebra*, 219(4):1062–1072, 2015.

[Joy85] A. Joyal. δ-anneaux et vecteurs de Witt. *C.R. Acad. Sci. Canada*, VII(3):177–182, 1985.

[Las05] Emanuel Lasker. Zur theorie der moduln und ideale. *Mathematische Annalen*, 60:20–116, 1905.

[LS] Aihua Li and Irena Swanson. Symbolic powers of radical ideals. *Rocky Mountain J. Math.* 36 (2006), no. 3, 997–1009.

[MnNnB19] Jonathan Montaño and Luis Núñez Betancourt. Splittings and Symbolic Powers of Square-free Monomial Ideals. *International Mathematics Research Notices*, 07 2019. rnz138.

[MS17] Linquan Ma and Karl Schwede. Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers. *arXiv:1705.02300*, 2017.

[M] Takumi Murayama. Uniform bounds on symbolic powers in regular rings. *arXiv:2111.06049*, 2021.

[Nag62] Masayoshi Nagata. *Local rings*. Interscience, 1962.

[Noe21] Emmy Noether. Idealtheorie in ringbereichen. *Mathematische Annalen*, 83(1):24–66, 1921.

[Sch85] Peter Schenzel. Symbolic powers of prime ideals and their topology. *Proc. Amer. Math. Soc.*, 93(1):15–20, 1985.

[Sec15] Alexandra Seceleanu. A homological criterion for the containment between symbolic and ordinary powers of some ideals of points in $\mathbb{P}^2$. *Journal of Pure and Applied Algebra*, 219(11):4857 – 4871, 2015.

[Zar49] Oscar Zariski. A fundamental lemma from the theory of holomorphic functions on an algebraic variety. *Ann. Mat. Pura Appl. (4)*, 29:187–198, 1949.

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Adélia Sequeira

by **João Janela*** and **Jorge Tiago****

Adélia Sequeira (https://www.math.tecnico.ulisboa.pt/~asequeir/) was born in Lisbon on 14 March, 1951. She is a Full Professor of Mathematics at the IST (Instituto Superior Técnico), Universidade de Lisboa, was Director of the Research Centre for Computational and Stochastic Mathematics — CEMAT/IST-ULisboa (2017–2021) and has been the Scientific Coordinator of CEMAT's Mathematical Modeling in Biomedicine Research Group (since 2010).

She has a PhD (thèse de 3ème cycle) in Numerical Analysis from École Polytechnique, Paris, in France (1981), and a second PhD in Mathematics from the Faculty of Sciences of Universidade de Lisboa (1985). She obtained a further degree (Habilitation) in Applied Mathematics and Numerical Analysis from IST in January 2001. She was awarded First Prize for the UTL/Santander Totta Scientific Awards for her research in the area of Pure and Applied Mathematics in 2010. In 2011 she received the Medal of Merit of the Faculty of Mechanical Engineering of the Czech Technical University in Prague. In November, 2018, she was elected as a corresponding member of the Lisbon Academy of Sciences, Class of Science, and in 2019 she was selected to be one of the *Women in Science* by *Ciência Viva* of the Portuguese Agency for Science and Technology (http://www.cienciaviva.pt/mulheresnaciencia/segunda-edicao/).

Currently, her research interests are in the area of cardiovascular mathematical modeling and simulations of closely connected problems of clinical relevance associated with vascular diseases, namely: patient-specific cerebral aneurysms progression and biomechanical and biochemical actions in blood vessels, with application to thrombosis and atherosclerosis processes. She also has a research interest in mathematical and computational fluid dynamics, with a particularly focus on viscoelastic non-Newtonian fluids, hemorheology and hemodynamics studies.

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Your career as a mathematician started in the late seventies, at the École Polytechnique, Paris, in France. What is your recollection of that period?

That was a wonderful time for me, one of the best in my life. I went to Paris on the 3rd of October 1977, by train, full of dreams and with many plans. This was a real adventure for me, in all perspectives. It provided me with the possibility to studying mathematics, specializing in the field of Numerical Analysis, where I had the opportunity to meet new people and experience living alone in a foreign country, ... I was 26 years old, and I had an open mind to everything that was new and challenging. I met my husband later on, in 1981, who had also gone to Paris to study for his PhD. We got married there in 1982, and our first daughter was born in Paris in 1983. I returned to Portugal in 1984, after having concluded my thesis at École Polytechnique in 1981 (thèse de 3ème cycle). Later, I concluded my PhD in Portugal in 1985. Overall, I have spent more than six years in Paris and my connections to Paris are so strong, that I consider Paris to be my second town.

Those were also changing times here in Portugal. How was it to be a Portuguese young woman in Paris, during that time?

I left Portugal in 1977, three years after the "25th of April" revolution, which was a period when it was naturally impossible not to experience political connections and hard to forget the incredible changes that happened to Portugal as a result of the Revolution. During my first years in Paris, I also had the opportunity to meet other Portuguese colleagues and we organized various meetings and established strong links with the association of Portuguese emigrants in Paris. In addition, many activities were held jointly with a similar association in Brussels to keep alive the spirit of "the 25th of April".

Was mathematics a passion since your high school times? Was it already an indisputable choice for the future?

Yes, as far as I remember, Mathematics has always been a real passion for me. I always had good grades, and I considered Mathematics to be one of the most fascinating subjects. However, when I finished high school, I wanted to study Medicine, as I was attracted to the idea of a career as a doctor. When it came to the time to choose what to study at university, my father thought medicine was not appropriate for a woman. Later on, he came to recognize that he was wrong, but by then it was too late. I then decided to follow my passion, and I went to the Faculty of Sciences of Universidade de Lisboa (FCUL), to study Mathematics. I graduated in 1973 and my first full job was as an Assistant at the Department of Mathematics of FCUL. Then the revolution took place in April 1974, all lectures were stopped, and I became involved in many political events. Interestingly, I was elected Chair of the Management Board of FCUL in 1975, along with 17 others (6 faculty, 6 students, and 6 members of staff). For two years the Faculty was 'governed' by our group. Although some lectures were resumed, during that period most of the time was spent in meetings and discussions. I was tired of spending the days doing that, and decided to change my life ... first I tried to enter the Faculty of Medicine, but this was not allowed for those who had already had a degree in another subject. Next, I was offered a scholarship from JNICT (now FCT), and decided to go to Paris to study for my PhD in Mathematics.

You are now internationally known for your contributions in computational modelling and simulation, with emphasis in cardiovascular modelling, a topic in which you have been awarded with several international projects. But it wasn't always like that. You had previously worked with more fundamental mathematics, before drawing your attention to, in your own words, "bloody mathematics". What motivated such a shift?

The start of my working with Numerical Analysis was in Paris, with P. A. Raviart, at the Laboratoire d'Analyse Numérique, Univ. Paris VI. This was an initiative of J.P. Carvalho Dias, who sent a group of young graduates to study for their PhD's in Paris and contacted renowned French mathematicians who were working in different fields. After studying for the DEA (Diplôme d'Éudes Approfondies) with P.A. Raviart, I then moved to CMAP-Centre de Mathématiques Appliquées, École Polytechnique, Palaiseau, where I worked with J.C. Nédélec, who became my thesis supervisor. First, I developed a numerical method to approximate the Stokes system in an exterior domain and obtained numerical results. After coming back to Portugal, I worked for a while with more fundamental mathematics, as you have observed, but always in the field of mathematical fluid mechanics, including Navier-Stokes equations and different types of non-Newtonian fluid models (inelastic and viscoelastic). As soon as I discovered that blood is a non-Newtonian fluid, I realized that I could apply my mathematical background to blood flow problems. For me, being able to apply Mathematics to Medicine was like turning a dream into reality!
This phase in my career took place during the nineties, however it was only after I obtained my Habilitation in Applied Mathematics and Numerical Analysis in 2001, that I started to contact clinical physicians. This was due to Prof. F. Ramôa Ribeiro, who was a member of the examination panel, and became very interested in my work. From then

onwards, right up to today, I have worked non-stop with "bloody mathematics", as I call it! I established international and national joint research agreements with many specialists from around the world, and realized that academia is confronted by so many fascinating problems in this field, that I will never change my research topic.

You were one the principal investigators in a very successful RTN European project -Haemodel - in the beginning of the millennium (2002 – 2006). Do you think that the fruitful scientific collaboration with the other partners (EPFL, INRIA, MOX, Imperial College, Univ Graz) established a point of no return towards Applied Mathematics and, more specifically, the mathematical modelling and simulation of the human cardiovascular system?

In fact, one can say that this truly was "a point of no return", as I'm still working on cardiovascular mathematics to this very day. I started some very important joint research in this field in collaboration with the RTN European Haemodel project, and under the auspices of this project I was able to contract several postdoctoral researchers to work with my group in Portugal. The leader was Alfio Quarteroni from École Polytecnique de Lausanne (EPFL) and Politecnico di Milano (MOX) and the project was entitled "Mathematical and Numerical Modelling in Haemodynamics". This project was followed by further very successful, national and international projects, all of which were funded by FCT, in particular the "Cardiovascular Imaging, Modeling and Simulation – SIMCARD" (2009-2012) project, carried out within the framework of the UT Austin|Portugal international partnership, with the group of Tom Hughes and C. Bajaj, and also the EXCL "Mathematical and Computational Modeling of Human Physiology – PHYSIOMATH" (2013-2016) FCT project. By means of all these projects, my group was able to establish strong national and international links with mathematicians, engineers, computer scientists and in particular clinical physicians, most of whom had outreached and proposed very challenging medical problems which required the interdisciplinary collaboration of my group to solve them.

You organized a few conferences and summer schools in collaboration with CIM, in the late nineties. Can you tell us something about these events and their importance in your scientific growth?

These events, consisting of three summer schools, took place during June-July 1999, under the umbrella title of "Thematic Term on Theoretical and Computational Fluid Dynamics": Summer School on Industrial Mathematics, IST, Lisbon, June 7-12; Summer School on Navier-Stokes Equations and Related Topics, IST, Lisbon, June 28 – July 3;

Adelia Sequeira with students and collaborators in Azores, 2011

Summer School on Computational Fluid Dynamics, campus of the Astronomical Observatory, University of Coimbra, July 12-17. Top level mathematicians from different countries were invited to lead study modules during these summer schools and they exercised a very strong scientific influence on the participants, especially young researchers and PhD students. In addition, some short research assignments (e.g., 'Research in Pairs' which lasted two or three months) also took place during the Thematic Term, which enabled fruitful international collaboration resulting in the publication of several common scientific papers. These activities were supported by CIM, with special funding from Fundação Calouste Gulbenkian.

You were elected a corresponding member of the Lisbon Academy of Sciences in March 2018. Do you see this as a recognition of the role of Applied Mathematics and, particularly, of computational modelling?

Yes, I was elected to be a corresponding member of the Lisbon Academy of Sciences (Mathematics Class of Sciences) in March 2018, in virtue of my experience in Applied Mathematics, being the first member ever elected from the area of computational modelling. For me this distinction was very meaningful, for apart from the personal recognition, this election also represented an appreciation for my research field, in particular computational modelling of cardiovascular flows.

The fact that you worked in different areas of Mathematics, allowed you to collaborate with an impressive number of mathematicians from different continents. Besides the obvious inspiration for new ideas in Mathematics, this should be also a very fruitful personal experience?

To be sure, I developed very deep personal relations with mathematicians from a great variety of countries who had experience of working with both fundamental and applied mathematical problems. I have maintained these friendships to this very day, and even if we don't see each other very often, or don't make contact very frequently, I can truly say that "good friends live forever". At this stage I would like to mention two outstanding mathematicians who have now sadly passed away, both of whom were very close friends: Jindřich Nečas and Olga Ladyzhenskaya.

Nowadays my work is entirely devoted to cardiovascular mathematics, however I still consider my colleagues whom I met 35 years ago, when I was working on mathematical fluid dynamics, to be good friends, and I know that I can always count on them, if needed. Furthermore, working on this subject for decades, quite often with colleagues from all over the globe (including Central and Eastern Europe, Northern

Africa, the United States, Japan, China, India, Brazil, and many other locations), has given me a broader vision of the world and of life itself.

As a mother of two, was it easy to comply with your family life?
We can say that all the travelling and sleepless nights made you stronger?

Balancing both responsibilities is a great challenge for any woman: to combine professional and family life, while trying to do one's the best in each case. I genuinely feel that I did my best, and I rest assured that my family recognized that. This makes me feel very happy, as we often think that we can always offer more to our children than we have done in practice. I attended a great many conferences and travelled a lot, however I always made a conscious effort to be 100% present when I was with them, even if some mathematical problem was puzzling me in the back of my head, or if a certain difficulty involving one of my colleagues at the university was keeping me awake at night. In fact, I'm used to sleeping just four to five hours a night, which is very little, I must say. Perhaps I will suffer from the consequences of sleep deprivation, but I must confess, I work well at night, and I'm already really used to it.

One of your daughters also pursued a career in research and is currently living abroad. Do you think your influence was determinant in her choice? Do you have mixed feelings about being now away from your daughter and grandchild?

Yes, I think it was. My husband and I both had a strong influence on our daughter's career. She is a biologist, not a mathematician like me, or an economist like my husband. In any case, I know that she appreciates my work, because she already said that. In particular, I will always remember when she made an emotional speech at a conference organized in Azores to celebrate my 60th birthday, when she expressed her appreciation in front of all my colleagues during the gala dinner. She is now working as a researcher in Leiden, in the Netherlands, at the Naturalis Biodiversity Center and has a 2 years old child. I'm missing them an awful lot, as you can imagine! My younger daughter has also been very successful in her career, and she is also a young mother, having given birth more recently, in July 2020. I am thus blessed to treasure two grandchildren, a boy and a girl, whom I greatly care about and love to help out!

You have recently received the award "Women in Science" from "Ciência Viva", the Portuguese agency for science. How is it to be a female scientist in Portugal today?

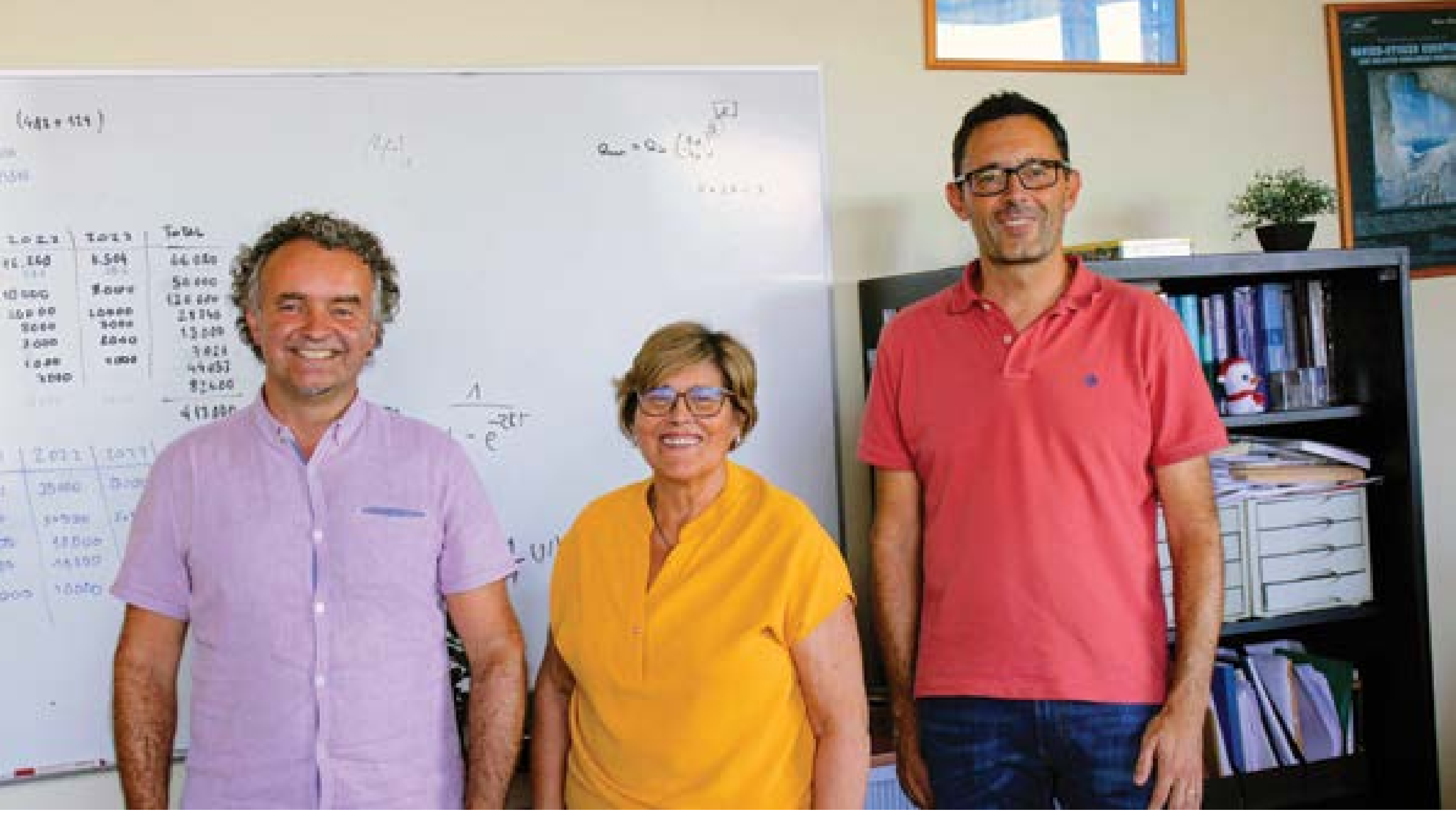
I received this distinction in 2019, and I am extremely honored to have been chosen in my role as an Applied Mathematician to join a group of women scientists who have been selected from among so many distinguished scientists from such diverse fields, who represent approximately 45% of all researchers in our country. It is not easy to be a female scientist in Portugal, or in any other country for that matter. As I mentioned previously, the fact that one is a woman means that one has to learn how to combine a private and a professional life, which is sometimes quite difficult, especially when small children are involved!

You have supervised an impressive number of PhD and MSc students, as well as postdoc researchers. Looking back, how does it feel to have been such a marking influence in so many researchers, not to mention being a pioneer in a new research field in Portugal?

This was part of my job as a professor. I have always felt that I could somehow help young students and researchers progress in their careers. For several years my research group was quite large, and it was most satisfying for us all to be able to participate in our regular meetings and to realize how scientific debate is an integral source of new ideas and progress for all those involved. Our group was pioneering, and in that sense it was unique in the field of cardiovascular mathematics. Several clinical physicians contacted me to put various questions and proposed groundbreaking important problems to solve, which in turn were immediately shared with the group or delegated to a particular student as a new research subject. I have always found supervisory responsibilities to be very rewarding, from various points of view.

Many students are now preferring to work in industry rather than pursuing an academic career. Well, this is a good sign of the country development; bright students may also have an important role in academia. What would you say to a young student to motivate her, or him, for a research career?

Now, that's a difficult issue. It's quite hard nowadays to convince second cycle students to pursue their studies towards a PhD, as they fail to comprehend the advantage gained from doing so. However, good students are naturally motivated, because they want to learn more and to progress in research in a certain field, without thinking much of anything else for the immediate future. In the cases of students who are hesitant about whether to progress in their studies or getting a job after their Master's, I would recommend to them that it is very important to carry out a PhD thesis and look for opportunities in the future. There is nothing better in life than to work on what you love to do. However, in spite of the expansion of research and academic opportunities, in particular with regards the possibility to combine studying for a PhD with a career in industry, I have to admit that nowadays it is very difficult for a young researcher to pursue a career such as mine, or people of my generation, mainly in our country, essentially because it is not easy to find a stable position. On the other hand, being a researcher has its good and bad moments, with the added necessity of being committed if one is to be able to overcome all these challenges. Either way, working in



Adelia Sequeira with João Janela and Jorge Tiago

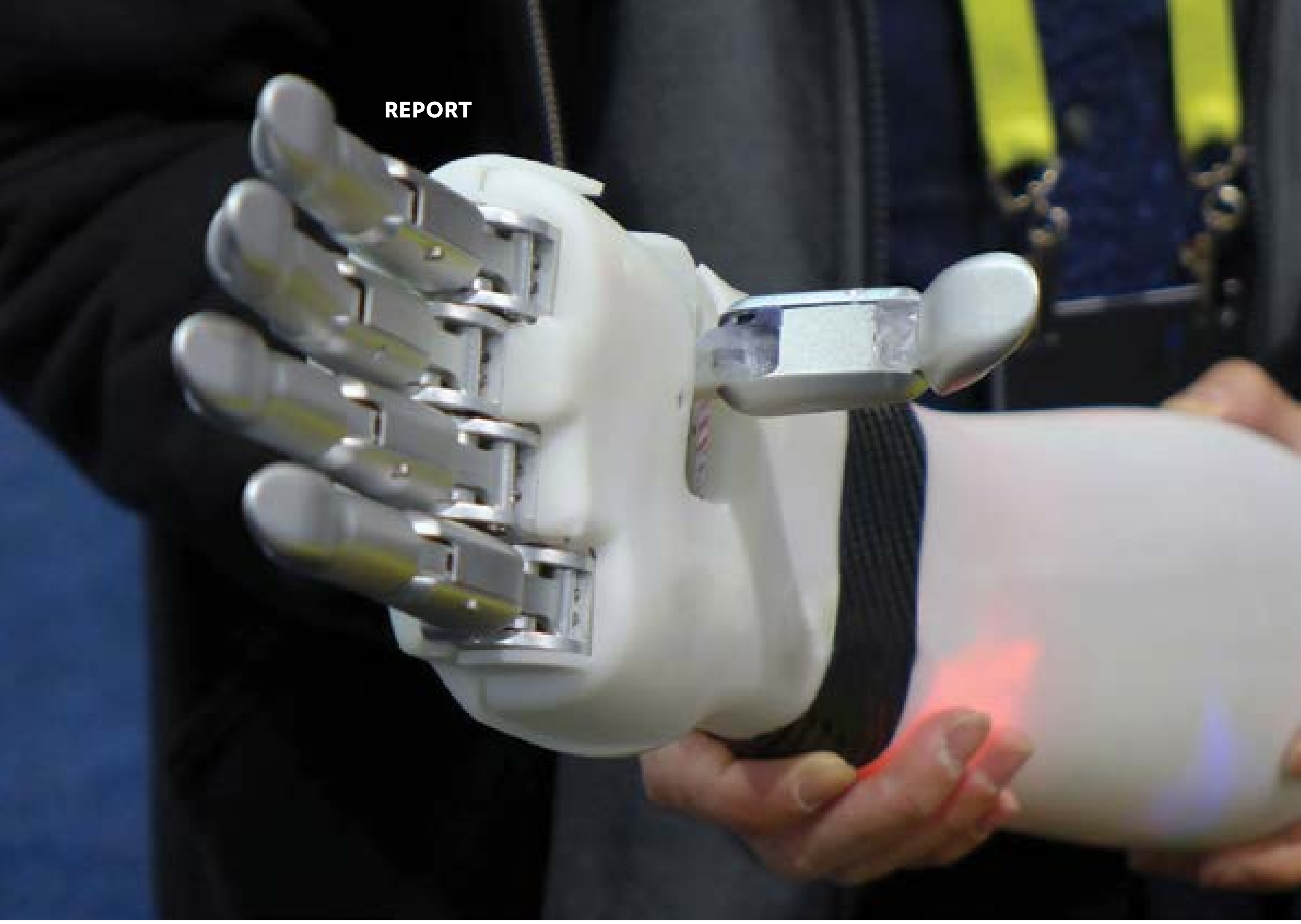
the field of our favorite subject provides us with a unique motivation.

You have recently co-authored with Antonio Fasano a book putting together your research in mathematical modeling of some of the most relevant physiopathologies of the vascular system. Is this the end of a chapter or just the start?

This book was a great challenge for me, or rather for both of us, since it took five years to write, from the moment that we conceived the idea. I remember that it was at an AIMS Conference in Orlando that we met and started thinking about writing the book. Even though it was extremely hard to write, but was also a most agreeable experience, not only from the point of view of our passion for the different topics covered in the book, but also because Antonio is an excellent mathematician and a dear friend. Towards the end, we worked under great pressure, and I am pleased to recount that the book was concluded successfully!

Your personal life and career have seen big changes very recently. You are a grandmother of two. At the same time, you have retired from your teaching duties at Instituto Superior Técnico. All of these during these strange times of pandemic constraints, fortunately now being more relaxed. A new Adélia was born, at the age of 70?... Now more seriously, what are your personal and academic perspectives for the following times? I know that you have started advising a new PhD student...

"A new Adélia was born" is a good expression! ... for I'm not getting any younger!... Anyway, I like to feel that I'm still full of energy, and I know that I want to pursue my research work for as long as possible. Currently I continue to write papers and evaluate research proposals. Furthermore, I continue to participate as a member of various national and international committees, and have others in the pipeline for the near future. I have also been invited several times to be a keynote speaker at international conferences, and I will be presenting at more very soon. In conjunction with colleagues, I am involved in the organization of the important ENUMATH 2021, conference at IST, which unfortunately had to be postponed until 2023 due to the pandemic. These are just a few aspects of my new life, with basically life going on as before, except for the teaching role and some fairly onerous administrative responsibilities ... Naturally, many other interesting things exist in life, and I now sincerely hope that I will be able to find more time to dedicate to both myself and my family, in particular to my grandchildren.



Workshop CIM & Enterprises

by **Ana Luísa Custódio***

On the 26th of November of 2021, the Workshop CIM & Enterprises was held virtually. It was promoted and organized by CIM, *the International Center for Mathematics*, in collaboration with PT-MATHS-IN, the *Portuguese Network of Mathematics for Industry and Innovation*.

Mathematics is an abstract science but also an undoubtably powerful tool to address challenging problems that arise in very different disciplines, such as physics, engineering, computer science, data science, pharmacy, medicine, biology, among many others. The main objective of this workshop was to promote communication, collaboration, and transfer of knowledge between research centers in mathematics and companies, through successful case studies of emerging mathematical applications. This event intended to potentiate opportunities for joint research or to promote new industry and

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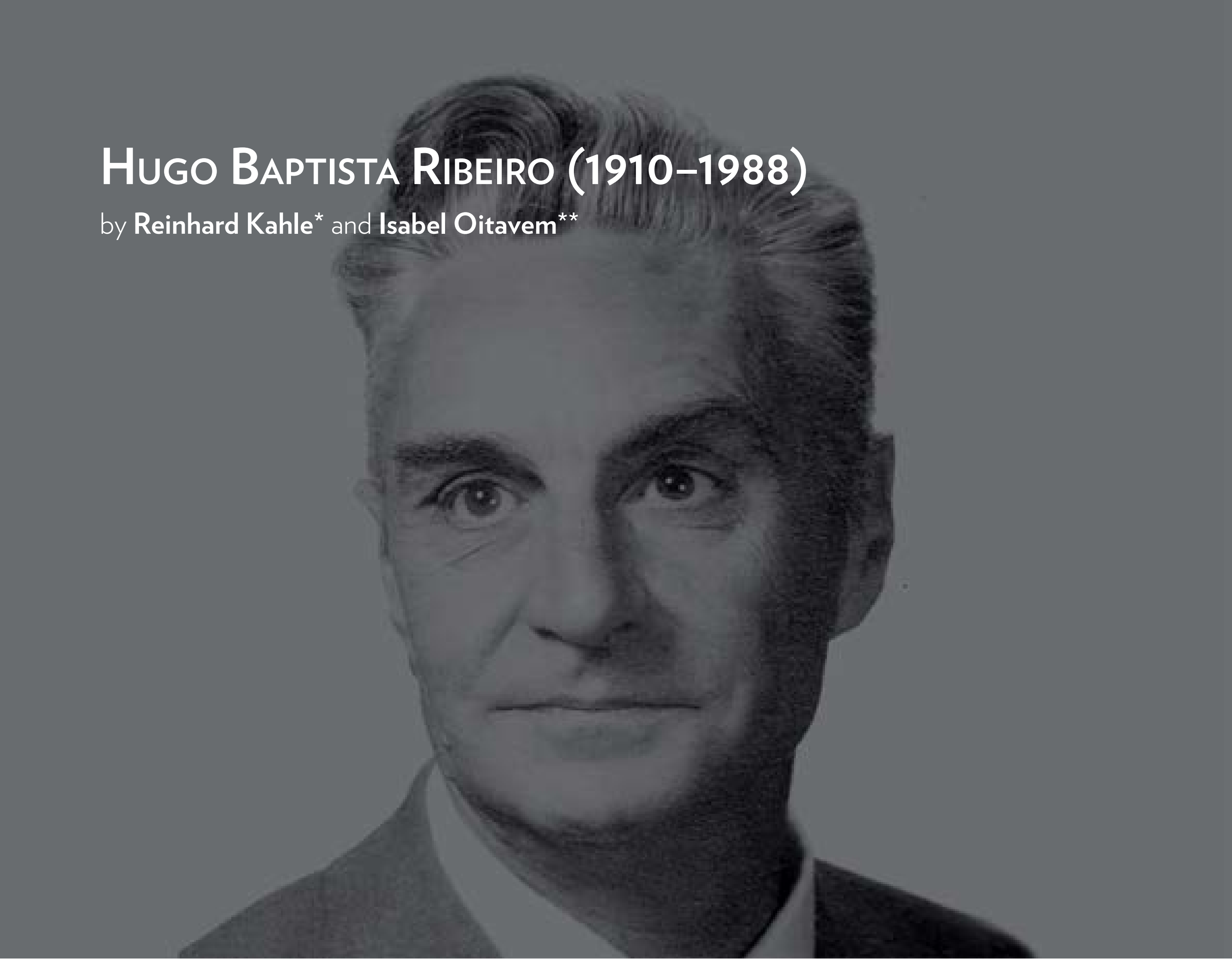
services careers for young researchers in mathematics.

With a target audience of researchers, technical staff, undergraduate and graduate students (PhD students or Postdoctoral fellows), with interest in challenging mathematical applications, the workshop counted with 35 participants, engaged in lively discussions with the speakers representing the seven invited enterprises, namely:

Marta Simões	*What can maths do for drug development?*	Bluepharma
Rui Correia	*Juggling speed, cost and quality on the road towards data excellence*	Defined.ai
Jacopo Bono	*Managing RiskOps with machine learning*	Feedzai
Hugo Penedones	*Scientific machine learning*	Inductiva Research Labs
Luís Pinheiro	*Structured products*	NOVA IMS
Jens Mühlsteff	*Applied mathematics in Philips research*	Philips
Ricardo Saldanha	*The role of applied mathematics in reducing costs and emissions in public transport: the SISCOG case*	SISCOG

Further information on the event can be found at https://www.cim.pt/agenda/event/220



Hugo Baptista Ribeiro (1910–1988)

by **Reinhard Kahle*** and **Isabel Oitavem****

Hugo Ribeiro

Hugo Baptista Ribeiro was a distinguished portuguese mathematician who made part of the *Geração de 40*, a generation of mathematicans which was responsible for renewal of Mathematics in Portugal in the decade 1936–1945. Members of this group were Bento de Jesus Caraça, Ruy Luis Gomes, Antóno Aniceto Monteiro, Manuel Zaluar Nunes and others.[1] The activites of this group included the foundation of the Portuguese Mathematical Society (*Sociedade Portuguesa de Matemática*, SPM) in 1940 and the publication of the journal *Portugaliae Mathematica* which succeeded to gain international recognition. Playing an important role in the promotion of modern Mathematics in Portugal, Hugo Ribeiro, however, spent his entire career as professor in the United States. The political circumstances allowed him to return to Portugal only after the revolution in 1974. He still lectured as retired professor at the University of Porto.

Hugo Ribeiro was born on May 16, 1910 in Lisbon. He graduated at the University of Lisbon in 1939. He obtained his PhD at the Swiss Federal Institute of Technology

[1] For a more detailed history of the *Geração de 40* see [5].

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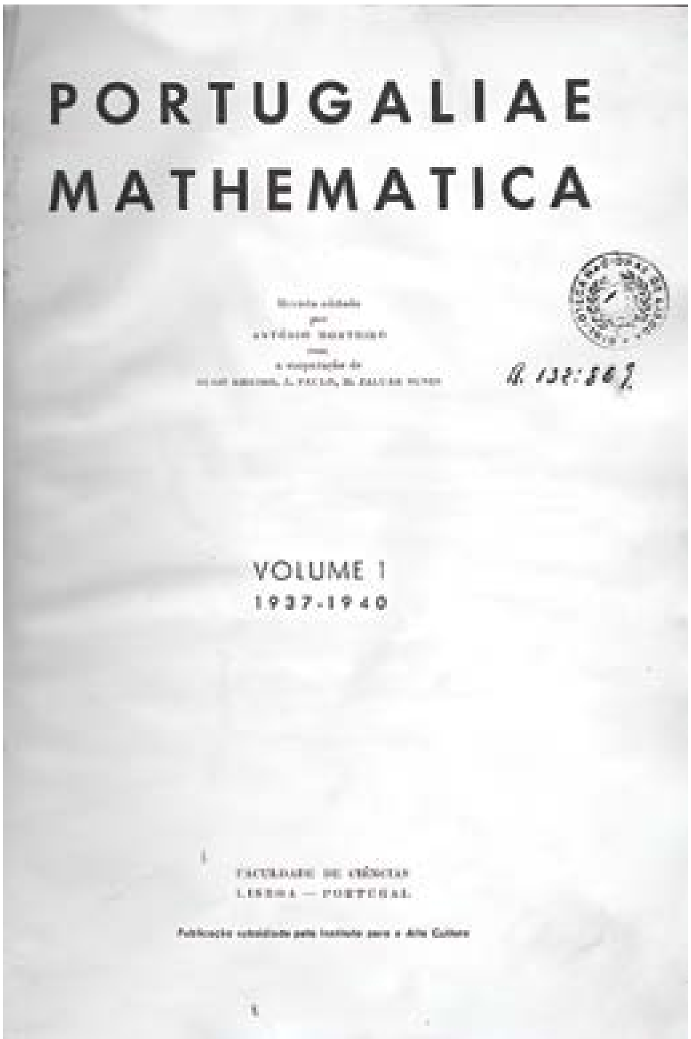
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Figure 1. Front page of the first volume of Portugaliae Mathematica.

(ETH) in Zurich, Switzerland. In 1944 he won the *Artur Malheiros* prize for Mathematics of the Lisbon Academy of Sciences. In 1947, however, the Salazar regime interfered at the universities with the result that most of the colleagues from the *Geração de 40* resigned or were expelled from their universities [5, p. 94], and Ribeiro emigrated to the United States. After three years as lecturer and instructor in Berkeley, he was appointed as associate professor of Mathematics at the University of Nebraska in Lincoln, and promoted to full professor in 1953. In 1961 he moved to Pennsylvania State University and he retired in 1975. Only then he returned to Portugal. On February 26, 1988 he passed away in Bicesse (Cascais).

The mathematical work of Hugo Ribeiro is described in a paper by Jorge Almeida [1]. Starting with research in Topology during his graduate times, Lattice Theory was the topic of his PhD thesis in Zurich, supervised by Paul Bernays. He also used Lattice Theory for a work on the foundation of Probability Theory which earned him the award of the Lisbon Academy of Sciences. In Berkeley, he joined the seminar of Alfred Tarski and started his work in Model Theory, and his subsequent research could probably be subsumed best under the topic of Universal Algebra.

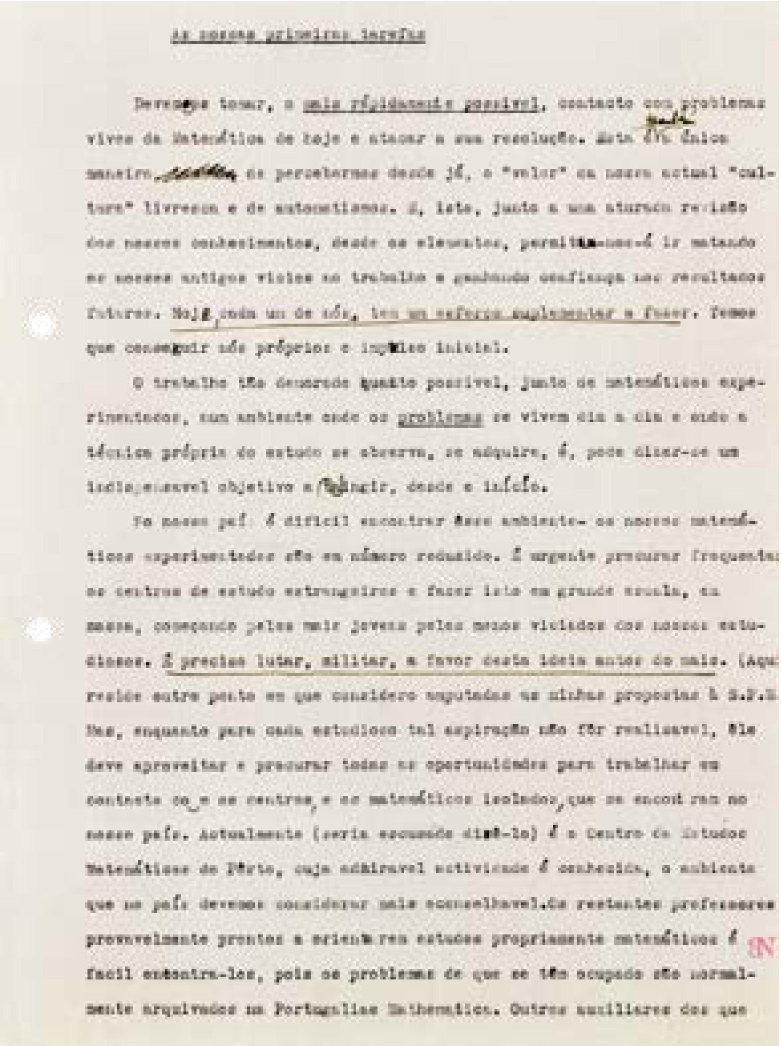
In the following, we will highlight two aspects of Hugo Ribeiro's personality: his engagement in the development of Mathematics in Portugal, and his interaction in a large international network of mathematicians, which included, in particular, his support for mathematicians — from Portugal and elsewhere — in the academic world.

Portuguese Mathematics

In 1937 members of the mentioned *Geração de 40* founded the journal *Portugaliae Mathematica* as a publication specialized in Mathematics [5, §4.1]. It provided a forum to publish work of portuguese mathematicians, but also to attract international scientists. The first editor was António Monteiro, but Hugo Ribeiro was mentioned on the front page of the first volume as collaborator along with José da Silva Paulo and Manuel Zaluar Nunes.

The journal was a great success. It published 87 papers with 1377 pages in total between 1937 and 1946. Hugo Ribeiro was contributing 5 papers as single author and more three with co-authors. But among the 36 authors of this decade, 23 were foreigners, including the names of John von Neumann, Maurice Fréchet and Heinz Hopf. In fact, the international recognition secured the survival of the journal even after several of the editors were excluded from portuguese university careers. After Monteiro emigrated to Brazil, Zaluar Nunes took over the position as managing editor, but Monteiro and Ribeiro continued contributing to the development of the journal from abroad. From the perspective of the ruling regime, the international scientific recognition which Portugal was gaining from the journal was probably more important than excluding further the exiled mathematicians. In consequence, *Portugaliae Mathematica* developed into the flagship of Portuguese Mathematics, withstanding the times of the dictatorship, and still today it is an established international journal in Mathematics.

Hugo Ribeiro was also one of the founding members


As nossas primeiras tarefas

Devemos tomar, o mais rápidamente possível, contacto com problemas vivos da Matemática de hoje e atacar a sua resolução. Esta é a única maneira de perdermos desde já, o "valor" da nossa actual "cultura" livresca e de automatismos. E, isto, junto a uma aturada revisão dos nossos conhecimentos, desde os elementos, permitir-nos-á ir matando os nossos antigos vícios de trabalho e ganhando confiança nos resultados futuros. Hoje, cada um de nós, tem um esforço suplementar a fazer. Temos que conseguir nós próprios o impulso inicial.

O trabalho tão demorado quanto possível, junto de matemáticos experimentados, num ambiente onde os problemas se vivem dia a dia e onde a técnica própria do estudo se absorve, se adquire, é, pode dizer-se um indispensável objetivo a atingir, desde o início.

No nosso país é difícil encontrar êsse ambiente— os nossos matemáticos experimentados são em número reduzido. É urgente procurar frequentar os centros de estudo estrangeiros e fazer isto em grande escala, em massa, começando pelos mais jovens pelos menos viciados dos nossos estudiosos. É preciso lutar, militar, a favor desta ideia antes de mais. (Aqui reside outro ponto em que considero esgotadas as minhas propostas à S.P.M.) Mas, enquanto para cada estudioso tal aspiração não fôr realizável, êle deve aproveitar e procurar todas as oportunidades para trabalhar em contacto com os centros e os matemáticos isolados, que se encontram no nosso país. Actualmente (seria escusado dizê-lo) é o Centro de Estudos Matemáticos do Pôrto, cuja admirável actividade é conhecida, o ambiente que no país devemos considerar mais aconselhável. Os restantes professores provavelmente prontos a orientarem estudos propriamente matemáticos é fácil encontrá-los, pois os problemas de que se têm ocupado são normalmente arquivados na Portugaliae Mathematica. Outros auxiliares dos que

Figure 2. Beginning of a typescript with tasks to modernize Mathematics in Portugal.

of the Portugese Mathematical Society (*Sociedade Portuguesa de Matemática*, SPM). The inaugural meeting took place on December 12, 1940 at the Faculty of Science of the University of Lisbon. While the Society was quite active at the beginning, it was not well received by the regime and it was not officially recognized. When, in 1946 and 1947, the suppression of the portuguese universities increased and most of the mathematicians had to leave their positions, the SPM practically stopped operating and it was only possible to continue the publications *Portugaliae Mathematica* and the *Gazeta de Matemática*.[2]

Ribeiro took active part in initiatives to modernaize Mathematics and Mathematical Education in Portugal. These efforts were cut short when he had to leave Portugal in 1947 after the interference of the government. While Mathematics in Portugal entered in decline, Ribeiro was following the situation from abroad and, consequently, was available to help reconstructing it after the revolution in 1974.

Zurich

For his PhD, Hugo Ribeiro was planning to go abroad. He had established, via John von Neumann, contact with Princeton and received a formal letter of acceptance on October 17, 1941. In parallel he applied for a grant from the Institute for High Culture (*Instituto para a Alta Cultura*, IAC) to support his stay abroad. Apparently, the support was already granted, but when the United States were dragged into World War II by the attack on Pearl Harbor in December 7, 1941, the Portuguese government suspended the application of the grant. In consequence, Ribeiro asked to go to Paris, and if that would not be allowed, to go to Zurich. While in April 1942, Paris was approved, four months later, the IAC changed the destination to Zurich, in Switzerland. He arrived there on September 30, 1942.

The stay in Zurich did not only result in Ribeiro's PhD [4], but it also extended his scientific network. Heinz Hopf's contribution in Portugaliae Mathematica in 1944 was clearly arranged by personal contact with Hugo Ribeiro. For his later life, the acquaintance with Henryk Schärf was to turn out crucial. Schärf, who had specialized in Actuarial Science,[3] had escaped to Zurich from Poland after the German occupation of his home country.

Through his supervisor Paul Bernays, Hugo Ribeiro came in contact with Alonzo Church, the founder of the *Journal of Symbolic Logic* (JSL). Church put a great effort in reviewing the published literature in logic, and Ribeiro wrote in sum 21 reviews for the JSL, in particular about papers in Romance languages. In 1944, when the postal service from the United States to Switzerland was interrupted, Church used Lisbon as a hub, and asked Ribeiro

[2] See the information at https://www.spm.pt/spm/historia/.

[3] See https://www.math.wustl.edu/schaerf.html.

Figure 3. Mario Dolcher, Paul Bernays, Maria Pilar, and Hugo Ribeiro in front of the ETH in Zurich in the 1940s.

to pass papers to Bernays in Zurich. Still in September 1945 postal services to Switzerland were not restored and Church again asked Ribeiro to pass mail to Bernays via Portugal.

United States

After returning to Portugal, it became soon clear that the political circumstances would not allow for a further career of Hugo Ribeiro in Portugal. It was at this moment, April 1947, that Schärf, who had already established himself in the United States, wrote to Ribeiro to consider to send applications to Harvard, Princetion and the University of California in Berkeley. With help of a recommendation of Schärf he obtained a lecturship in Berkeley, and soon joined the logic seminar of Tarski.

His first permanent position was at the University of Nebraska in Lincoln. There is a letter preserved from a 16 years old high school student of Omaha, who thanks Ribeiro as the only one who replied to him when searching for mathematical literature. This student was Saul Kripke, a nowadays famous logician who made his appearence in the scientific world when he published a semantics for modal logic at the age of 18. It is likely that this work had profited from the help Kripke received from Ribeiro.

In 1961 he was appointed full professor at Pennsylvania State University, a university of highest reputation where he had, for instance, Haskell Curry as another logician at his side.

Colleagues and Students

The *Geração de 40* was, to a large extent, forced to leave Portugal, but their members managed to keep contact and to maintain a successful network — especially by the continuation of *Portugaliae Mathematica*. Hugo Ribeiro was not only actively involved in this network,[4] but he was also an example of an academic professor who helped

[4] See [3].

Figure 4. José Morgado and Hugo Ribeiro

colleagues and students when they approached him with all kind of problems. We already mentioned Saul Kripke as a high school student. Ribeiro also helped J. Richard Büchi for his move to the United States in 1949. Büchi was, as Ribeiro, a PhD student of Paul Bernays, and on Bernays' suggestion he published his thesis in *Portugaliae Mathematica*. He became a Professor at Purdue University in Lafayette, Indiana, as expert in Formal Languages, a branch of Mathematical Logic which shaped Theoretical Computer Science.

One can find many other examples in his correspondence. We may only add two students from Portugal who asked him for advice. In the late 1960s, Catarina Kiefe, graduated in Porto, approached him about possibilities to study Mathematical Logic, and subsequently Celestial Mechanics, in the Netherlands or the United States, because it would not be possible to study it in Portugal. A long correspondence started and Kiefe finished a PhD in Mathematics in 1973 at Stony Brook, State University of New York. She went on to obtain a Medical Doctor at University of California, San Francisco and she works nowadays as professor for Biometical Research at the University of Massachusetts. Already back in Portugal Hugo Ribeiro also replied to requests of Fernando Ferreira, today full professor for Mathematical Logic at the University of Lisbon, when Ferreira was about to move to Penn State University to study with Stephen Simpson.

Maria do Pilar Ribeiro

An appreciation of Hugo Ribeiro would be incomplete without an acknowledgement to his wife, Maria do Pilar Ribeiro (1911–2011).[5] Unusual for a woman at that time in Portugal, she studied Mathematics at University of Lisbon and finished her *licenciatura* in 1933, when she also got married with Hugo Ribeiro. She accompanied her husband to Switzerland and to the United States, but kept her mathematical activities. Before the move to Switzerland, she was working as a high school teacher

[5] See the information at https://www.spm.pt/maria_pilar_ribeiro/.

Figure 4. Maria Pilar and Hugo Ribeiro at Penn State (around 1970).

in Lisbon. At Penn State she was also lecturer for Mathematics, and after the return to Portugal she was lecturing at the University of Porto.

In 1951, together with José da Silva Paulo, she translated into portuguese the famous book *Grundlagen der Geometrie* of David Hilbert [2]. She was also engaged in Hugo Ribeiro's translation of van der Waerden's *Moderne Algebra* [6]. She corresponded with van der Waerden in the late 1940s about the unclear copyright situation as the victorious powers had nullified the copyright of German publishing houses.

She was one of the founding members of the Portuguese Mathematical Society and served as *first secretary* (i.e., vice secretary) of the Society in 1941/1942 and 1946/1947. At the time of the legal registration of the Society in 1977, Maria Pilar was registered as *member nº 1*.

References

[1] Jorge Almeida. The Mathematician Hugo Ribeiro. *Portugaliae Mathematica*, 52(1): 1–14, 1995.

[2] David Hilbert. *Fundamentos da Geometria*. Instituto para a Alta Cultura, 1952. Translated by Maria Pilar Ribeiro and José da Silva Paulo.

[3] José Morgado. Hugo Baptista Ribeiro. Matemático portugués que só pôde ensinar numa Universidade portuguesa depois do 25 de Abril. *Boletim da Sociedade Portuguesa de Matemática*, 12: 31–42, 1989.

[4] Hugo Ribeiro. "Lattices" des groupes abéliens finis. *Commentarii Mathematici Helvetici*, 23:1–17, 1949.

[5] Luis M. R. Saraiva. A década prodigiosa da Matemática Portuguesa: Os Começos da Sociedade Portuguesa de Matemática (1936– 1945). *Revista Brasileira de História da Matemática*, 11(23): 73–98, 2011. Anais do IX Seminário Nacional de História da Matemática. English title: The wonder decade of Portuguese Mathematics: The beginnings of the Portuguese Society of Mathematics (1936–1945).

[6] B. L. van der Waerden. *Álgebra moderna*. Sociedade Portuguesa de Matemática, 1956. Translated by Hugo Baptista Ribeiro.

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Semilinear Elliptic Problems: Old and New

by **Hugo Tavares***,**

Elliptic partial differential equations are a very import class of equations with obvious connections to applied sciences (e.g. physics, biology, chemistry and engineering) as well as to other fields of Mathematics such as Differential Geometry, Functional Analysis and Calculus of Variations. Because of these facts they are a quite fascinating topic and an increasingly active field of research. In this article we focus our attention on semilinear problems of type $\Delta u = f(u)$, more specifically on the Lane-Emden equation, as a mean to explain some of the tools and methods (mostly topological or variational) that are available to treat elliptic problems. The topics addressed concern existence and multiplicity of solutions, as well as their qualitative properties such as sign and symmetry. The goal is not to provide a complete state-of-the-art (which would not fit in a few pages), but rather to present some relevant and interesting questions and, whenever possible, to explain which ones we cannot answer yet.

1 Introduction

Many problems can be modelled with the aid of elliptic partial differential equations[1]. One of the most well known examples is the classical Poisson equation: given a bounded regular domain $\Omega \subset \mathbb{R}^n$, we take

$$-\Delta u = f \text{ in } \Omega.$$

Its solutions may represent the shape of an elastic membrane in equilibrium subject to a vertical load $f : \Omega \to \mathbb{R}$ ($u(x)$ corresponds to the vertical displacement at the point x); an electrostatic potential (for $f = \rho/\varepsilon$, where $\rho(x)$ is the volume charge density and ε the permittivity of the medium), a gravitational potential (for $f = -4\pi G\rho$, where ρ is the density of the object and G the gravitational constant), or the stationary solutions for the heat equation (in this case u represents a temperature, and f is a heat source or sink). Here $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$ is the Laplace operator (the trace of the Hessian matrix). To obtain existence and uniqueness of solution, one couples the equation with boundary conditions: *Dirichlet boundary conditions* ($u = g$ on $\partial\Omega$) or *Neumann boundary conditions* ($\partial u/\partial \nu := \nabla u \cdot \nu = g$ on $\partial\Omega$, where $\nu = \nu(x)$ is the exterior normal at $x \in \partial\Omega$) are typical examples arising in applications. Linear problems are very well understood and can be found in classical textbooks (see for instance [14, 26]), while current research aims at a good understanding of nonlinear problems. Among the wide class of possible nonlinear problems, the simplest to treat (although already quite rich mathematically, as we will see) are semilinear ones, where $f : \mathbb{R} \to \mathbb{R}$, $f = f(u)$, is a nonlinear function, that is, the nonlinearity occurs at the level of the zero order terms. Let us see some examples.

Example 1.— The stationary Fisher equation: $-\Delta u = au(b - u^2)$. Solutions are equilibrium points of the evolutionary equation

$$v_t - d\Delta_x v = v(a - bv), \qquad (x,t) \in \Omega \times \mathbb{R}^+,$$

where a, b are positive constants, and $v = v(x,t)$ represents the density of a population at the point x and time t, subject to diffusion (modelled by the Laplacian

[1] Is is out of the scope of this article to give a more general definition of what is an elliptic problem, but we can briefly explain the nomenclature. It comes from an analogy with the classification of conics, and from the classification of a general linear second order PDE in dimension 2: $a(\partial^2 u/\partial x_1^2) + 2b(\partial^2 u/\partial x_1 \partial x_2) + c(\partial^2 u/\partial x_2^2) + \text{lower order terms} = f$ is called elliptic if $b^2 - ac < 0$. A typical example is the case $a = c = 1$, $b = 0$, which corresponds to having the Laplace operator in dimension 2.

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term) and following a logistic law of growth (compare with the well known ordinary differential equation version of it: $u' = u(a - bu)$).

Example 2.— The Lane-Emden equation: $-\Delta u = |u|^{p-2}u$, which appears in astrophysics. If $n = 3$, and u is *radially symmetric* and positive, then $\theta(|x|) = u(x)$ solves the *Lane-Emden equation of index* $p - 1$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^{p-1} = 0, \quad \theta(0) = 1,\ \theta'(0) = 0$$

which can be used to model self-gravitating spheres of plasma such as stars. Up to constants, we have that θ is the temperature, θ^p is the pressure, the first root r_1 of θ is the star's radius, and $\int_0^{r_1}\theta(r)^{p-1}r^2\,dr$ is the total mass [12]. We will meet again this equation in Section 3.

Example 3.— The time independent nonlinear Schrödinger equation: $-\Delta u + (V(x) + \lambda)u = \mu|u|^2u$ appears naturally when looking for standing wave solutions of the Schrödinger Equation $i\Phi_t = -\Delta\Phi + V(x)\Phi - \mu|\Phi|^2\Phi$, that is, solutions whose modulus is time-independent: $\Phi(x,t) = e^{i\lambda t}u(x)$. This evolutionary equation appears in quantum mechanics, nonlinear optics and in the study of Bose-Einstein condensation.

Example 4.— The Yamabe Problem (Differential Geometry):

$$-\frac{4(n-1)}{n-2}\Delta_{\mathscr{M}}u + S_g(x)u = S_{g_0}u^{2^*-1}, \tag{1}$$

where $2^* = 2n/(n-2)$. Here $\mathscr{M}$ is a compact manifold of dimension $n \geq 3$ with a Riemannian metric g and scalar curvature S_g. Yamabe in 1960 made the conjecture that there always exist a conformal metric g_0 with constant scalar curvature (actually he *proved it*, but a mistake in the proof was found by Trudinger in 1968). The final proof was given in the 1984 after the contributions of Yamabe, Trudinger, Aubin and Schoen (see a detailed account in [21]). The previous equation has a positive solution u and, for the metric $g_0 := u^{4/(n-2)}g$, the manifold has constant scalar curvature S_{g_0}. The exponent 2^* is called the Sobolev exponent and plays a crucial role in the theory of Sobolev spaces and weak solutions (we will meet them briefly in Section 2). In local coordinates, (1) reads

$$-\frac{1}{a(x)}\operatorname{div}(A(x)\nabla u_i) + S_g(x)u = S_{g_0}u^{2^*-1}$$

in an open bounded domain Ω,

$$a(x) = \frac{n-2}{4(n-1)}\sqrt{|g(x)|} := \frac{n-2}{4(n-1)}\sqrt{\det g_{ij}(x)}$$

and

$$A(x) = \sqrt{g(x)}(g^{ij}(x))_{ij},$$

where $(g^{ij}(x))_{ij}$ is the inverse matrix of the metric $(g_{ij}(x))_{ij}$.

The purpose of this article is to briefly explain some of the questions that mathematicians working in this field try to answer. To fix ideas, we focus on the equation introduced in Example 2, as it is one of the simplest prototypical situations. Whenever is possible and not too complicated to do, we will leave some open problems for the interested reader. Before entering into more recent and sophisticated material, it is helpful to review some classical one for the Poisson equation. We do it in the next section.

2 Weak Solutions. The Variational Method

Let us deal with the Poisson equation with zero Dirichlet boundary conditions:

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2}$$

where $f = f(x) : \overline{\Omega} \to \mathbb{R}$ is a regular function (observe that the function f depends only on the x variable, not on the solution itself like in Example 2). A classical solution is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the equalities in (2) pointwise. It is typically not easy to find classical solutions *directly*, being common to take the variational point of view: if u is a $C^2(\overline{\Omega})$ solution, then it is not hard to prove that u is a solution to the minimization problem

$$\inf\left\{\mathscr{E}(u)\ :\ u \in C^2(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega\right\},$$

where

$$\mathscr{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu. \tag{3}$$

This is called the Dirichlet's Principle, and it is connected with one of the physical interpretations of the problem: for an elastic membrane, we are minimizing the total potential energy [26, §6.1]. This may look like a very good way of finding solutions, however it is not as simple as it sounds: there are situations where there are no minimizers (there are famous counterexamples by Weierstrass (1870) and F. Prym (1871)). In a nutshell, the problem is that there is a natural norm present:

$$u \mapsto \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{1/2}, \tag{4}$$

however the function space $C^2(\overline{\Omega})$ is not complete when equipped with it. This, among other things, led to Sobolev spaces (mid 1930's), defined as the closure of $C^2(\overline{\Omega})$ for (4) or, equivalently, as[2]

$$
\begin{aligned}
&H^1(\Omega) = W^{1,2}(\Omega) = \\
&= \left\{ u \in L^2(\Omega) \, : \, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right\} = \\
&= \left\{ u : \Omega \to \mathbb{R} \, : \, \int_\Omega u^2, \int_\Omega \left(\frac{\partial u}{\partial x_i} \right)^2 < \infty \ \forall i \right\}.
\end{aligned} \tag{5}
$$

We also denote by $H^1_0(\Omega)$ the set of functions in $H^1(\Omega)$ which are zero at the boundary (in the sense of traces). We advice the reader to check for instance [26, §7] or [14, §5] for the details. This leads to the minimization problem

$$\inf \left\{ \mathscr{E}(u) \, : \, u \in H^1_0(\Omega) \right\},$$

for $\mathscr{E} : H^1_0(\Omega) \to \mathbb{R}$, as defined in (3).

The functional $\mathscr{E}$ is differentiable in $H^1_0(\Omega)$. It is standard (within the field of Calculus of Variations) to prove that minimizers exist; moreover, if u is a minimizer, then, for every $v \in H^1_0(\Omega)$,

$$
\begin{aligned}
\mathscr{E}'(u)[v] &:= \tfrac{d}{dt} \mathscr{E}(u + tv)|_{t=0} = \\
&= \textstyle\int_\Omega \nabla u \cdot \nabla v - \int_\Omega f v = 0.
\end{aligned}
$$

Definition 1.— We say $u \in H^1_0(\Omega)$ is a *weak solution* (or *variational solution*) of (2) if

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega f v = 0 \qquad \forall v \in H^1_0(\Omega).$$

Therefore u is a weak solution of (2) if, and only if, u is a critical point of $\mathscr{E}$. We have also seen that minimizers (which exist) provide weak solutions. On the other hand, if u is sufficiently regular, then the notion of classical and weak solution coincide. This theory allows to treat separately the existence of solutions and their regularity, and the existence part is translated into finding critical points of a certain functional. This leads to the subject *Variational Methods/Critical Point Theory*, which can be used to tackle not only linear problems, but also semilinear problems like the ones presented in the Introduction.

3 Study of a model problem: the Lane-Emden equation

3.1 Statement of the problem and some technical background

Let $n \geq 3$, $p > 1$ and let Ω be a bounded, regular, connected open set. We work from now on with the prototypical problem considered in Example 2, under Dirichlet boundary conditions

$$-\Delta u = |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{6}$$

Clearly $u \equiv 0$ is always a solution, but we are interested in nontrivial ones. Based on what we have seen in the previous section, a natural definition of weak solution is

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega |u|^{p-2} u v = 0 \qquad \forall v \in H^1_0(\Omega),$$

and (at least formally, for now) weak solutions correspond to critical points of the functional

$$
\begin{aligned}
&\mathscr{I} \, : \, H^1_0(\Omega) \to \mathbb{R}, \\
&\mathscr{I}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p
\end{aligned} \tag{7}
$$

(observe that $(|t|^p)' = p|t|^{p-2} t$ for every $t \in \mathbb{R}$, $p > 1$). To make these statements precise and correct, actually we need a restriction on the exponent p: the integral $\int_\Omega |u|^p$ is not always finite for $u \in H^1_0(\Omega)$. One needs to recall *Sobolev inequalities*: for $1 \leq 2^* = 2n/(n-2)$ there exists $C_{n,p} > 0$ such that

$$\left(\int_\Omega |u|^p \right)^{1/p} \leq C_{n,p} \left(\int_\Omega |\nabla u|^2 \right)^{1/2}$$

for every $u \in H^1_0(\Omega)$, which amounts to say that the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. The number 2^* is the critical Sobolev exponent, which we already met in Example 4. Therefore, in conclusion, (7) is defined only for $p \leq 2^*$. In this case, in order to look for weak solutions of the problem (6), we may try to find critical points of $\mathscr{I}$.

Now the question is: how can we find a critical point of $\mathscr{I}$? And how many of them are there? The answer depends on p: not only the geometry of $\mathscr{I}$ changes from $p < 2$ to $p > 2$, but also the situations

[2] Observe (or recall) that the meaning of $(\partial u/\partial x_i) \in L^2(\Omega)$ is not obvious at all for a function u in $L^2(\Omega)$. It means that the first order *weak derivatives* of u are $L^2(\Omega)$ — functions; in other words, for every $i = 1, \dots, N$, there exists $g_i \in L^2(\Omega)$ (which we call $\partial u/\partial x_i$) such that

$$\int_\Omega g_i \varphi = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \qquad \forall \varphi \in C^\infty(\Omega) \text{ with compact support.}$$

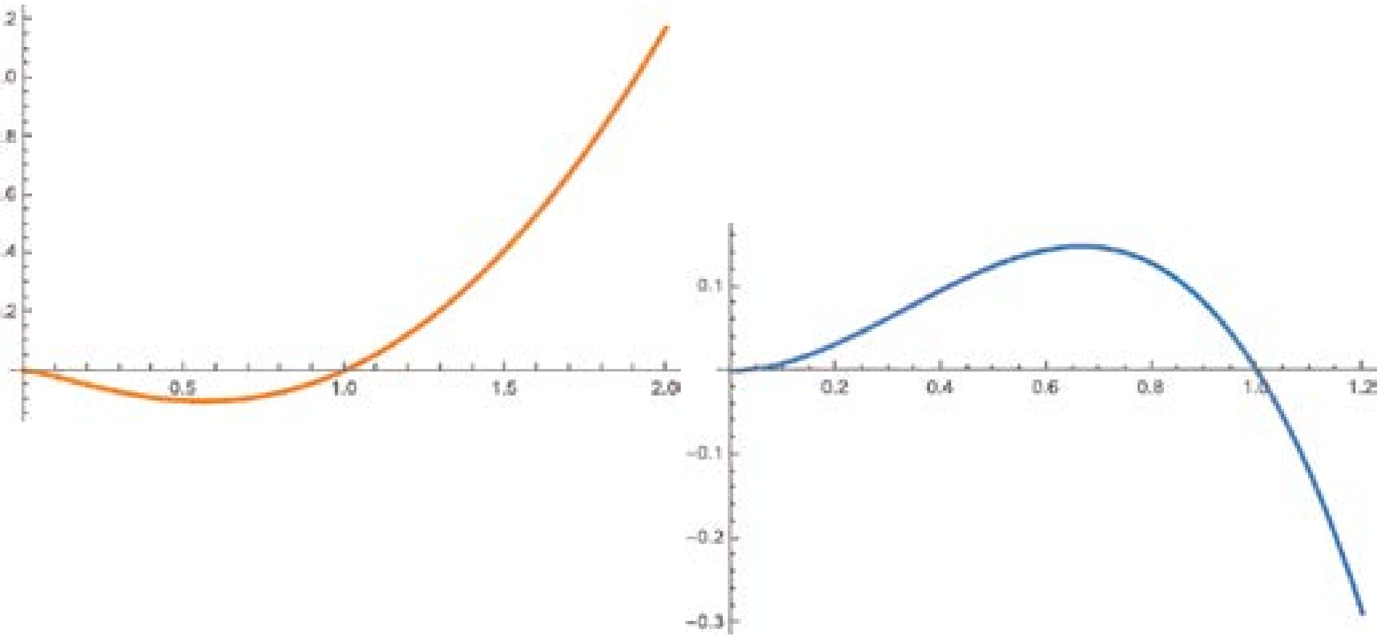

Figure 1.—Given $u \in H_0^1(\Omega)$, the shape of the map $t \in \mathbb{R}_0^+ \mapsto \mathcal{J}(tu)$ for $p < 2$ and $p > 2$ is represented on the left and right figures respectively.

$p < 2^*$ and $p = 2^*$ are very different: the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$ is compact only for $1 \leq p < 2^*$. The discussion of the case $p > 2^*$ is much harder and less is known, being out of the scope of this article.

3.2 The linear case $p = 2$.

Before going nonlinear, let us analyse what happens in the linear case $p = 2$, that is: $-\Delta u = u$ in Ω, $u = 0$ on $\partial\Omega$. This problem may or may not have a solution; what we are asking in other words is if $\lambda = 1$ is an eigenvalue of the operator $A := -\Delta$ with Dirichlet boundary conditions. In this context, indeed, $\lambda \in \mathbb{R}$ is called an eigenvalue whenever $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ admits a nontrivial (weak) solution. From the spectral theory of compact operators (using the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$), we deduce that the eigenvalues of $-\Delta$ (counting multiplicities) form a nondecreasing sequence

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \to \infty$$

with associated eigenfunctions $(v_n)_{n\in\mathbb{N}}$ which form a Hilbert base of $H_0^1(\Omega)$. Exactly as for eigenvalues of a matrix, the eigenvalues admit a variational formulation, namely

$$\begin{aligned} &\lambda_1(\Omega) = \min\left\{\mathcal{R}(u) \,:\, u \in H_0^1(\Omega) \setminus \{0\}\right\}, \\ &\lambda_k(\Omega) = \min_{V\subset H_0^1(\Omega), \dim V = k}\ \max_{u\in V\setminus\{0\}} \mathcal{R}(u), \quad (k \geq 2) \end{aligned} \tag{8}$$

where $\mathcal{R}(u) = \int_\Omega |\nabla u|^2 / \int_\Omega u^2$ is called the Rayleigh quotient. The details can be found in [26, §6], for instance. Therefore, the question of whether (6) in the case $p = 2$ admits a nontrivial solution or not depends on the domain: the answer is affirmative only for domains for which $1 = \lambda_i(\Omega)$ for some i.

3.3 The sublinear case $0 < p < 2$.

This is called the *sublinear* case. It is quite easy to see that $\mathcal{J}$ has a minimum in each direction: for a fixed $w \in H_0^1(\Omega)$, this corresponds to study the real function $f(t) = \mathcal{J}(tw)$, which has the form $at^2 - b|t|^p$ for some $a, b > 0$ (see the left picture on Figure 1). Using Sobolev inequalities and the direct method of Calculus of Variations [14, §8.2], one shows that $\mathcal{J}$ admits a negative global minimum in $H_0^1(\Omega)$: the level

$$\inf\{\mathcal{J}(u) \,:\, u \in H_0^1(\Omega)\} < 0$$

is achieved, providing a nontrivial solution (which is called a least energy solution). We know a lot about minimizers. First of all, they are signed: either $u > 0$ in Ω or $u < 0$ in Ω (consequence of the inequality $\mathcal{J}(|u|) \leq \mathcal{J}(u)$ and the strong maximum principle [17, §2.2]). Positive solutions are unique [20]. This uniqueness property also implies symmetry properties in symmetric domains: for instance if the domain is radially symmetric (ball or annulus), the solution is radially symmetric (working in the space $\{u \in H_0^1(\Omega) \,:\, u(x) = u(|x|)\ \forall x \in \Omega\}$ provides a positive solution). More generally, we can consider the situation of a domain Ω which is invariant for a subgroup G of $O(N)$.

In the previous paragraph we described properties of minimizers. Does $\mathcal{J}$ admit more critical points (i.e., solutions of the problem (6))? The answer is affirmative (see e.g. [7]): there exists a sequence of critical points $(v_k)_k$, which satisfy

$$\mathcal{J}(v_k) < 0, \qquad \mathcal{J}(v_k) \to 0.$$

This is a consequence of the $\mathbb{Z}_2$–symmetry of the problem (the functional is invariant under the map $u \mapsto -u$); solutions can be found as saddle points of $\mathcal{J}$, characterized via min-max methods in an analogous way to what happens for higher eigenvalues (recall (8)). Observe that, since positive (and negative) solutions are unique, the previous multiplicity result yields the existence of infinitely many *sign-changing* solutions. The next step is then to understand them as better as possible. The study of the zero-set of sign-changing solutions (the *free-boundary* set $\Gamma = \{x \in \Omega : u(x) = 0\}$) has been done recently: up to a set with small Hausdorff dimension, Γ is a regular hypersurface [27, 28]. Moreover, one may also ask if, among all sign-changing solutions, there is one that minimizes the energy functional $\mathcal{J}$, that is, if the level

$$c_{nod} = \inf\{\mathcal{J}(u) : u \text{ is a sign-changing critical point of } \mathcal{J}\}$$

is achieved. The answer is affirmative, as shown recently in [9]. On radial domains this solution is *not* radially symmetric, but only axially symmetric. In this last paper it is also shown, quite remarkably, that the type of critical point we find depends on the domain: there are domains where the least energy nodal solution is a local minimizer of $\mathcal{J}$, and others where it is a saddle point. A complete understanding of how the domain influences the type of critical point is open.

3.4 The superlinear–subcritical case $2 < p < 2^*$.

For the case $p > 2$, in each direction the functional looks like the picture on the right in Figure 1. Using Sobolev inequalities, one can show that:

- the origin $u = 0$ is a strict local minimum;
- $\mathcal{J}$ is unbounded from below and from above.

In this case, to obtain solutions we cannot simply minimize (nor maximize) the functional in the whole $H_0^1(\Omega)$. Based on the geometry of the functional, we can use the following version of the celebrated result by Ambrosetti and Rabinowitz [3].

Theorem 2 (Mountain Pass Theorem).— Let H be a Hilbert space and let $\mathcal{J} : H \to \mathbb{R}$ be a $\mathscr{C}^{1,1}$ functional satisfying

- $\mathcal{J}(0) = 0$;
- there exists $r > 0$ such that $\mathcal{J}(0) \le \mathcal{J}(u)$ for every $\|u\| \le r$ and $\inf\{\mathcal{J}(u) : \|u\| = r\} > 0$;
- there exists v such that $\mathcal{J}(v) < 0$.

Let

$$\Gamma := \{\gamma \in C^1([0,1]; H_0^1(\Omega)) : \gamma(0) = 0,\ \mathcal{J}(\gamma(v)) < 0\},$$

and

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} \mathcal{J}(u).$$

Then there exists a sequence $(u_k) \subset H$ such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$.

The proof of this result uses deformation lemmas and the study of steepest descending flows (a simple proof can be found in [14, §8]). The existence of a sequence (u_k) such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$, by itself, does not imply the existence of a critical point (take the counterexample $H = \mathbb{R}$, $u_k = -k$ and $\mathcal{J}(x) = e^x$). A new concept regarding compactness is needed:

> A functional $\mathcal{J} \in C^1(H, \mathbb{R})$ satisfies the *Palais-Smale condition at c* if, whenever we have a sequence (u_k) such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$, then there exists a subsequence (u_{k_j}) of (u_k) and $u \in H$ such that $u_{k_j} \to u$ in H. In particular, $\mathcal{J}'(u) = 0$.

Using the compactness of the Sobolev embeddings for $p < 2^*$, one proves that $\mathcal{J}$ defined before in (7) satisfies this condition, and the Mountain-Pass theorem provides the existence of a critical point of $\mathcal{J}$, hence a solution of (6). What can we now say about this solution? It is also a *least energy solution* (a.k.a. ground state), in the sense that

$$c = \inf\{\mathcal{J}(u) : u \in H \setminus \{0\},\ \mathcal{J}'(u) = 0\}.$$

Exactly as in the sublinear case, the solution can be shown to be signed: it is either stricly positive or strictly negative in Ω. However, uniqueness of positive solutions does not hold in general, as an effect of the topology of the domain (there are multiplicity results in annular domains) or of the geometry (dumbbell shaped domains). There is a long standing conjecture by Kawohl (1985) and Dancer (1988) stating that, if the domain is convex, then there is uniqueness of positive solution of (6) for $2 < p < 2^*$. A good review of the state-of-the-art regarding this can be found in the introduction of [16]. What about the symmetry in radial domains? When the domain is a ball, positive solutions are radially symmetric (consequence of a the so called Moving Plane Method[3], which uses many types of maximum principles, see [15] or [17, §2.6]). However, if Ω is an annulus, the solutions (at least for large p) loose one axis of symmetry, being just axially symmetric [6]. As we can see, there are some key changes between the cases $p < 2$ and $p > 2$.

Regarding multiplicity of solutions, again by the $\mathbb{Z}_2$—invariance of the functional, there exist infinitely many (sign-changing) solutions; however, unlike the sublinear case, this time we can find a sequence of solutions (u_k) such that $\mathcal{J}(u_k) \to \infty$. A long standing open question is whether the symmetry is necessary to obtain multiplicity results, with partial results obtained over the years by Bahri, Beresticky, Struwe, Rabinowitz, Bolle, Ghoussoub, Tehrani, Lions, Ramos, T., Zou, among many others. The study of the regularity of the zero-set of sign changing solutions is actually simpler in the superlinear case $p > 2$ than in the sublinear one $p < 2$ (although in any case it is not at all simple); this is as a consequence of the map $f(t) = |t|^{p-2}u$ being of class C^1 for $p > 2$ [19, 22].

The critical case $p = 2^* = 2n/(n-2)$

In this case, we are dealing with

$$-\Delta u = |u|^{2^*-2}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

and the associated functional $\mathcal{J}$ does not satisfy the Palais-Smale condition for all levels c. The question of whether there are (nontrivial) solutions or not for $p = 2^*$ or $p > 2^*$ depends strongly on the domain. When Ω is star-shaped, for instance, there are no solutions (by the Pohozaev identity, see for instance [4, Theorem 3.4.26]); however there are examples of contractible domains where solutions do exist. This shows that the topology of the domain is not enough to characterize the situation, although it has influence: if, for some positive d, the homotopy group of Ω with $\mathbb{Z}_2$ coefficients is non trivial, $\mathcal{H}_d(\Omega, \mathbb{Z}_2) \neq \{0\}$, then we have a positive solution [5]. Multiplicity results are much more recent (and challenging); recent contributions are due to Clapp, Ge, Musso, Pistoia, Weth, among others.

In order to show how delicate the situation is in the critical case, we make two remarks:

1. If the domain is not bounded but instead the whole $\mathbb{R}^n$, then we have (explicit!) solutions:
$$U_{\delta,\xi} = (n(n-2))^{(n-2)/4} \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}},$$
for $\delta > 0$, $\xi \in \mathbb{R}^n$.

2. If we consider a linear perturbation of the problem:
$$-\Delta u = \lambda u + |u|^{2^*-2}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$
the situation changes: this problem has positive solutions for $\lambda \in (0, \lambda_1(\Omega))$ and $n \geq 4$ (the problem is commonly known as the Brezis-Nirenberg problem [11]). The topology of the domain, in this situation, also influences multiplicity results: there exist at least $\mathrm{cat}_\Omega(\Omega)$ solutions, where the (Lyusternik-Schnirelmann) category of Ω is the least integer d such that there exists a covering of Ω by d closed contratible sets. As $\lambda \to 0$, the solutions tends to concentrate and blowup at certain points which depend on geometric properties of Ω [18, 24].

4 Recent directions of Research

In the previous section we reviewed some old and new results regarding the Lane-Emden equation with Dirichlet boundary conditions. This is still a very active field of research and there are still many interesting questions left open. Although we described quite a few results, there would clearly be a lot more to be said. In this section, instead, we point out new directions of research that poped up more recently. One is the case of other boundary conditions such as the Neumann problem:

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

The difficulty of the problem is related with the fact that, in the associated functional $\mathcal{J}$, we have the presence of only $\|\nabla u\|_{L^2}$, which is a norm in $H^1_0(\Omega)$ but not in $H^1(\Omega)$. The study of least-energy solutions for

[3] The argument is based on a method developed by Alexandrov circa 1958 to establish that spheres are the only embedded compact hypersurfaces of R^n with constant mean curvature.

$p < 2^*$ (existence, study of its symmetry in symmetric domains, ...) is mostly recent [23, 25]. Quite remarkably, for $p = 2^*$, there are solutions in all regular domains, one of the few things that has been known for a while [13]. An open question is whether there is a solution for every $p > 2^*$ (or at least for $p \in (2^*, 2^* + \varepsilon)$ for sufficiently small ε).

Another direction of research is the case of systems. Here one might consider

$$-\Delta u_i = f_i(u_i) + u_i \sum_{j \neq i} \beta_{ij} u_j^2 \text{ in } \Omega$$

under a symmetric interaction $\beta_{ij} = \beta_{ji}$. From a physical point of view, this is connected with the search of standing wave solutions in systems of nonlinear Schrödinger type equations (coming from Bose-Einstein condensation and nonlinear optics). Mathematically speaking, this is a good prototype of a gradient system (the interaction term is the gradient of the potential $H(u_1, \dots, u_k) = (\sum_{j<i} \beta_{ij} u_i^2 u_j^2)/2)$. Again, one might study existence, multiplicity and classification of solutions, concentration results in the critical case and symmetry questions. These issues are more complex for systems due to the possibility of different types of interaction between components, see for instance [30] and references.

Other variational systems (not of gradient type) are *Lane-Emden systems*:

$$-\Delta u = |v|^{q-2} v, \qquad -\Delta v = |u|^{p-2} u \text{ in } \Omega$$

(the reader might take a look at [8] for a recent survey).

5 Conclusion and recommended readings

In this short article we motived the study of some elliptic problems, starting from some classical material, introducing the concept of weak solutions and speaking about some variational methods, concluding with recent directions of research. With the exception of the fourth section and part of the third, everything is by now already included in introductory books. For Sobolev spaces, weak Solutions, and the linear theory of elliptic equations, the recommendation is [10, 14, 26] (the author of these lines uses a combination of these three books whenever he teaches a PDE course at the master level). For a gentle introduction to semilinear theory and the use of variational methods, we recommend [4], while [1, 2, 29, 31] contains more advanced material.

References

[1] Antonio Ambrosetti and Andrea Malchiodi. *Perturbation methods and semilinear elliptic problems on* $\mathbf{R}^n$, volume 240 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2006.

[2] Antonio Ambrosetti and Andrea Malchiodi. *Nonlinear analysis and semilinear elliptic problems*, volume 104 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[3] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[4] Marino Badiale and Enrico Serra. *Semilinear elliptic equations for beginners*. Universitext. Springer, London, 2011. Existence results via the variational approach.

[5] A. Bahri and J.-M. Coron. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. *Comm. Pure Appl. Math.*, 41(3):253–294, 1988.

[6] Thomas Bartsch, Tobias Weth, and Michel Willem. Partial symmetry of least energy nodal solutions to some variational problems. *J. Anal. Math.*, 96:1–18, 2005.

[7] Thomas Bartsch and Michel Willem. On an elliptic equation with concave and convex nonlinearities. *Proc. Amer. Math. Soc.*, 123(11):3555–3561, 1995.

[8] D. Bonheure, E. Moreira dos Santos, and H. Tavares. Hamiltonian elliptic systems: a guide to variational frameworks. *Port. Math.*, 71(3-4):301–395, 2014.

[9] D. Bonheure, E. Moreira dos Santos, E. Parini, H. Tavares, and T. Weth. Nodal solutions for sublinear-type problems with dirichlet boundary conditions. *International Mathematics Research Notices*, in press, https://doi.org/10.1093/imrn/rnaa233.

[10] Haïm Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[11] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[12] S. Chandrasekhar. *An introduction to the study of stellar structure*. Dover Publications, Inc., New York, N. Y., 1957.

[13] Myriam Comte and Mariette C. Knaap. Solutions of elliptic equations involving critical Sobolev exponents with Neumann boundary conditions. *Manuscripta Math.*, 69(1):43–70, 1990.

[14] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[15] B. Gidas, Wei Ming Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[16] Massimo Grossi, Isabella Ianni, Peng Luo, and Shusen Yan. Non-degeneracy and local uniqueness of positive solutions to the lane-emden problem in dimension two. arXiv:2102.09523 (2021).

[17] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[18] Zheng-Chao Han. Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 8(2):159–174, 1991.

[19] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. Critical sets of solutions to elliptic equations. *J. Differential Geom.*, 51(2):359–373, 1999.

[20] M. A. Krasnosel'skii. *Positive solutions of operator equations*. P. Noordhoff Ltd. Groningen, 1964. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron.

[21] John M. Lee and Thomas H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[22] Fang-Hua Lin. Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 44(3):287–308, 1991.

[23] Enea Parini and Tobias Weth. Existence, unique continuation and symmetry of least energy nodal solutions to sublinear Neumann problems. *Math. Z.*, 280(3-4):707–732, 2015.

[24] Olivier Rey. Proof of two conjectures of H. Brézis and L. A. Peletier. *Manuscripta Math.*, 65(1):19–37, 1989.

[25] Alberto Saldaña and Hugo Tavares. Least energy nodal solutions of Hamiltonian elliptic systems with Neumann boundary conditions. *J. Differential Equations*, 265(12):6127–6165, 2018.

[26] Sandro Salsa. *Partial differential equations in action*, volume 99 of *Unitext*. Springer, [Cham], third edition, 2016. From modelling to theory, La Matematica per il 3+2.

[27] Nicola Soave and Susanna Terracini. The nodal set of solutions to some elliptic problems: singular nonlinearities. *J. Math. Pures Appl. (9)*, 128:264–296, 2019.

[28] Nicola Soave and Tobias Weth. The unique continuation property of sublinear equations. *SIAM J. Math. Anal.*, 50(4):3919–3938, 2018.

[29] Michael Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.

[30] Hugo Tavares, Song You, and Wenming Zou. Least energy positive solutions of critical schrödinger systems with mixed competition and cooperation terms: the higher dimensional case. arXiv:2109.14753 (2021).

[31] Michel Willem. *Minimax theorems*, volume 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.

Screenshot of one of the moments of the event.

Dynamic Control and Optimization

International Conference On the occasion of Andrey Sarychev's 65th birthday, University of Aveiro, February 3–5, 2021

by **Eloísa Macedo*** and **Tatiana Tchemisova****

The international conference Dynamic Control and Optimization (DCO 2021) was organized by the Department of Mathematics of the University of Aveiro, the Center for Research & Development in Mathematics and Applications from the University of Aveiro, and the Center for Applied Mathematics and Economics from the University of Lisbon. CIM — International Center of Mathematics supported the organization of this event. Due to the context of the COVID-19 pandemic, the conference was held virtually on February 3-5, 2021.

The conference was organized on the occasion of the 65th anniversary of Professor Andrey Sarychev from the University of Florence, Italy, an expert on nonlinear dynamical control systems, optimal control, calculus of variations, and propagation of acoustic waves in elastic media, who was teaching at the University of Aveiro from 1993 to 2002.

The scientific program consisted of 11 plenary talks with leading invited speakers from Europe and the USA, and 36 contributed presentations divided into three

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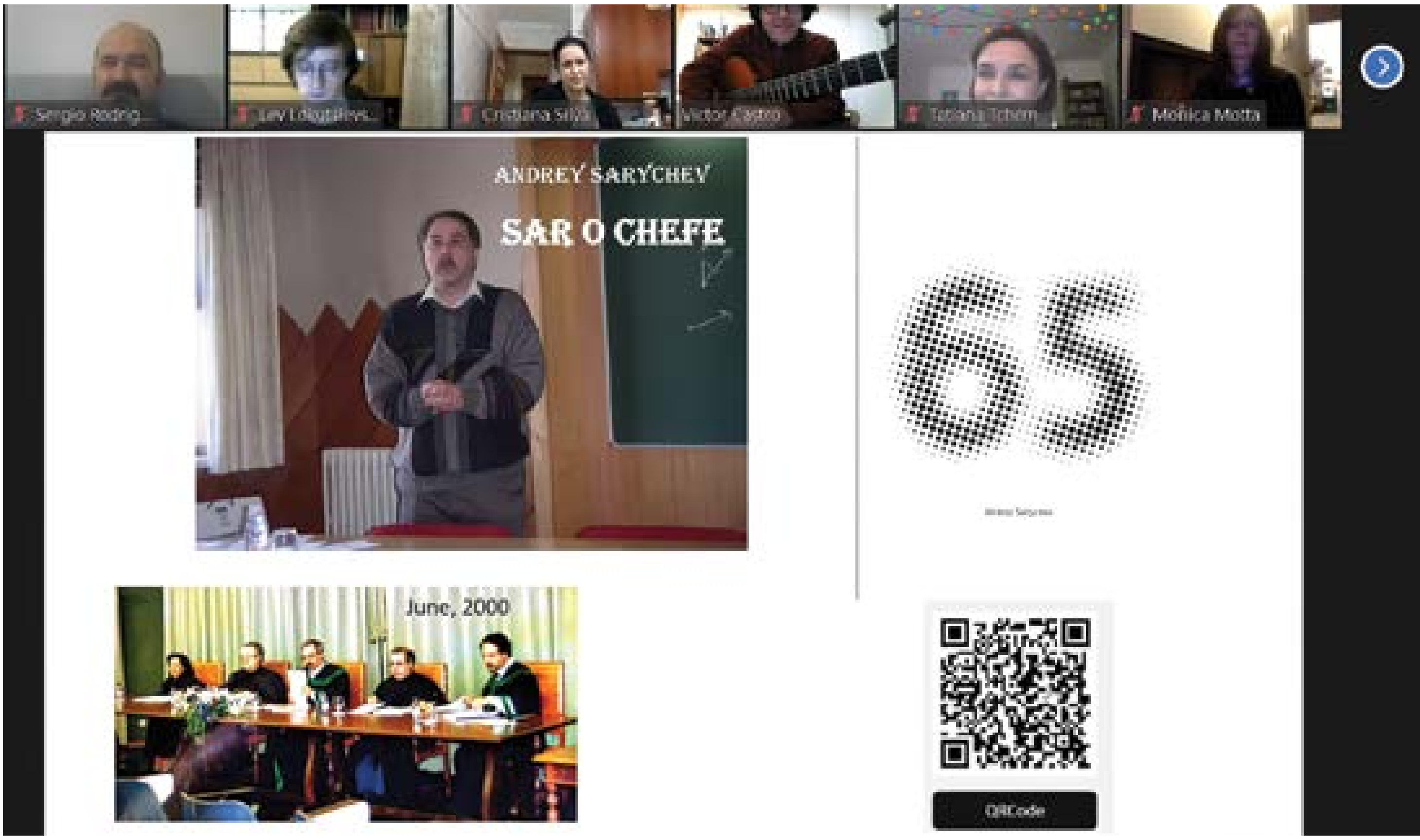

Screenshot of the presentation of the tribute book to Professor Sarychev.

main streams: Dynamic Control, Optimization, and Applications of Control and Optimization. The DCO 2021 accounted for more than 70 participants from 19 countries, and it was an opportunity to discuss recent scientific research results and developments in the field.

Thanks to the support of the sponsors, various Ph.D. students were able to participate in the conference. Besides the scientific part, although the event was held online, the program included a social session dedicated to Professor Andrey Sarychev, with a musical moment with Mr. Victor Castro, one of the best classical guitar players in Portugal, and a presentation of the tribute book.

Further information on the event can be found at https://sites.google.com/view/dco2021/dco-2021.

REPORT

Portuguese Mathematical Typography: 1496–1987

by José Francisco Rodrigues*

* CMAFcIO_Universidade de Lisboa [jfrodrigues@ciencias.ulisboa.pt]

Figure 1.—*Moinho do papel* is a modern museum on the left bank of the river Lis, in the city of Leiria, where in 1411 there was the first paper mill in Portugal.

The *Tipografia Matemática Portuguesa: 1496–1987* is a unique, rare and eloquent exhibition, which first edition took place from July 1st to October 31st 2021 in the hexacentenary *Moinho de Papel* in Leiria, a city with an ancient castle since the 12th century in the center of Portugal, residence of kings and setting of several cortes (medieval parliaments). The city gave the name to a famous pine forest (*Pinhal de Leiria*), wood supplier of the ships used in the Portuguese navigations of the 15th and 16th centuries. The exhibition was an initiative of the city of Leiria in partnership with the CIM and the Polytechnic Institute of Leiria.

The *Moinho de Papel* is an historical building on the river Lis, the first paper mill established in 1411 in Portugal, which may well have influenced the fact that Leiria was also one of the first Portuguese cities to have a typography and where 525 years ago the first scientific book, which was instrumental for navigation in the age of discoveries, was printed in the country. The second date of the title of this unprecedented exhibition corresponds to the publication, coincidently 500 years after the first book printed in Portugal, of the first volume of *Portugaliae Mathematica* electronically composed in *TeX*. With 32 significant original works, this exhibition traverses the History of Mathematical Sciences in Portugal, through military engineering, essential in the wars of Restoration (1640–1668) after the end of the Iberian Union, through the successive reforms of Colleges, Military Academies and Universities (1772 and 1911) and through scientific research in the 20th century.

The publication in 1496, in Leiria, of the *Almanach perpetuum*, with the astronomical tables of the Sephardi scholar Abraão Zacuto, referred to the year 1473 and translated and edited by the Portuguese José Vizinho, took place a few years after a first edition of the *Torah* of 1487, in a Hebrew typography in Faro, in the south of Portugal, and the *Tratado de Confissom* of 1489, which is the first Christian text in Portuguese language that was printed in Chaves, in the north of the country. It should be noted that the Gutenberg Bible, the first book printed in Europe, dates from 1455 and the first printing of Euclid's Elements in 1482 was made by the printer Erhard Ratdolt, in Venice, in a Latin edition containing the first geometric diagrams of the press.

That *Almanach* is a landmark of the beginning of the culture of Mathematical Sciences in Portugal through the influence and use of the art and knowledge of navigation, namely in the first ocean voyages of Vasco da Gama to India and Pedro Álvares Cabral to Brazil. It was used in the following century in the preparation of *Reportórios dos Tempos*, the popular time calendars and almanacs also used in astrology, one of which had the collaboration of a certain Gaspar Nicolas. This Portuguese mathematician published in 1519, in Lisbon, a *Tratado da pratica Darismetyca*, which is a book of a technical and utilitarian nature about the rules of arithmetic, also "for overseas

Figure 2.— A glimpse of the nine showcases and the posters containing the 32 books and the eight biographies of Portuguese mathematicians of the Leiria exhibition.

Figure 3.—*Almanach Perpetuum*, by Abraham Zacut (1452-1515), printed in 1496 in Leiria by Abraham d'Ortas.

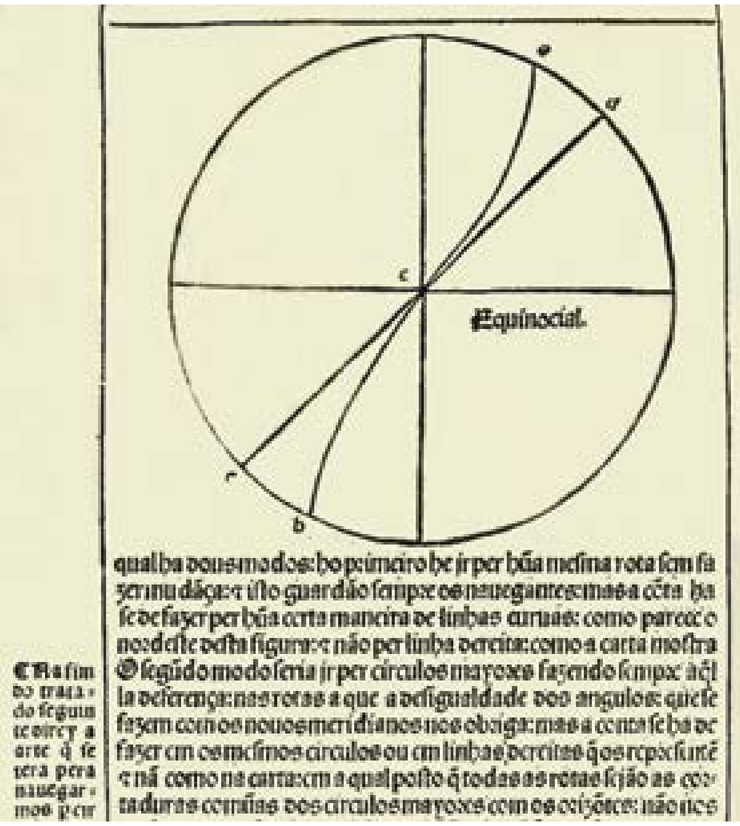

Figure 4.—The Pedro Nunes' representation of the rhumb line *acb* and of the great circle dce was published in 1537 in the ***Tratado da Sphera***, printed in Lisbon by Germão Galhardo.

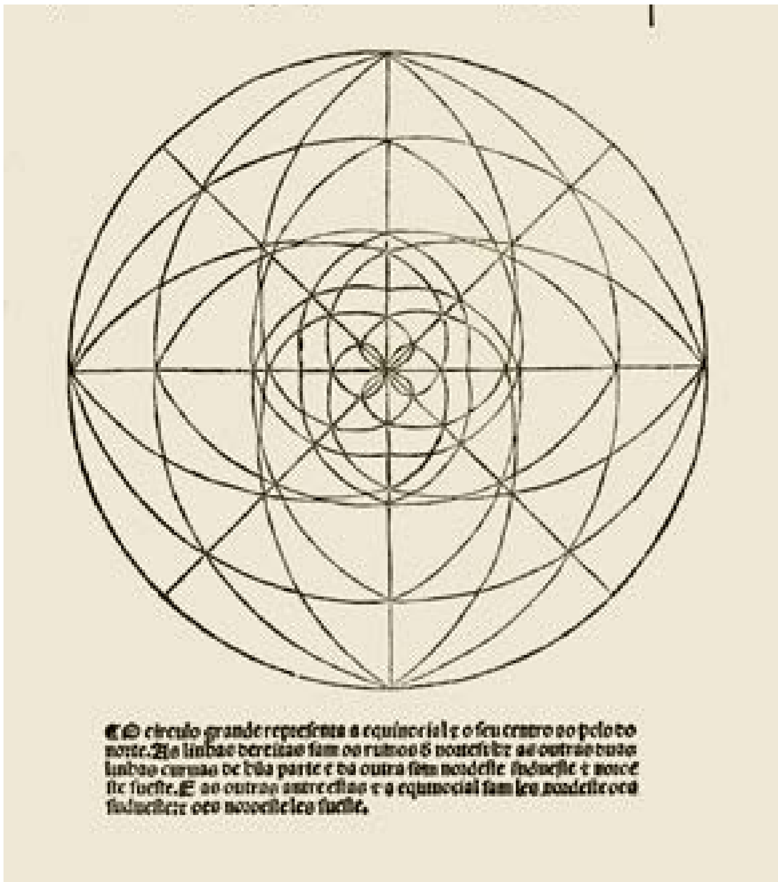

Figure 5.—The first polar projection of the rhumb line in the Pedro Nunes' ***Tratado da Sphera*** is the beautiful rosette composed of projections of loxodromes with azimuths with angles of 45º and 67.5º.

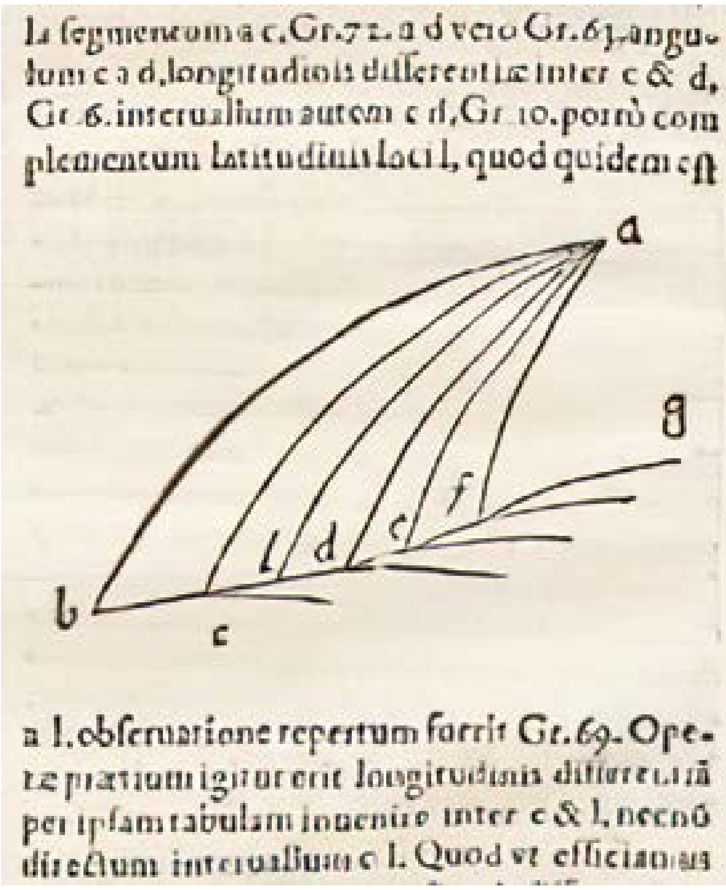

Figure 6.—The broken line bcdef (noniodrome) is the 1566's approximation of the loxodrome proposed by Pedro Nunes for the construction of nautical tables used by Mercator in his Mappa-mundi of 1569 and Edward Wright in 1596, which is in page 28 of the second edition of ***De arte atque ratione navigandi***, printed in Coimbra, by António Mariz.

trade", and which had ten reeditions in the following two centuries.

Printing also played an important role in the development of the mathematical theory of navigation through the works of Pedro Nunes, with the edition in Portuguese of his *Tratado da Sphera*, in 1537, which presented the first mathematical conceptualisation of the rhumb line, later called loxodrome. With the enlarged reedition of Nunes' fundamental work *De arte atque ratione navigandi*, where the Portuguese mathematician developed original methods, in particular for the approximation of the rhumb line, which laid the foundations for the elaboration of nautical tables and the cartographic projection made in 1569 by Mercator. That broken line invented by


PRINCIPIOS MATHEMATICOS
PARA INSTRUCÇAO
DOS ALUMNOS DO COLLEGIO
DE
SAÕ LUCAS,
DA REAL CASA PIA DO CASTELLO
DE
SAÕ JORGE:
OFFERECIDOS
AO SERENISSIMO SENHOR
D. JOAÕ,
PRINCIPE DO BRAZIL:
COMPOSTOS PELO DOUTOR
JOSÉ ANASTACIO DA CUNHA,
DE ORDEM DO DESEMBARGADOR DO PAÇO
DIOGO IGNACIO DE PINA MANIQUE,
Intendente Geral da Policia da Côrte, e Reino, &c, &c, &c.

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PRINCIPIOS MATHEMATICOS.

LIVRO VIIII.

Definiçaõ I.

SERIE convergente chamam os Mathematicos aquella, cujos termos saõ semelhantemente determinados, cada hum pelo numero dos termos precedente, de sorte que sempre a serie se possa continuar, e finalmente venha a ser indifferente o continua-la ou naõ, por se poder desprezar sem erro notavel a somma de quantos termos se quizesse ajuntar aos já escritos ou indicados: e estes ultimos indicam-se escrevendo &c. depois dos primeiros dois, ou tres, ou quantos se quizer: he porém necessario que os termos escritos mostrem como se poderia continuar a serie, ou que isso se saiba por outra via.

Proposiçaõ I.

SErie de proporcionaes continuas he convergente, se o primeiro he maior que o segundo. Sejam A e B duas quaesquer grandezas homogeneas, mas $A > B$; e $A + B + B\frac{B}{A} + B\frac{B}{A} \times \frac{B}{A} + B\frac{B}{A} \times \frac{B}{A} \times \frac{B}{A} +$ &c. a serie. Escreva-se e em lugar de $\frac{B}{A}$, a serie será $A\ (1 + e + ee + eee + eeee +$ &c.$)$. Seja O huma grandeza $< A$, e tal que se possa desprezar sem erro notavel. Na serie $1 +$

PRINCIPIOS MATHEMATICOS.

LIVRO XV.

Definiçoens.

Advertencias.

Figure 7.— The ***Principios Mathematicos***, by José Anastácio da Cunha, printed in 1790, in Lisbon by António Rodrigues Galhardo, contains the first modern definitions of convergence of a series, in Chapter 8, and of the differential, in Chapter 15, respectively.

Pedro Nunes in 1566, called the *noniodrome* and chosen as the logo of the exhibition, is the natural way to approximate the loxodrome that corresponds to the method of Euler to integrate differential equations and was used numerically by Edward Wright in the secants' method to construct nautical tables in 1596.

Printing was instrumental also in the teaching and practical use of mathematics for military architecture, navigation and artillery in Europe and, in particular in Portugal with the *Methodo Lusitanico* (1680) and *O Engenheiro Portuguez* (1728), after the Restoration of Portuguese independence. Mathematics book printing continued in Portugal for the navy and the army schools, throughout the 18th and 19th centuries, among others, with the translations of the notable books by Lagrange (1798) and Lacroix (1812), the latter at the *Impressão Régia* (Royal Printing House) in Rio de Janeiro, which was the capital of the kingdom from 1808 until 1821, as well as with other original texts, such as the interesting *Carta Físico-Mathematica sobre a theórica da pólvora em geral e o comprimento das peças em particular* (1769), by José Anastácio da Cunha.

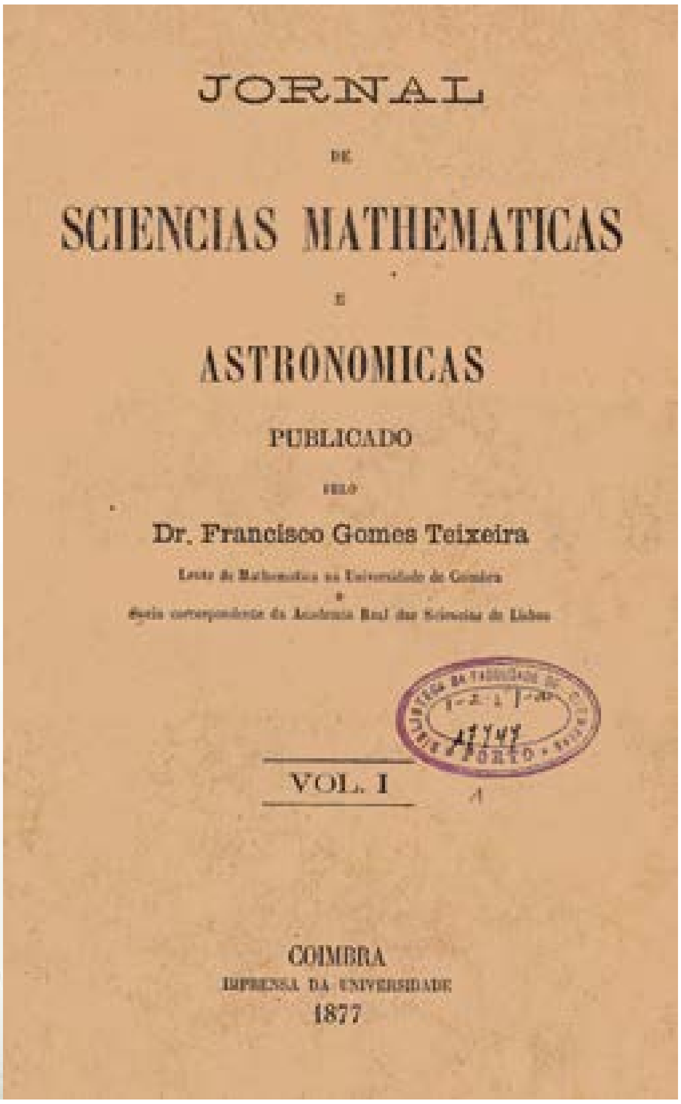
After the reforms of Marquis of Pombal, head of the government in the age of the Portuguese Enlightenment, the printing press would serve teaching with the publication of higher mathematics textbooks, initially with translations of foreign authors, such as Euclid's *Elementos* (1768), for the *Colégio dos Nobres* in Lisbon, several Bézout's textbooks for the first Faculty of Mathematics of the University of Coimbra, reformed in 1772, and, for the following century, the *Curso Completo de Mathematicas Puras* (1838 and 1839) by Francoeur. The first exception is the original and remarkable *Princípios Mathematicos* (1790), by José Anastácio da Cunha, the military mathematician and "*lente penitenciado*", penitentiated professor at the University of Coimbra, whose innovations place him among the eminent predecessors of the 19th century reform of the Infinitesimal Calculus, namely with the modern criteria of convergence of series and the rigorous definitions of infinitesimal and differential, more than three decades before Cauchy.

The publication of university original textbooks by Portuguese authors would only continue a century later in Porto, with the *Curso de Analyse Infinitesimal* (1887), by F. Gomes Teixeira, which, reedited and enlarged, became the reference Portuguese treatise at the beginning of the 20th century. Only from the middle of that century did this university practice resume, illustrated by the classical *Curso de Álgebra Superior* by J. Vicente Gonçalves (Coimbra, 1933) and the modern *Lições de Análise Infini-*


MEMORIAS
DA
ACADEMIA REAL
DAS SCIENCIAS DE LISBOA
DESDE O SEU ESTABELECIMENTO
EM 1780 ATÉ 1788.

SOLUÇÃO GERAL
DO
PROBLEMA DE KEPLER
Sobre a Medição das Pipas, e Toneis,
POR JOSÉ MONTEIRO DA ROCHA.

(1.) AS vaſilhas, que ordinariamente ſervem para guardar, e tranſportar toda a eſpecie de licores, são conſtruidas de maneira, que podem ſenſivelmente tomar-ſe por ſolidos de revolução, compoſtos de dous troncos iguaes, e ſemelhantes, os quaes
Tom. I. A ſe


JORNAL
DE
SCIENCIAS MATHEMATICAS
E
ASTRONOMICAS
PUBLICADO
PELO
Dr. Francisco Gomes Teixeira
Lente de Mathematica na Universidade de Coimbra
e
Socio correspondente da Academia Real das Sciencias de Lisboa

VOL. I

COIMBRA
IMPRENSA DA UNIVERSIDADE
1877

Figure 8.—The ***Memórias*** of the Academy of Sciences of Lisbon, started their publication in 1797, by its own typography, with an interesting applied mathematics article on the Kepler's problem on the volume of barrels.

Figure 9.—The first periodic Portuguese research journal, ***Jornal de Scienciass Mathematicas e Astronómicas***, was published by Francisco Gomes Teixeira and was printed in 1877 by the press of the University of Coimbra.

tesimal by F.R. Dias Agudo (Lisbon, 1973), among others.

The Lisbon Academy of Sciences, created in late 1779 and endowed with a printing press, began publishing its *Memórias* in 1797 with an article on applied mathematics, worthy of its motto "*Nisi utile est quod facimus stulta est gloria*" (If what we do is not useful, glory is in vain), continuing in a new series after the period of *Regeneração*, in the 1850s, attempted to develop the country economically and modernise it, and creating the first Portuguese scientific journal, the *Jornal de Sciencias Mathematicas Physicas e Naturaes* (1867), where few articles of mathematics appeared, including original articles by Daniel da Silva.

The first Portuguese periodical exclusively dedicated to mathematics, the *Jornal de Sciencias Mathematicas e Astronomicas* by F. Gomes Teixeira, started publication in 1877 in Coimbra, at the University Press. Later its publication was transferred to Porto and was integrated in 1905 in the *Annaes Scientificos da Academia Polytechnica do Porto*, and only more than thirty years later Portugal had a new mathematical journal. Francisco Gomes Teixeira was the most active and fruitful Iberian mathematician of the 19th century, who corresponded with numerous European mathematicians of his time. He was also the author of the remarkable and unsurpassable *Traité des Courbes Spéciales Remarquables Planes et Gauches*, in three of the seven volumes of its *Obras* with over 1300 pages, and he was the first rector of the University of Porto, between 1911 and 1918.

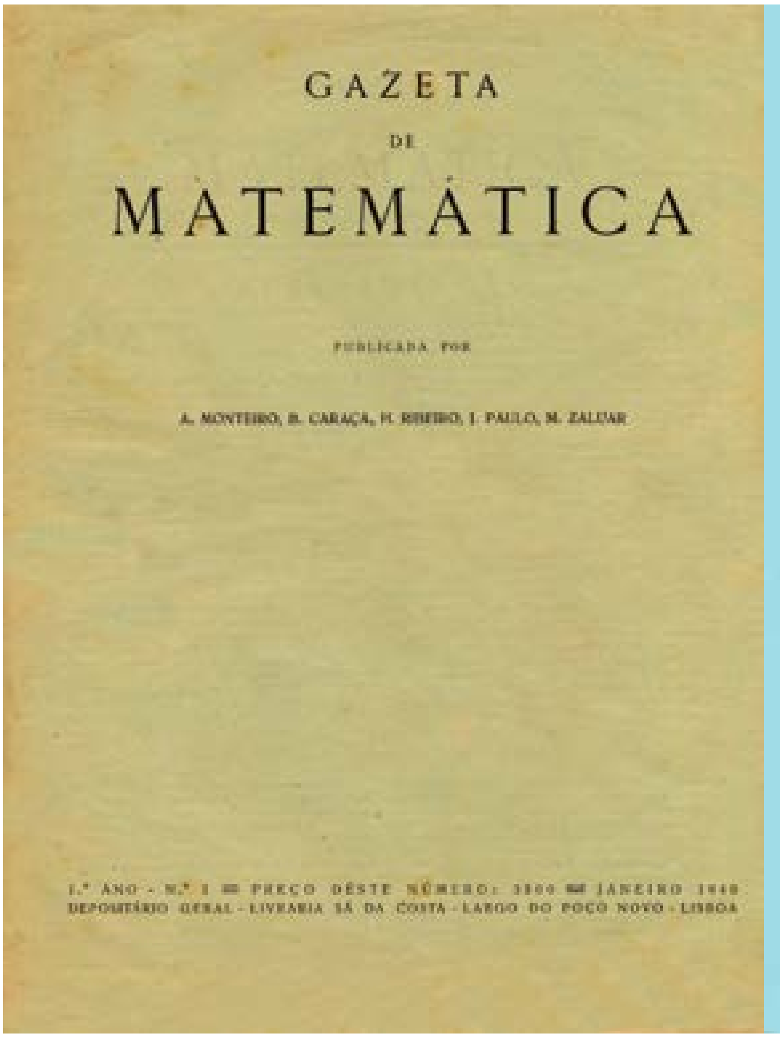
The foundation of the scientific journal *Portugaliae*

Figure 10.—The first original book on the popularisation of mathematics, ***Conceitos Fundamentais da Matemática***, had several edition after its first publication in Lisbon in 1941, integrated in the relabel collection of scientific culture Biblioteca Cosmos, funded by Bento de Jesus Caraça.

Figure 11.—The first volume of ***Portugaliae Mathematica***, published by António Monteiro, started to be printed in Porto in 1937 and it was completed in 1940 in Lisbon.

Mathematica in 1937 in Lisbon, by António Aniceto Monteiro, who returned the year before from Paris where he had completed his doctoral degree under Maurice Fréchet, started a modernist movement also in Science with a certain "mathematical effervescence", which lasted a short decade in Portugal. Is was followed by the *Gazeta da Matemática*, in 1940, which was later printed by the *Tipografia Matemática*, and by the creation of the first research mathematical center in the country, the *Centro de Estudos Matemáticos de Lisboa*, also in that year. That unique typography was established in 1945 and had a remarkable, although limited, activity in the Portuguese mathematical press for more than three decades.

During that short decade some mathematical activity flourished in Portugal. For example, the remarkable little book *Conceitos Fundamentais da Matemática*, by B. J. Caraça, had a first edition in 1941 and was the first Portuguese book aimed at the popularization of mathematics in a perspective of the "integral culture of the individual". More advanced publications, like two doctoral theses printed at the *Tipografia Matemática*, are illustrated in the exhibition with the *Publicação #18* of the *Centro de Estudos Matemáticos do Porto*, by A. Pereira Gomes, which was the first modern PhD thesis in a Portuguese university, or *As Funções Analíticas e a Análise Funcional*, by José Sebastião e Silva, both published in *Portugaliae Mathematica*, respectively in 1946 and in 1950. In this work, J. Sebastião e Silva starts deep contributions to Functional Analysis, which will lead him to introduce in 1955 an important class of the locally convex spaces as



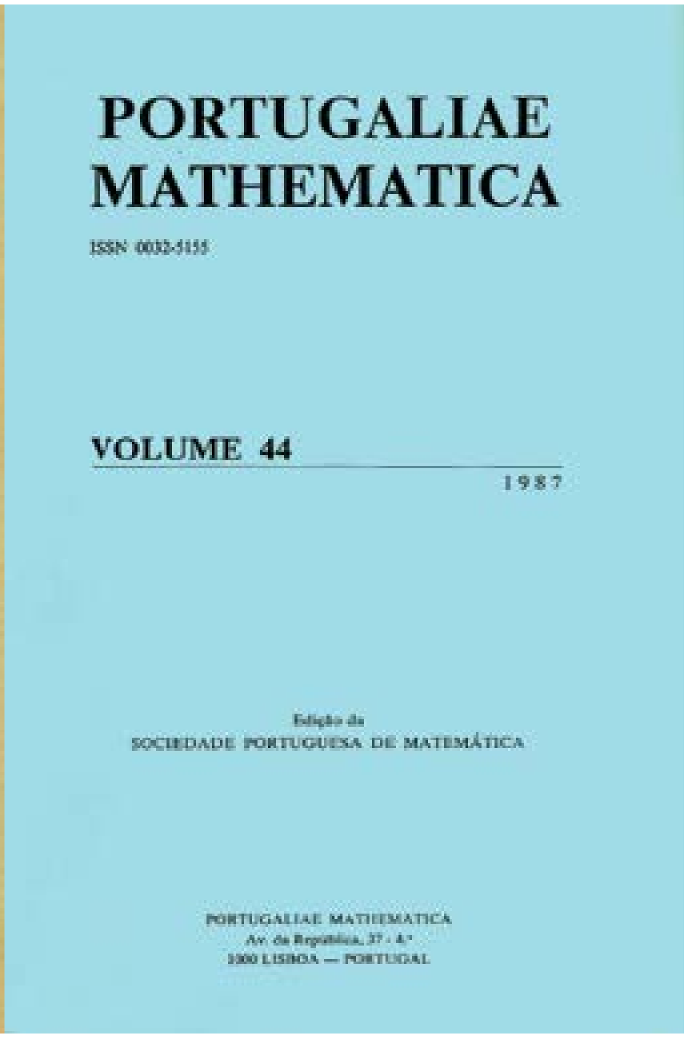


Figure 12.—The first issue of ***Gazeta de Matemática***, published in Lisbon in 1940, was mainly dedicated to the first years university students in science, engineering and economics and became later the Mathematics Magazine of the Portuguese Mathematical Society

Figure 13.—The volume 44 of ***Portugaliae Mathematica***, was published in its fiftieth anniversary already by the Portuguese Mathematical Society in 1987 in Lisbon, and it was already composed in ***TeX***.

inductive limits of an increasing sequence of normed spaces with compact inclusions, later called the Silva LN*-spaces.

After five centuries of existence, the mathematical typography of movable type, with its specific and distinctive aspects, such as tables, figures, diagrams and mathematical formulae, gave way to electronic publishing driven by the *TeX* program, created by Donald Knuth in 1978. This electronic composition system was adopted by the American Mathematical Society five years later and was used in 1987 by the *Sociedade Portuguesa de Matemática* (SPM) in the publication of the fiftieth anniversary volume 44 of *Portugaliae Mathematica*, the last book of this exhibition.

This first edition of this exhibition, with these 32 significant works of Portuguese mathematical typography, many of them being rare books and containing unknown or still poorly known very interesting pages, grouped in nine thematic showcases, allowed for an illustrated visit to the main milestones of five centuries of the History of Mathematics in Portugal. It is expected that more editions of the *Tipografia Matemática Portuguesa: 1496-1987* will be organised in the next years by the Universities of Porto, Coimbra and Lisbon in collaboration with the CIM and the SPM.

EXPOSITOR 1

TABELAS MATEMÁTICAS PARA A NAVEGAÇÃO/REGRAS ARITMÉTICAS PARA O COMÉRCIO ULTRAMARINO

1 — 1496 - **Almanach Perpetuum**, *Abraão Zacuto*
2 — 1519 - **Tratado da pratica Darismetyca**, *Gaspar Nycolas*

EXPOSITOR 2

MATEMÁTICAS DA NAVEGAÇÃO, DA OBSERVAÇÃO DOS CÉUS E DA CARTOGRAFIA

3 — 1537 - **Tratado da Sphera**, *Pedro Nunes*
4 — 1542 - **De Crepusculis**, *Pedro Nunes*
5 — 1573 - **De arte atque ratione navigandi**, *Pedro Nunes*

EXPOSITOR 3

ENSINO E USO PRÁTICO/MILITAR DA MATEMÁTICA ANTES/DEPOIS DA RESTAURAÇÃO

6 — 1634 - **Elementos Mathematicos**, *Ignacio Stafford*
7 — 1680 - **Methodo Lusitanico**, *Luis Serrão Pimentel*
8 — 1728 - **O Engenheiro Portuguez**, *Manuel de Azevedo Fortes*
9 — 1838 - **Carta Fisico-Mathematica** de 1769, *José Anastácio da Cunha*

EXPOSITOR 4

ENSINO NO COLÉGIO DOS NOBRES/UNIVERSIDADE/COLÉGIO DE SÃO LUCAS/ACADEMIAS MILITARES

10 — 1768 - **Elementos**, *Euclides*
11 — 1773 - **Elementos de Arithmetica**, *É. Bézout*
12 — 1790 - **Principios Mathematicos**, *José Anastácio da Cunha*
13 — 1798 - **Theorica das Funções Analyticas**, *J. L. Lagrange*
14 — 1812 - **Tratado Elementar de Calculo Differencial**, *S. F. Lacroix*

EXPOSITOR 5

SOBRE A UTILIDADE/FUNDAMENTO DAS CIÊNCIAS MATEMÁTICAS PARA A "GLÓRIA NÃO SER VÃ"

15 — 1797 - **Memórias da Academia Real das Sciencias de Lisboa**, Tomo 1
16 — 1851 - **História e Memórias da Academia R. das Sciencias de Lisboa**, vol.III
17 — 1867 - **Jornal de Sciencias Mathematicas Physicas e Naturaes**, Tomo I, nº3

EXPOSITOR 6

PUBLICAÇÕES PERIÓDICAS COM E DE ARTIGOS DE CIÊNCIAS MATEMÁTICAS

18 — 1856 - **O Instituto**, vol. 4, *Ensaio sobre Os Principios da Mechanica*, *J. Anastácio da Cunha*
19 — 1877 - **Jornal de Sciencias Mathematicas e Astronomicas**, vol. 1, publicado por F. Gomes Teixeira
20 — 1955 - **Revista da Faculdade de Ciências, Universidade de Lisboa**, 2ª série, A, vol. IV

EXPOSITOR 7

MANUAIS DE MATEMÁTICAS SUPERIORES NA TRANSIÇÃO DO SÉC. XIX PARA O XX

21 — 1838 - **Curso Completo de Mathematicas Puras**, tomo primeiro, *L.-B. Francoeur*
22— 1887 - **Curso de Analyse Infinitesimal**, *Francisco Gomes Teixeira*
23 — 1933 - **Curso de Álgebra Superior**, *José Vicente Gonçalves*
24 — 1973 - **Lições de Análise Infinitesimal**, II. Cálculo Integral em R^n, *Fernando R. Dias Agudo*

EXPOSITOR 8

DA COMPILAÇÃO HISTÓRICO-EXPOSITIVA, À DIVULGAÇÃO E À INVESTIGAÇÃO MATEMÁTICA

25 — 1909 - **Traité des Courbes Spéciales Remarquables**, Tomo II, *F. Gomes Teixeira*
26 — 1926 - **Fundamentos de Geometria Diferencial**, *Aureliano de Mira Fernandes*
27— 1941 - **Conceitos Fundamentais da Matemática**, vol.1, *Bento de Jesus Caraça*
28— 1946 - **Publicações do Centro de Estudos Matemáticos do Porto**, nº 18

EXPOSITOR 9

PERIÓDICOS DA SOCIEDADE PORTUGUESA DE MATEMÁTICA

29 — 1940 - **Portugaliae Mathematica**, vol.1, Fundada por António Monteiro
30 — 1937 - **Gazeta da Matemática**, 1º ano nº1
31 — 1950 - **As Funções Analíticas e a Análise Funcional**, *José Sebastião e Silva*
32 — 1987 - **Portugaliae Mathematica**, vol. 44, 1º composto em TeX

Tipografia
Matemática
Portuguesa
1496-1987

Figure 14.—The list of the 32 original books of the Leiria exhibition was organised in nine thematic topics, one per showcase: https://www.ipleiria.pt/sdoc/tipografia-matematica-portuguesa-1496-1987/
The online Catalogue, in Portuguese, can be read in
https://online.anyflip.com/yuqqw/btnw/mobile/index.html?1624885994427

Encontro Nacional da SPM

by **João Araújo*** and **Mário Bessa****

One of the aftermaths due to the pandemic crisis was the revocation of the Encontro Nacional da SPM (ENSPM) 2020 that should have been done in the beautiful city of Tomar, in July 2020. After advances and setbacks and numerous doubts faced by the newly elected directive board of the SPM, in September 2020, finally it was established that 2021 would not end without a remarkable ENSPM! Nevertheless, it was getting clearer that a face-to-face meeting was out of the question and SPM gathered all its strength, will and energy to make an unforgettable meeting with the presence of outstanding scientists. Despite the fact that the distance learning skills grew in each day it was clear that a meeting of this magnitude needed professional advice and so, avoiding at any cost, the emergence of murphy's law statement: "Anything that can go wrong will go wrong!". This was precisely the moment when most of the Mathematical Research Centers all over Portugal joined efforts providing an essential help in order to make our dream come true. One of these Centers was Centro Internacional de Matemática, which reinforced the already solid partnership with SPM. The meeting unfolded from 12-16 july 2021, with almost 800 participants and 19 Invited Parallel Sessions organized mainly by mathematicians working in the country and developing research with significant impact in the world. There were almost 50 Parallel Sessions proposals with a quite active participation of the national community, but also with a substantial interest from foreign researchers. Several courses, to be attended by secondary school teachers, made this meeting transversal to the entire community at all levels of education. Anyway, the cherry on top of the cake were the plenary talks where 18 world class scientists from a wide range of different areas of mathematics, from pedagogy, computer science and history of science, gifted us with amazing lectures. In overall, it will be forever in our memory the jingles before plenary talks, the graphic environment, the bouncy emojis emphasizing our excitement after a thrilling explanation and the feeling of closeness with personalities with huge impact in science nowadays treading the path of the future! In 2022 there will be a face-to-face ENSPM in Tomar, we will be together again but a seed was planted and the impact of the online meeting ENSPM 2021 certainly caused a paradigm change and things will no longer be the same.

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What's in a Circle Action?

by **Leonor Godinho***

Given a compact connected Lie group G and a closed manifold M it is natural to ask if M admits a nontrivial action of G and, if yes, how many different actions it can have. The existence of even the simplest case of a circle action already imposes strong restrictions on the topology of the manifold. We will explore some of these restrictions, illustrating how the simple existence of a circle symmetry already provides much information on the underlying manifold.

1 Introduction

Given a compact connected Lie group G and a closed manifold M it is natural to ask if M admits a nontrivial action of G and, if yes, how many different actions it can have. The existence of even the simplest case of a circle action already imposes strong restrictions on the topology of the manifold. For instance, in the 1970's, Petrie proved that if M is homotopy equivalent to a complex projective space and admits a circle action with isolated fixed points, then its Pontrjagin classes are determined by the representations at the fixed points [27]. Based on this, he formulated what is known as the *Petrie conjecture*: if M is homotopy equivalent to a complex projective space and admits a circle action with isolated fixed points then its Pontrjagin classes are the same as those of the projective space. This was proved in many situations [7, 13, 19, 24, 25, 33, 34] but it is still open in general.

If we consider an almost complex manifold, the existence of a circle action again restricts many topological (or almost complex) invariants of the manifold. A simple example is the Euler characteristic. Indeed, if M^{2n} admits a circle action with isolated fixed points that preserves the almost complex structure, then the number of fixed points coincides with the Euler characteristic of M [11, Section 3]. If, in addition, M is symplectic[1] and the action is Hamiltonian[2] then the fixed points are the critical points of the Hamiltonian function (a perfect Morse function) and so, as the Morse inequalities become equalities, the number of fixed points is equal to the sum of the even Betti numbers of M (all critical points have an even index). Since the classes $[\omega^k] \in H^{2k}(M;\mathbb{R})$ are non zero for $k = 0, \ldots, (\dim M)/2$, the number of fixed points, and consequently the Euler characteristic of M, is at least $n + 1$.

Another topological invariant of an almost complex manifold that is determined by the circle action is the Chern number $c_1 c_{n-1}[M]$, where $c_j \in H^{2j}(M;\mathbb{Z})$ is the degree-$2j$ Chern class of TM, for $j = 0, \ldots, n$. Salamon [30] showed that this Chern number can be obtained from the Hirzebruch genus[3] of M as

$$c_1 c_{n-1}[M] = 6 \left.\frac{d^2 \chi_y(M)}{dy^2}\right|_{y=-1} + \frac{n(5-3n)}{2}\, \chi_{-1}(M). \tag{1}$$

[1] A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed non-degenerate 2-form on M called a *symplectic form*.

[2] A symplectic circle action on (M, ω) is said to be Hamiltonian if there exists an S^1-invariant function $\psi : M \to \mathbb{R}$ (called the *moment map* or *Hamltonian function*) such that $d\psi = -\iota(\xi^\sharp)\omega$, where $\xi^\sharp$ is the vector field generated by the circle action.

[3] The Hirzebruch genus $\chi_y(M)$ is the genus corresponding to the power series $Q_y(x) = (x(1 + ye^{-x(1+y)}))/(1 - e^{-x(1+y)})$ (cf. [16]).

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Since the Hirzebruch genus is rigid for almost complex manifolds admitting a circle action, this Chern number is determined by a set of integers N_i defined by the action as follows:

THEOREM 1.— [11, Theorem 1.2] Let (M^{2n}, J) be a closed almost complex manifold with an S^1-action that preserves the almost complex structure J and has isolated fixed points. For every $i = 0, \dots, n$, let N_i be the number of fixed points with exactly i negative weights[4]. Then

$$c_1 c_{n-1}[M] = \sum_{i=0}^{n} N_i \left[6i(i-1) + \frac{5n - 3n^2}{2} \right]. \qquad (2)$$

In the following sections we will see some interesting applications of this result. The goal is to illustrate how the simple existence of a circle symmetry can provide so much information on the underlying manifold.

2 Possible weights for a circle action

The collection of possible weights for a circle action on an almost complex manifold must satisfy many strong conditions imposed, for instance, by the Localization Theorem in equivariant cohomology [2, 5]. Using (2) we can obtain additional linear relations through an algorithm constructed in [11].

These relations are very powerful. In particular, when M is symplectic of dimension smaller than 10 and the action is Hamiltonian with a **minimal** number of fixed points, it is possible to determine all families of weights, proving a symplectic generalization of the Petrie conjecture proposed by Tolman [32]:

CONJECTURE 1 (SYMPLECTIC PETRIE CONJECTURE).— If a symplectic manifold (M, ω) satisfying $H^{2i}(M; \mathbb{R}) = H^{2i}(\mathbb{CP}^n; \mathbb{R})$ for all i admits a Hamiltonian circle action, then $H^j(M; \mathbb{Z}) = H^j(\mathbb{CP}^n; \mathbb{Z})$ for all j. Moreover, the total Chern class $c(TM)$ is completely determined by the cohomology ring $H^*(M; \mathbb{Z})$.

In dimension 4, the weights obtained by the algorithm agree with those of the standard circle action on the complex projective plane, and so do the (equivariant) cohomology ring and Chern classes of the manifold.

In dimension 6 we recover previous results of Ahara [1] and Tolman [32].

THEOREM 2.— [32, Theorem 1] Let (M^6, ω) be a closed symplectic manifold with a Hamiltonian circle action with 4 fixed points. Then one of the following holds:

1. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^4)$, and $c(TM) = 1 + 4x + 6x^2 + 4x^3$;
2. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 2y, y^2)$, and $c(TM) = 1 + 3x + 8y + 4xy$;
3. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 5y, y^2)$, and $c(TM) = 1 + 2x + 12y + 4xy$;
4. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 22y, y^2)$, and $c(TM) = 1 + x + 24y + 4xy$;

(where, in all cases, x has degree 2 and y has degree 4).

In (1) the weights agree with those of the standard circle action on $\mathbb{CP}^3$. In (2) they agree with those of a circle action on the Grassmannian of oriented 2-planes[5] $Gr_2^+(\mathbb{R}^5)$ as a subgroup of $SO(5)$. In (3) and (4) they are the same as those of circle actions on the Fano manifolds V_5 and V_{22} [26].

In dimension 8 the algorithm yields the following result [11, 21].

THEOREM 3.— Let (M^8, ω) be a closed symplectic manifold with a Hamiltonian S^1-action with 5 fixed points. Then the weights agree with those of the standard circle action on $\mathbb{CP}^4$ as well as the cohomology ring and Chern classes, i.e.

$$H^*(M; \mathbb{Z}) = \mathbb{Z}[y]/(y^5) \quad \text{and} \quad c(TM) = (1 + y)^5,$$

where y has degree 2.

3 Lower bounds for the number of fixed points

Theorem 1 imposes several restrictions on the possible number of fixed points of a circle action. If M

[4] Given a fixed point $p_i \in M$, the S^1-representation on $T_{p_i}M$ is determined by a multiset of integers $\{w_{i1}, \dots, w_{in}\}$ called the weights of the action at p_i and we can equivariantly identify $T_{p_i}M$ with $\mathbb{C}^n$ with a circle action given by $\lambda \cdot (z_1, \dots, z_n) = (\lambda^{w_{i1}} z_1, \dots, \lambda^{w_{in}} z_n)$, for $\lambda \in S^1$.

[5] An $SO(5)$ coadjoint orbit.

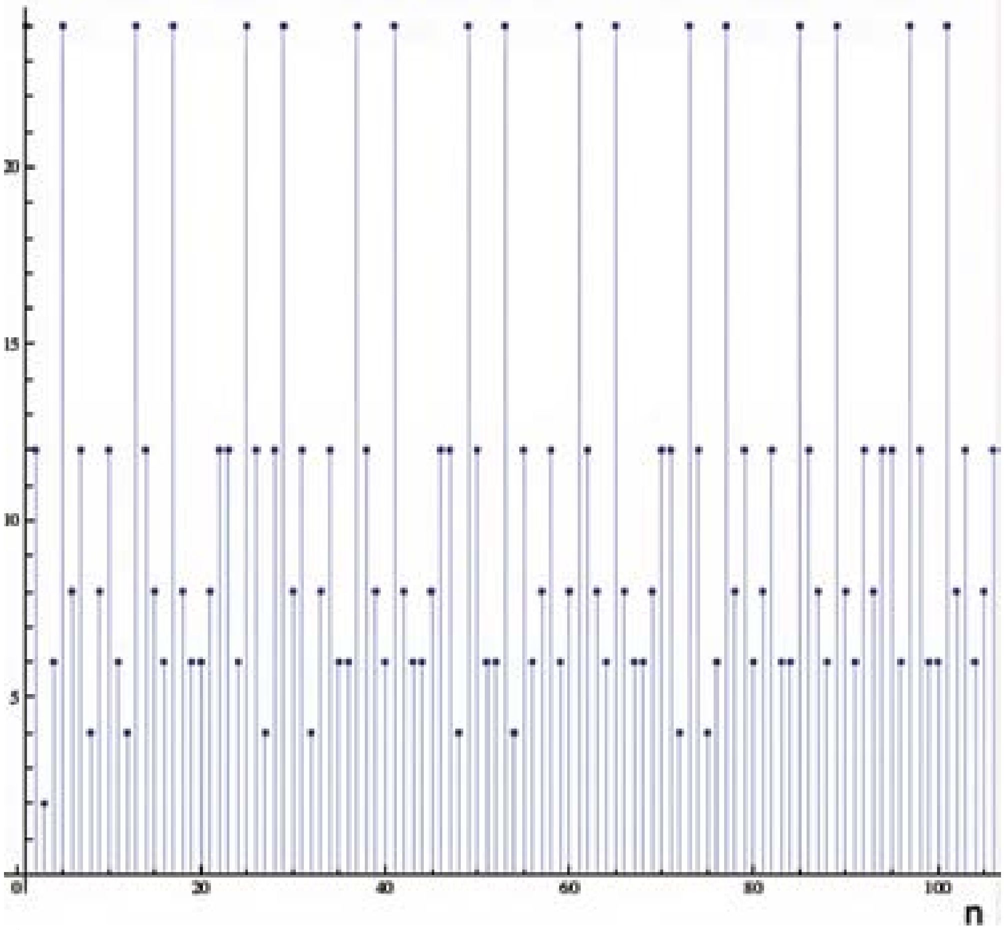

Figure 1.—Lower bounds for the number of fixed points, $n \le 100$.

is symplectic and the action is Hamiltonian we have seen that there are at least $n+1$ fixed points. Moreover, for general unitary S^1-manifolds[6] there exists an open conjecture stated by Kosniowsky [22].

Conjecture 2 (Kosniowski).— There exists a linear function f such that, for every $2n$-dimensional compact unitary S^1-manifold M with isolated fixed points which is not equivariantly unitary cobordant with the empty set, the number of fixed points is greater than $f(n)$. In particular, $f(x) = x/2$ should satisfy this condition, implying that the number of fixed points is expected to be at least $\lfloor n/2 \rfloor + 1$.

Using information from a non-vanishing Chern number, several lower bounds have been obtained (see for example [14, 29, 23, 6, 20]). If, on the other hand, we have $c_1c_{n-1}[M] = 0$, which is satisfied, for example by all symplectic Calabi-Yau manifolds, then Theorem 1 provides additional lower bounds (see Figure 1) in the case of an almost complex manifold and, in particular, for symplectic circle actions [12, Theorem B]. This requires the use of classical number theory results involving polygonal numbers originally stated by Fermat and proved later by Legendre, Lagrange, Euler, Gauss and Ewell [8, 9]. In some cases the lower bounds obtained are stronger than those conjectured by Kosniowski. However, our bounds are at most 24 in all dimensions, and so they do not provide evidence of a lower bound that depends linearly on the dimension, as proposed by Kosniowski.

Still when $c_1c_{n-1}[M] = 0$, Theorem 1 provides strong divisibility conditions that must be satisfied by the number of fixed points. These improve the existing divisibility results for the Euler characteristic of almost complex manifolds satisfying $c_1c_{n-1}[M] = 0$ obtained by Hirzebruch in [15], adding that the Euler characteristic must be divisible by 3, whenever the dimension of the manifold is not a multiple of 6.

Theorem 4.— [12, Theorem A] Let (M^{2n}, J) be a closed connected almost complex manifold equipped with a J-preserving circle action with nonempty, discrete fixed point set M^{S^1} and such that $c_1c_{n-1}[M] = 0$. Let m be such that $n = 2m$ $(m \ge 1)$ when n is even, and $n = 2m+3$ $(m \ge 1)$ when n is odd. If

[6] Unitary S^1-manifolds are smooth manifolds with a fixed S^1-invariant complex structure on the stable tangent bundle.

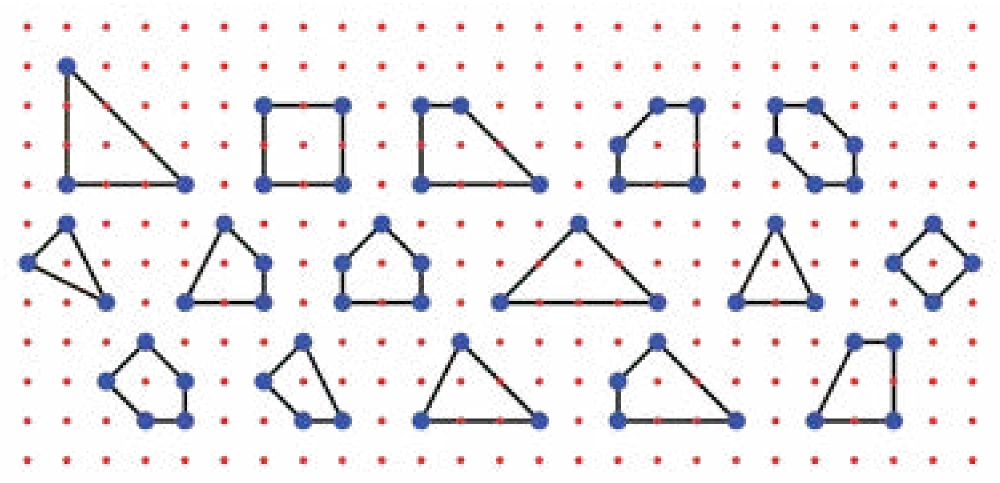

Figure 2.—Reflexive Polygons.

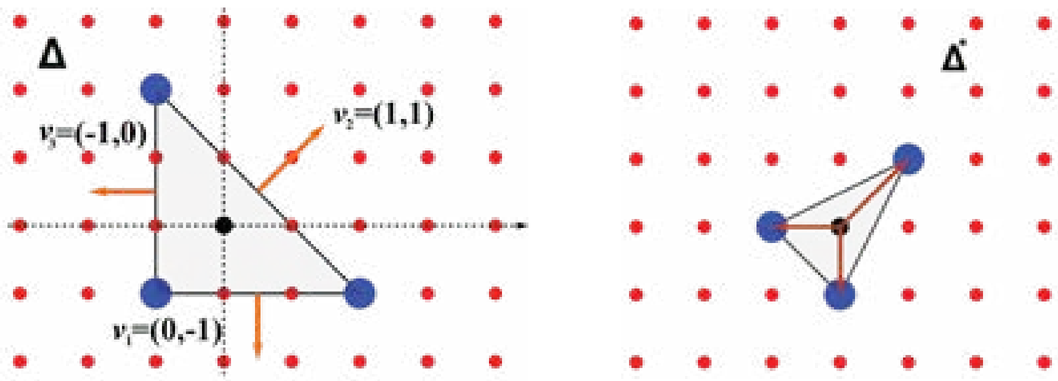

Figure 3.—A reflexive triangle Δ and its polar dual Δ*.

$r = \gcd(m, 12)$, then

$$|M^{S^1}| \equiv 0 \mod \frac{12}{r} \quad \text{if } n \text{ is even}$$

and

$$|M^{S^1}| \equiv 0 \mod \frac{24}{r} \quad \text{if } n \text{ is odd.}$$

If we restrict to Hamiltonian actions, keeping the hypothesis that $c_1 c_{n-1}[M] = 0$, we can improve the existing lower bound of $n + 1$.

Theorem 5.— [12, Theorem 2.8] Let M be a $2n$-dimensional closed connected symplectic manifold with $c_1 c_{n-1}[M] = 0$. Then the number of fixed points of a Hamiltonian circle action on M is at least

- $(n+1)(n+2)$, if n is even;
- $n^2 + 6n + 17 + \dfrac{24}{\gcd(\frac{n-3}{2}, 12)}$, if $n > 3$ is odd.

4 Reflexive polytopes

Another interesting application of Theorem 1 concerns reflexive polytopes. A polytope Δ is called *reflexive* if it is integral, contains the origin in the interior and can be written as

$$\Delta = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \le 1\},$$

where $v_i \in \mathbb{Z}^n$ are the primitive outward normal vectors to the hyperplanes supporting the facets of Δ (see Figure 2).

They were first defined by Batyrev [3], play an important role in mirror symmetry and satisfy many special combinatorial properties. For example, they have only one interior lattice point (the origin) and their polar duals are also reflexive. Moreover,

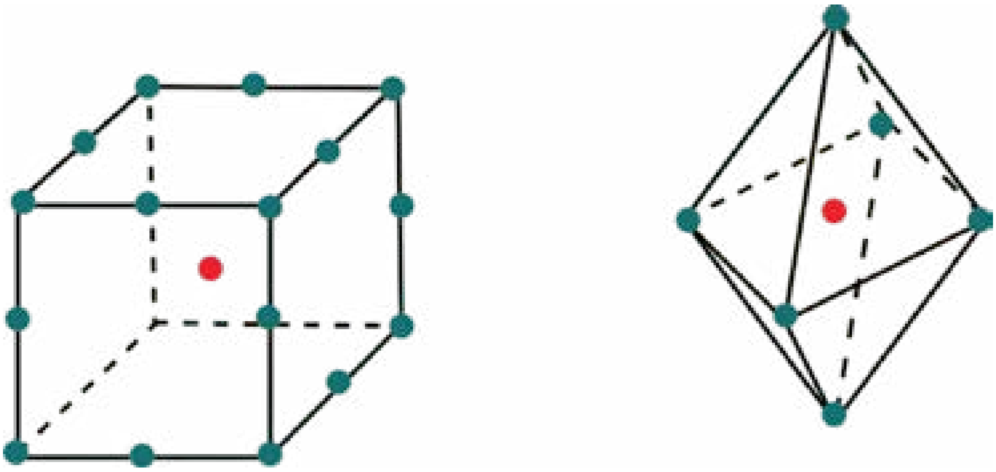

Figure 4.—The reflexive cube and its dual.

they satisfy the following property in dimensions 2 and 3, involving the relative (lattice) length of their edges and of their polar duals.

Theorem 6.— (12 and 24 -Theorem) Let Δ be a reflexive polytope of dimension n with edge set E.

- If $n = 2$ then
$$\sum_{e\in E} l(e) + \sum_{f\in E^*} l(f) = 12\,;$$
- If $n = 3$ then
$$\sum_{e\in E} l(e)l(e^*) = 24\,,$$

where E^* denotes the edge set of the dual polytope Δ^*, the edge $e^* \in E^*$ is dual to the edge $e \in E$ and $l(e)$ is the relative length of e.

One can prove this theorem in many ways. For example, since there exists only a finite number of reflexive polytopes in each dimension (up to a lattice isomorphism) one can prove it by exhaustion. In dimension two, there are other proofs [28, 17], involving modular forms, toric geometry and certain relations in $SL(2,\mathbb{Z})$. In dimension three, this result was obtained by Dais and Batyrev [4, Corollary 7.10] using toric geometry. A combinatorial proof is presented in [18, Section 5.1.2].

Surprisingly, we can use Theorem 1 to generalize Theorem 6 to all Delzant[7] reflexive polytopes, i.e. those arising as moment map images of closed symplectic toric manifolds (M,ω) with $c_1 = [\omega]$. In particular, we have the following result.

Theorem 7.— [10, Theorem 1.2] Let Δ be a Delzant reflexive polytope of dimension n with edge set E. Then
$$\sum_{e\in E} l(e) = 12 f_2 + (5-3n) f_1, \tag{3}$$
where f_k is the number of faces of Δ of dimension k.

For $n = 2$, the Delzant reflexive polygons are depicted in the first line of Figure 2. Moreover, in this dimension, Theorem 6 is equivalent to the property that the sum of the relative lengths of the edges of Δ and the number of vertices of Δ is always equal to 12 (see Fgure 2). Indeed, all the edges of the dual polygon Δ^* have length equal to 1 (as Δ is Delzant) and the number of edges of Δ^* is equal to the number of vertices of Δ. On the other hand, the relation in (3) tell us that the sum of the relative lengths of the edges of Δ is equal to $12 - f_1$ or, equivalently, to $12 - f_0$ (as the number of edges of a polygon is equal to the number of vertices) and so the two theorems agree.

When $n = 3$, the relation in (3) tells us that the sum of the integer lengths of the edges of a Delzant reflexive polytope is equal to $12f_2 - 4f_1$. Using the Euler relation $f_0 - f_1 + f_2 = 2$ and the fact that $3f_0 = 2f_1$ (as the polytope is simple), we obtain that this sum is always 24. This agrees with Theorem 6 since the length of every edge of the dual of a Delzant reflexive polytope is always 1 (see, for example, Figure 4 for the reflexive cube and its dual). Note that, in this dimension, the sum of the relative lengths of the edges of Δ has a nice geometric interpretation: it is the Euler characteristic of a Calabi Yau surface (a

[7] A polytope of dmension n is said to be *Delzant* if it is simple (each vertex is the intersection of exactly n edges), rational (the n edges that intersect at a vertex v are contained in affine lines of the form $v + \langle u_i \rangle$ with $u_i \in \mathbb{Z}^n$) and smooth (for each vertex, the edge vectors $u_i \in \mathbb{Z}^n$ can be chosen so that $\langle u_1, \ldots, u_n\rangle_{\mathbb{Z}} = \mathbb{Z}^n$).

$K3$ surface) obtained from Δ through a construction described, for example, in [3]. To prove Theorem 7 using Theorem 1, we consider the symplectic toric manifold (M_Δ, ω, ψ) corresponding to the Delzant polytope $\Delta = \psi(M_\Delta)$ (where ψ is the toric moment map) and the preimage $\mathcal{S} := \psi^{-1}(E)$ of the edge set. Then $\mathcal{S}$ is a union of smoothly embedded 2-spheres $\mathcal{S} = \cup_{e\in E} S_e^2$ and is Poincaré dual to c_{n-1}. Hence, we have

$$c_1 c_{n-1}[M_\Delta] = \sum_{S_e^2\in\mathcal{S}} c_1[S_e^2] = \sum_{S_e^2\in\mathcal{S}} [\omega]([S_e^2]) =$$
$$= \sum_{S_e^2\in\mathcal{S}} \mathrm{Vol}_\omega(S_e^2) = \sum_{e\in E} l(e),$$

and so this Chern number is the sum of the relative lengths of the edges of Δ. Taking a generic subcircle of the torus acting on M_Δ and the corresponding N_i (the number of fixed points of this circle action with exactly i negative weights which, in turn, is equal to the Betti number $b_{2i}(M_\Delta)$), and expressing the Betti numbers of M_Δ in terms of the face numbers of Δ [31], we obtain the relation in (3).

This result can also be proved without any symplectic or toric geometry, using only the combinatorial properties of Delzant reflexive polytopes (see [10] for details).

References

[1] K. Ahara, 6-dimensional almost complex S^1-manifolds with $\chi(M) = 4$, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.*, **38** (1991), 47–72.

[2] M. F. Atiyah and R. Bott, The moment map and equivariant cohomology, *Topology*, **23** (1984), 1–28.

[3] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535.

[4] V. Batyrev and D. I. Dais, Strong McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry, *Topology* **35** (1996), 901–929.

[5] N. Berline and M. Vergne, Classes caractéristiques équivariantes, formule de localisation en cohomologie équivariante *C.R. Acad. Sci. Paris* **295** (1982) 539–541.

[6] H. W. Cho, J. H. Kim, and H. C. Park, On the conjecture of Kosniowski, *Asian J. Math.* **16** (2012), 271–278.

[7] I. J. Dejter, Smooth S^1-manifolds in the homotopy type of $\mathbb{CP}^3$, *Michigan Math. J.* **23** (1976), 83–95.

[8] L. E. Dickson, *History of the Theory of Numbers*, Vol. 2, Chelsea, New York, 1952.

[9] J. A. Ewell, On sums of triangular numbers and sums of squares, *Amer. Math. Monthly*, **99** (1992) 752–757.

[10] L. Godinho, F. von Heymann and S. Sabatini, 12, 24 and beyond, *Adv. in Math.* 319 (2017), 473–521.

[11] L. Godinho and S. Sabatini, New tools for classifying Hamiltonian circle actions with isolated fixed points, *Found. Comput. Math.* **14** (2014), 791–860.

[12] L. Godinho, A. Pelayo and S. Sabatini, Fermat and the number of fixed points of periodic flows, *Commun. Number Theory Phys.*, **9** (2015), 643–687.

[13] A. Hattori, Spin^c-structures and S^1-actions, *Invent. Math.* **48** (1978), 7–31.

[14] A. Hattori, S^1-actions on unitary manifolds and quasi-ample line bundles, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.*, **31** (1984), 433–486.

[15] F. Hirzebruch, On the Euler characteristic of manifolds with $c_1 = 0$. A letter to V. Gritsenko, *Algebra i Analiz* **11** (1999), 126–129; translation in *St. Petersburg Math. J.* **11** (2000), 805–807.

[16] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics, E20, Vieweg, (1992).

[17] L. Hille and H. Skarke, Reflexive polytopes in dimension 2 and certain relation in $SL_2(\mathbb{Z})$, *J. Algebra Appl.* **1** (2002), 159–173.

[18] C. Haase, B. Nill and A. Paffenholz, Lecture Notes on Lattice polytopes, Preprint.

[19] D. M. James, Smooth S^1-actions on homotopy $\mathbb{CP}^4$'s, *Michigan Math. J.* **32** (1985), 259–266.

[20] D. Jang, Symplectic periodic flows with exactly three equilibrium points, *Ergodic Theory Dynam. Systems* **34** (2014), 1930–1963.

[21] D. Jang and S. Toman, Hamiltonian circle actions on eight-dimensional manifolds with minimal fixed sets, *Transf. Groups*, DOI: 10.1007/s00031-016-9370-0

[22] C. Kosniowski, C., *Some formulae and conjectures associated to circle actions*, Topology Symposium, Siegen 1979 (Prof. Symps., Univ. Siegen, 1979), pp 331–339. Lecture Notes in Math **788**, Springer, Berlin 1980.

[23] P. Li and K. Liu, Some remarks on circle actions on manifolds, *Math. Res. Letters*, **18** (2011), 435–446.

[24] M. Masuda, Integral weight system of torus actions on cohomology complex projective spaces, *Japan. J. Math.* **9** (1983), 55–86.

[25] O. R. Musin, Actions of the circle on homotopy complex projective spaces, *Mat. Zametki* **28** (1980), 139–152.

[26] D. McDuff, Some 6-dimensional Hamiltonian S^1-manifolds, *J. Topology* **2** (2009), 589–623.

[27] T. Petrie, Smooth S^1-actions on homotopy complex projective spaces and related topics, *Bull. Math. Soc.* **78** (1972), 105–153.

[28] B. Poonen and F. Rodriguez-Villegas, Lattice polygons and the number 12, *Am. Math. Mon.* **107** (2000), 238–250.

[29] A. Pelayo, A. and S. Tolman, Fixed points of symplectic periodic flows, *Erg. Theory and Dyn. Syst.* **31** (2011).

[30] S. M. Salamon, *Cohomology of Kähler manifolds with* $c_1 = 0$, Manifolds and geometry (Pisa, 1993), 294–310, Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge, (1996).

[31] R. Stanley, The number of faces of a simplicial convex polytope, *Adv. Math.* **35** (1980), 236–238.

[32] S. Tolman, On a symplectic generalization of Petrie's conjecture, *Trans. Amer. Math. Soc.* **362** (2010), 3963–3996.

[33] K. Wang, Differentiable circle group actions on homotopy complex projective spaces, *Math. Ann.* **214** (1975), 73–80.

[34] T. Yoshida, On smooth semifree S^1 actions on cohomology complex projective spaces, *Publ. Res. Inst. Math. Sci.* **11** (1976), 483–496.