Overview of Lie theory for Poisson brackets

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Abstract.—Poisson brackets are a central notion in a variety of topics related to mathematical-physics. In this note, we provide an overview of geometric and Lie theoretic aspects behind Poisson brackets and mention some recent ramifications and applications. In particular, we describe the notion of symplectic groupoids viewed as integrations of Poisson manifolds, explicit constructions of these integrations, the general theory of multiplicative structures on Lie groupoids, and applications to quantization and geometric numerical methods.

1 Introduction

Poisson brackets became a central notion in a variety of areas motivated by mathematical-physics (see e.g. the textbooks [30, 39] and more references therein). Their initial role in the description of mechanical systems (see [1, 4]) was latter complemented by their relevance in the connection between classical and quantum mechanics (see [5, 6, 39]), spreading into the corresponding modern mathematical theories. Presently, the applications of Poisson brackets range from more classical ones in Hamiltonian dynamics and integrable systems to symbol calculus and non-commutative algebra (e.g. [35, 47, 42, 45]), passing through topics like Lie theory and quantum groups (e.g. [38, 44, 46]).

This short overview is intended to provide a brief description of the *Lie theory associated with Poisson brackets* (see main results in [28, 29]) as well as to highlight some recent developments in the area, applications and ramifications. The central topics grow from the idea that, just as Lie algebras correspond to infinitesimal version of Lie groups, Poisson brackets correspond to the infinitesimal version of so-called *(local) symplectic groupoids*. This viewpoint is largely motivated by both the general study of the *Poisson geometry* behind these brackets and by the study of their *quantization*. We thus intend to summarize the main ideas related to this generalization of classical Lie theory and to mention some of their recent applications.

The structure of this paper is as follows. In Subsection 1.1 we recall the main definitions and features of Poisson brackets and of the underlying Poisson geometry. In Section 2, we provide an overview of the corresponding Lie theory in which the role played by transformation groups in classical Lie theory is now played by symplectic groupoids. We also include a summary of the Lie theory for general groupoids and algebroids (Section 2.1) as well as a description of explicit integration procedures (Section 2.3). Finally, we review some applications and ramifications in Section 3. In particular, we describe the general theory of multiplicative structures on Lie groupoids (Section 3.1), the link to the quantization of Poisson brackets (Section 3.2) and mention some recent work applying the general Lie theory to the construction of geometric numerical methods (Section 3.3).

1.1 Poisson brackets

Let us first give the basic definition and then comment on the underlying dynamical and geometric fea-

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tures. We observe that such general Poisson brackets can arise from more standard ones via quotient or reduction by symmetries, as well as by the presence of dynamical constraints, and that the axioms capture their most characteristic properties. (See more details in [1, 30].)

DEFINITION 1.— Let M be a smooth manifold. A *Poisson bracket* on M is a $\mathbb{R}$-bilinear operation

$$\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$$

such that: for any $f_1, f_2, f_3 \in C^\infty(M)$,

1. it is a derivation in each argument,
$$\{f_1, f_2 f_3\} = \{f_1, f_2\}f_3 + f_2\{f_1, f_3\},$$
and analogous for $\{f_1 f_2, f_3\}$;
2. it defines a *Lie bracket*, namely, it is skew symmetric $\{f_1, f_2\} = -\{f_2, f_1\}$ and satisfies the Jacobi identity
$$\{f_1,\{f_2, f_3\}\} = \{\{f_1, f_2\}, f_3\}+\{f_2,\{f_1, f_3\}\}.$$

A *Poisson map* $\varphi : (M_1, \{,\}_1) \to (M_2, \{,\}_2)$ is a smooth map that preserves brackets as follows,

$$\varphi^*\{f_1, f_2\}_2 = \{\varphi^* f_1, \varphi^* f_2\}_1,\ \forall f_1, f_2 \in C^\infty(M_2).$$

We now describe the significance of these axioms, starting with their role in Hamiltonian dynamics. The derivation property allows one to promote any function $H \in C^\infty(M)$, called the *Hamiltonian*, to an ODE on M given by the corresponding *Hamiltonian vector field* $X_H \in \mathfrak{X}(M)$. Seen as a derivation, X_H is defined by

$$X_H(f) = \{H, f\}.$$

It then follows that the evolution law for any *observable* given by a function $f \in C^\infty(M)$ along the flow φ_t^H of X_H is determined by the Poisson brackets,

$$\dot f := \frac{d}{dt}(f\circ\varphi_t^H) = \{H, f\}\circ\varphi_t^H.$$

The skew symmetry axiom thus implies the so-called *conservation of energy*, $\dot H = \{H, H\} = 0$, so that the solutions restrict to the level sets of H. The Jacobi identity implies that φ_t^H defines a Poisson autormorphism of $(M, \{,\})$ for any Hamiltonian $H \in C^\infty(M)$.

We end this section by discussing the underlying *Poisson geometry* of such $(M, \{,\})$. The derivation axiom together with skewsymmetry are equivalent to the fact that the brackets are determined by a tensor called *Poisson bivector*, $\pi \in \Gamma\Lambda^2 TM$, via

$$\{f_1, f_2\}|_x = \pi_x(df_1|_x, df_2|_x),$$

for $x \in M$ and $f_1, f_2 \in C^\infty(M)$.

One can think of π_x as a skew-symmetric bilinear form on each cotangent space T_x^*M, smoothly depending on $x \in M$. Hamiltonian vector fields are obtained as $X_H = \pi(dH,\cdot)$ so that they are in the range of the bilinear form π. The Jacobi identity is equivalent to a quadratic PDE for π which can be written as $[\pi,\pi] = 0$, for $[,]$ the Schoutens bracket extension of the Lie bracket of vector fields to the exterior algebra $\Gamma\Lambda^\bullet TM$. We shall call (M,π) a *Poisson manifold*.

REMARK 1.— (The symplectic case) When the bilinear form π_x is non-degenerate for every $x \in M$, the Poisson structure is called *symplectic*. In this case, the inverse bilinear form at each point defines a 2-form $\omega = \pi^{-1} \in \Omega^2(M)$ which is non-degenerate and closed, $d\omega = 0$. In this way, one recovers the usual description of symplectic structures as a particular case of Poisson structures. One important example is $M = \mathbb{R}^n \times \mathbb{R}^n \ni (q, p)$ with[1] $\omega = dq^i \wedge dp_i$. In this case, the Hamiltonian vector field X_H of $H(q, p) = |p|^2/2m + V(q)$ yields the standard Hamilton's equations in mechanics. This example generalizes to cotangent bundles $M = T^*Q$ of any manifold Q endowed with a *canonical symplectic structure* $\omega_c \in \Omega^2(T^*Q)$. (See more in [1, 4].)

A fundamental structural result in Poisson geometry is *Weinstein's splitting theorem*, [43]. It asserts that, locally around any point $x_0 \in M$, the Poisson structure is isomorphic to a product $(M_1,\pi_1)\times(M_2,\pi_2)$ with π_1 being non-degenerate (i.e. symplectic, as in the remark above) and with $\pi_2|_{x_0} = 0$. One consequence is that every Poisson manifold (M,π) has an underlying partition $M = \coprod L$ into (immersed) symplectic submanifolds (L,ω_L) corresponding to the symplectic factor in the local model. This determines a singular foliation with leaves of varying dimension (given by the rank of π at each point) which is called *the symplectic foliation* of (M,π). This foliation also admits a dynamical characterization: the leaf through $x \in M$ is given by the all the points that can be reached from x following a fininte number of Hamiltonian flows $\varphi_{t_j}^{H_j}$.

We finish by noting that there are more geometric structures induced by (M,π), see [30, 43]. The linearization of the degenerate local factor π_2 in Weinstein's splitting gives rise to a Lie algebra $\mathfrak{g}_x$ for each $x \in M$ called *isotropy Lie algebra*. These are isomorphic for points on the same symplectic leaf. Similarly,

[1] Sum over repeated indices is understood throughout this paper.

higher order terms of π_2 define a *transverse structure* to the symplectic foliation. In this way, Poisson geometry can be seen as the branch of differential geometry which studies such rich set of structures defined by (M, π). Some natural problems in this context are: linearization of the structure around a leaf and stability of leaves under deformation.

Example 1.— (A non-symplectic Poisson structure from rigid body dynamics) Let us consider $M = \mathbb{R}^3$ with the bracket $\{f_1, f_2\}|_x = (\nabla f_1|_x \times \nabla f_2|_x) \cdot x$, where the formula indicates the cross product of the gradients followed by inner product with $x \in \mathbb{R}^3$. By chosing $H(x)$ a suitable quadratic form (the "inertia tensor"), the corresponding Hamiltonian vector field X_H yields Euler's equation for the angular momentum of a free rigid body (see [1]). The symplectic foliation $M = \coprod S^2_r$ is given by spheres S^2_r with center at the origin $x = 0$ and radius r. Note that the origin itself is a singular leaf of dimension zero, while the rest are leaves of dimension 2 for $r > 0$. The symplectic structure ω_r on S^2_r is given by $1/r$ times the standard area form. The isotropy Lie algebra $\mathfrak{g}_0$ at the origin is given by infinitesimal rotations $\mathfrak{so}(3)$ while $\mathfrak{g}_x$ is one-dimensional for $x \neq 0$. From this geometric description, it follows that typical trajectories of Euler's equation X_H are given by the intersection of a sphere S^2_r and the ellipsoid $H^{-1}(H(x(t_0)))$. This Poisson structure generalizes to $M = \mathfrak{g}^*$, for any Lie algebra $\mathfrak{g}$, endowed with a naturally induced *linear Poisson structure* $\pi_x = (1/2)x([e^i, e^j])\partial_{x^i} \wedge \partial_{x^j}$, for (e^i) a basis for $\mathfrak{g}$ and $x \in \mathfrak{g}^*$ with dual linear coordinates (x^i). The present example corresponds to the three dimensional $\mathfrak{g} = \mathfrak{so}(3)$.

Other interesting examples come from *Poisson-Lie groups* (G, π) in which a Lie group G is endowed with a compatible Poisson structure. These arise as classical limits of so-called quantum groups (see [39, 44] and references therein). Other examples include: duals of Lie algebroids, Poisson homogeneous spaces, polynomial Poisson structures on vector spaces and Poisson groupoids, among others.

2 Symplectic groupoids as integrations

We now move in the direction of describing the Lie theory behind Poisson brackets, see [27, 38, 44]. Let us first recall some facts from classical Lie theory for latter comparison, see [32] for more details. If $G \subset Gl(n)$ is a matrix Lie group and g_t is a smooth curve through the identity $1 \in G$ at $t = 0$, then

$$g_t = 1 + t\xi + O(t^2)$$

with the possible first order parameters $\xi \in \mathfrak{g} \subset \mathfrak{gl}(n)$ spanning the underlying Lie algebra. These ideas define the differentiation functor Lie from Lie groups to Lie algebras. Conversely, the main results in classical Lie theory say that the infinitesimal parameters $\mathfrak{g}$ define a connected Lie group $G \in Lie^{-1}(\mathfrak{g})$ uniquely up to covering and isomorphism; when G is 1-connected, Lie algebra morphisms $\mathfrak{g} \to Lie(G')$ integrate uniquely to Lie group morphisms $G \to G'$. Lastly, the restriction of a Lie group to a neighborhood $U \subset G$ of the identity can be intrinsically characterized as a *local* Lie group. The Lie functor factors through any such restriction and the structure of U can be thought of as arising from small enough parameters $\xi \sim 0 \in \mathfrak{g}$. Given a Lie algebra $\mathfrak{g}$, one can construct a natural local Lie group structure integrating $\mathfrak{g}$ on a neighborhood $U \subset \mathfrak{g}$ of 0 via the Baker-Campbell-Hausdorff formula.

2.1 Lie groupoids and algebroids

We now review the general notions of Lie groupoids and algebroids, see [41]. A *groupoid* is an algebraic structure, denoted $\mathcal{G} \rightrightarrows M$, which can be described as the data of a category, with all morphisms given by a set $\mathcal{G}$ and all objects given by a set M, and with the special property that every morphism is invertible. The two arrows in the notation correspond to the source and target maps $s, t : \mathcal{G} \to M$, respectively, so that we think of $g \in \mathcal{G}$ as an arrow between the corresponding objects in M,

$$y = t(g) \xleftarrow{g} x = s(g).$$

We adopt the following notational convention for composition of morphisms or "groupoid multiplication" in $\mathcal{G}$,

$$m : \mathcal{G}^{(2)} = \{(g_1, g_2) : s(g_1) = t(g_2)\} \to \mathcal{G},$$

$$(g_1, g_2) \mapsto m(g_1, g_2) \equiv g_1 g_2.$$

In this way, $\mathcal{G} \rightrightarrows M$ can be seen as an equivalence relation $y \sim x$ between the objects $y, x \in M$ which has been enhanced by the information of a set $\mathcal{G}_{yx} = t^{-1}(y) \cap s^{-1}(x)$ parameterizing different possible ways in which y is equivalent to x. The underlying equivalence class L_x of x is called the *orbit* through $x \in M$, and autoequivalences $\mathcal{G}_{xx}$ is called

[2]To incorporate interesting examples coming from foliations, it is customary to allow $\mathcal{G}$ to be a non-Hausdorff manifold but with s-fibers being

the *isotropy* at x.

A *Lie groupoid* is an algebraic groupoid $\mathcal{G} \rightrightarrows M$ together with given smooth manifold structures[2] on $\mathcal{G}$ and M such that s, t are smooth submersions (so that $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ has a natural manifold structure) and that all the other category structure maps are smooth. A *morphism of Lie groupoids* is a functor $F : (\mathcal{G}_1 \rightrightarrows M_1) \to (\mathcal{G}_2 \rightrightarrows M_2)$ which is also a smooth map $\mathcal{G}_1 \to \mathcal{G}_2$. Orbits of $\mathcal{G} \rightrightarrows M$ define a singular foliation $M = \coprod L$ by (immersed) submanifolds of varying dimensions and isotropies $\mathcal{G}_{xx}$ define a Lie group for each $x \in M$.

Some typical examples of Lie groupoids are the following. The pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$ in which arrows $g = (y, x)$ are completely determined by their source and target, and composition is defined in the obvious way for the underlying equivalence relation to be transitive. A Lie group G can be seen as a Lie groupoid $\mathcal{G} = G \rightrightarrows *$ over an abstract 1-point set; this groupoid is pure isotropy. The fundamental groupoid $\mathcal{G} = \Pi_1(M) \rightrightarrows M$ is given by paths in the manifold M modulo homotopies with fixed endpoints and composition is given by composition of paths. The orbits are connected components of M and isotropies $\mathcal{G}_{xx} = \pi_1(M, x)$ are the fundamental groups at different points. Other examples include: holonomy groupoid of a foliation in M, total spaces of vector bundles, the "gauge" groupoid associated to a principal bundle $P \to M$; among others.

We now describe briefly the infinitesimal version of a Lie groupoid $\mathcal{G} \rightrightarrows M$, namely, its underlying *Lie algebroid*. Following what we recalled from classical Lie theory for groups, if we consider a smooth curve $g_t \in \mathcal{G}$ through an identity arrow 1_x, $x \in M$, and with fixed source $s(g_t) = x$, we will have

$$g_t = 1_x + t\xi_x + O(t^2) \qquad (1)$$

from which we could identify the possible first order parameters ξ_x in a linear space $A_x \subset T_{1(x)}\mathcal{G}$ normal to $1(M)$. By analogy with the classical case, suitably linearizing the structure of $\mathcal{G}$ one should arrive to the relevant Lie algebroid structure on $A = \coprod_x A_x$. Indeed, this procedure works as expected and one is led to the following definition: a Lie algebroid $(A \to M, \rho, [,])$ is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$, called the *anchor*, and a Lie bracket $[,]$ on the space of sections ΓA such that the following derivation property holds $[a, fb] = f[a, b] + (L_{\rho(a)}f)b$, $a, b \in \Gamma A$, $f \in C^\infty(M)$.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, the corresponding Lie algebroid $Lie(\mathcal{G}) = (A \to M, \rho, [,])$ is defined by $A = Ker(Ds)|_{1(M)}$, $\rho = Dt$ and $[,]$ determined by the identification of sections of A with the Lie algebra of right-invariant vector fields on $\mathcal{G}$. When $Lie(\mathcal{G}) \simeq A$, we say that $\mathcal{G} \rightrightarrows M$ *integrates* the Lie algebroid A. Two of the key features of classical Lie theory generalize directly to this setting: when $\mathcal{G} \rightrightarrows M$ has connected source fibers, the algebroid $A = Lie(\mathcal{G})$ determines $\mathcal{G} \rightrightarrows M$ uniquely up to covering of its source-fibers and isomorphism; when the source-fibers of $\mathcal{G}$ are 1-connected, algebroid morphisms $A \to Lie(\mathcal{G}')$ integrate uniquely to Lie groupoid morphisms $\mathcal{G} \to \mathcal{G}'$. The generalized theory deviates, though, from the classical one in terms of integrability, as remarked next.

Remark 2.— (Integrability and local Lie groupoids) As opposed to classical Lie theory, not every Lie algebroid comes from some Lie groupoid. When it does, we say that A is *integrable*. The general theory of obstructions to integrability was developped in [28]. On the other hand, every Lie algebroid can be integrated by a *local* Lie groupoid. Local Lie groupoids correspond to the intrinsic structure inherited by neighborhoods $U \subset \mathcal{G}$ of the identities $1(M)$, just as local Lie groups are to neighborhoods of $1 \in G$. This can be formalized (see [18, 27]) by considering that each structure map has a domain given by a neighborhood containing the underlying identities and each axiom also has an open domain where it holds. Given a Lie algebroid A, its integrability admits the following alternative characterizations: suitable discreteness of underlying monodromy groups ([28]), by the existence of complete actions ([28, 2]), and by complete associativity properties of the associated germ of local Lie groupoids ([34]).

We mention some examples: $Lie(M \times M \rightrightarrows M) = Lie(\Pi_1(M) \rightrightarrows M)$ is given by the *tangent algebroid* $A = TM$ with $\rho = id$ and $[,]$ the standard Lie bracket of vector fields; when $\mathcal{G} = G \rightrightarrows *$ is a Lie group then $Lie(\mathcal{G}) = Lie(G) = \mathfrak{g}$ is the standard Lie algebra seen as a bundle over an abstract point. Other examples include an involutive regular distribution $D \subset TM$ and the gauge (or "Atiyah") algebroid TP/G associated with a principal G-bundle P. We end this subsection with the most relevant case for our purposes: every Poisson manifold $(M, \{,\} \equiv \pi)$ determines a *cotangent Lie algebroid* structure de-

Hausdorff, see [30, 41].

noted $T^*_\pi M$ on $A = T^*M \to M$. The anchor map is determined by $\rho(dH) = X_H$, the Hamiltonian vector field, and the bracket is uniquely defined by $[df_1, df_2] = d\{f_1, f_2\}$.

2.2 Symplectic groupoids and Poisson brackets

In this subsection, we shall see that Poisson brackets can be seen as the infinitesimal structure behind so-called symplectic groupoids, see [27, 38, 41, 44, 46]. Indeed, we will also see that these are the integration of the Lie algebroid $T^*_\pi M$ associated with a Poisson manifold $(M, \{,\} \equiv \pi)$.

Definition 2.— A *symplectic groupoid* $(\mathcal{G} \rightrightarrows M, \omega)$ is defined by a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a symplectic form $\omega \in \Omega^2(\mathcal{G})$ satisfying the following compatibility condition,

$$m^*\omega = pr_1^*\omega + pr_2^*\omega, \qquad (2)$$

where $m, pr_1, pr_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ denote the multiplication map and the two projections $(g_1, g_2) \mapsto g_j$, for $j = 1, 2$, respectivelly.

The compatibility condition can be understood by saying that ω defines a 1-cocycle in a suitable complex and any form $\omega \in \Omega^k(\mathcal{G})$ satisfying the above equation is called *multiplicative* (see [7]). Moreover, the above equation for a symplectic 2-form turns out to be equivalent to the fact that the graph of the multiplication map m, given by triples $(g_1, g_2, m(g_1, g_2))$ with $(g_1, g_2) \in \mathcal{G}^{(2)}$, defines a Lagrangian submanifold[3] in $(\mathcal{G}, -\omega) \times (\mathcal{G}, -\omega) \times (\mathcal{G}, \omega)$, see [27].

Let us summarize some key properties of a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \omega)$. The identities

$$1 : M \hookrightarrow (\mathcal{G}, \omega)$$

define a Lagrangian submanifold and, thus, $\dim(\mathcal{G}) = 2\dim(M)$. The inversion map $\mathrm{inv} : \mathcal{G} \to \mathcal{G}$, given by $g \mapsto g^{-1}$ is anti-symplectic, $\mathrm{inv}^*\omega = -\omega$.

More importantly, there exists a unique Poisson structure $\{,\} \equiv \pi$ on M such that the source map $s : (\mathcal{G}, \omega) \to (M, \pi)$ is a Poisson morphism. We say that $(\mathcal{G} \rightrightarrows M, \omega)$ *integrates* the underlying Poisson manifold (M, π) and that s defines a *symplectic realization* of (M, π). We also observe that the target map is automatically anti-Poisson, namely, a Poisson map of the form $t : (\mathcal{G}, \omega) \to (M, -\pi)$.

In terms of the Lie theory for groupoids and algebroid recalled in Section 2.1, one can show that when $(\mathcal{G} \rightrightarrows M, \omega)$ integrates (M, π) in the sense above, then $\mathcal{G} \rightrightarrows M$ integrates the cotangent Lie algebroid $T^*_\pi M$ associated with (M, π), namely, $Lie(\mathcal{G}) \simeq T^*_\pi M$ naturally as Lie algebroids. Conversely, given (M, π) Poisson, if a Lie groupoid $\mathcal{G} \rightrightarrows M$ integrates $T^*_\pi M$ and its source fibers are 1-connected, there exists a unique symplectic $\omega \in \Omega^2(\mathcal{G})$ turning $(\mathcal{G} \rightrightarrows M, \omega)$ into a symplectic groupoid integrating (M, π) (see also Section 3.1 below). In this way, the two notions of integration introduced above coincide.

Let us also observe that the symplectic geometry and the groupoid structure are deeply interconnected. For any $f \in C^\infty(M)$ the symplectic Hamiltonian flow $\phi_t^{s^*f}$ of s^*f in $(\mathcal{G}, \omega)$ is left-invariant for m,

$$\phi_t^{s^*f}(g) = m(g, \phi_t^{s^*f}(1(s(g))).$$

In particular, one can deduce that the *symplectic realization data* given by $(\mathcal{G}, \omega, s, 1)$ completely determines the germ of the other groupoid structure maps around units [27]. In terms of families of arrows $g_t \in \mathcal{G}$ of eq. 1 above, we have that the first order generating parameters are, in this case, covectors $\xi_x \in T^*_x M$ on M. We can understand these as a point-based, finite-dimensional version of the 1-forms dH which integrate to the flows φ_t^H of the Hamiltonian vector fields $X_H = \pi(dH, \cdot)$ and which generate a complicated infinite dimensional group. Thus, working with group*oids* instead of with groups can be seen as a structural price paid in exchange of a finite dimensional treatment of Hamiltonian flows.

Some examples are given as follows. When $\pi = 0$, then $\mathcal{G} = T^*M$ with $s = t = q : T^*M \to M$ the bundle projection and $m(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2$ defines a Lie groupoid which, together with the standard symplectic structure $\omega_c \in \Omega^2(T^*M)$, is such that $(T^*M \rightrightarrows M, \omega_c)$ defines a symplectic groupoid integrating $(M, \pi = 0)$. If $(M, \pi = \omega_M^{-1})$ is already symplectic then the pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$ endowed with $\omega = (-\omega_M) \times \omega_M$ defines a symplectic groupoid integrating (M, ω_M^{-1}). The Poisson manifold $(M = \mathbb{R}^3, \pi)$ of Example 1 is integrated by a symplectic groupoid $(\mathcal{G} = T^*SO(3) \rightrightarrows \mathfrak{so}(3)^* \simeq \mathbb{R}^3, \omega_c)$ where ω_c is the canonical symplectic structure and the underlying groupoid $T^*SO(3) \rightrightarrows \mathfrak{so}(3)^*$ is given by a so-called *cotangent lift* of the Lie group structure on $SO(3)$, see [41]. This example generalizes to $(T^*G \rightrightarrows \mathfrak{g}^*, \omega_c)$ integrating the linear Poisson manifold $(M = \mathfrak{g}^*, \pi)$ of that example associated with any Lie group G. For Poisson-Lie groups (G, π), the integration is given by so-called Drinfeld double symplec-

[3] Let us recall that a Lagrangian submanifold $i : \Sigma \hookrightarrow (S, \omega)$ in a symplectic manifold (S, ω) is a submanifold such that $i^*\omega = 0$ and satisfies the maximality condition $dim(\Sigma) = dim(S)/2$.

tic groupoids (see e.g. [41]) generalizing the previous case.

Remark 3.— (Integrability and local integrations in the Poisson case) Not every Poisson manifold is integrable by a symplectic groupoid, see the specialized theory of integrability in [29]. A simple example is given by a minor modification of Example 1 on $M = \mathbb{R}^3$, $\{f_1, f_2\}^a|_x := a(|x|)(\nabla f_1|_x \times \nabla f_2|_x) \cdot x$ where the original bracket has been multiplied by a function $a(r) > 0$ of the radius $r = |x|$. Simple choices of a lead to non-integrable brackets $\{,\}^a$, see [29, Sec. 7.2]. In general, one can verify that (M, π) is integrable by a symplectic groupoid iff the cotangent algebroid $T^*_\pi M$ is integrable by a Lie groupoid (see Section 3.1 below). On the other hand, every Poisson manifold is integrable to a *local* symplectic groupoid, see [19, 27]. Namely, a local Lie groupoid $U \rightrightarrows M$ as in Remark 2 together with a symplectic form $\omega \in \Omega^2(U)$ which satisfies the multiplicativity condition locally around units $(1(x), 1(x)) \in U^{(2)}$. The main properties of symplectic groupoids still hold in the local case and, in particular, such a local symplectic groupoid $(U \rightrightarrows M, \omega)$ induces a unique Poisson structure on M which is then said to be integrated by the former.

2.3 Explicit constructions

We finish this section by outlining explicit constructions of (local) symplectic groupoids $(U \rightrightarrows M, \omega)$ for a given Poisson manifold (M, π). The more concrete ones produce local integrations in the sense of Remark 3 above.

We first mention a more abstract yet universal and global construction based on hommotopy classes of adapted paths, see [28, 29, 24]. A *cotangent path* is a path $\alpha : I = [0,1] \to T^*M$ over $x : I \to M$ such that

$$\dot{x}(t) = \pi(\alpha(t), \cdot)|_{x(t)} \in T_{x(t)}M.$$

One can understand this equation as the map $TI \to T^*_\pi M, (t, \partial_t) \mapsto \alpha(t)$ defining a Lie algebroid morphism from the tangent algebroid of the interval I. The idea is that the covector $\alpha(t)$ guides the base path $x(t)$ through the Poisson structure. A *cotangent homotopy* with fixed endpoints is defined as a Lie algebroid morphism $T(I \times I) \to T^*_\pi M$ with appropriate boundary conditions. Then, one considers $W(M) = \{\alpha : TI \to T^*_\pi M\}/\sim \rightrightarrows M$ the space of cotangent paths modulo cotangent homotopies, which inherits a groupoid structure from composition of paths, called the *Weinstein groupoid* associated with (M, π). In general, it is just a topological groupoid but one can show the highly non-trivial result that (M, π) is integrable iff $W(M) \rightrightarrows M$ inherits a *Lie* groupoid structure, see [28, 29]. In this case, $W(M)$ has 1-connected source fibers. Moreover, it inherits a symplectic structure $\omega \in \Omega^2(W(M))$ via symplectic reduction from the canonical one on the space of all paths $PT^*M := \{I \to T^*M\}$ (see [29, 24]) and, endowed with it, $(W(M) \rightrightarrows M, \omega)$ integrates the original Poisson manifold (M, π). This construction provides a sort of universal answer to the integration problem but it is rather abstract and hard to construct in concrete cases.

We now move on to more explicit constructions intended to yield *local* symplectic groupoids integrating a given (M, π). These can be understood as *gauge fixing* slices $U \subset T^*M \hookrightarrow PT^*M$ which select particular cotangent paths $\alpha(t)$ in the corresponding homotopy classes via an auxiliary choice (of a *spray*, see below). The resulting structure is only local since these choices only work for *small* paths and break down for *large* ones.

The key auxiliary piece of data is called *Poisson spray* (see [31, 19]). It is given by an ODE on the total space of T^*M, namely, a vector field $V \in \mathfrak{X}(T^*M)$ such that

$$Dq(V|_\alpha) = \pi(\alpha, \cdot) \in T_{q(\alpha)}M, \ \alpha \in T^*M,$$

with $q : T^*M \to M$ the bundle projection, and that it is homogeneous of degree 1, $m^*_\lambda V = \lambda V$ for $m_\lambda : T^*M \to T^*M$ fiber multiplication by $\lambda \in \mathbb{R}_+$. Note that the first condition is the infinitesimal version of the cotangent path equation, so that trajectories of V are indeed cotangent paths. The 1-homogeneous condition implies that, in a chart of canonical coordinates (x^j, p_j), we have $V = \pi^{ij}(x)p_i\partial_{x^j} + \Gamma^{ij}_k(x)p_ip_j\partial_{p_k}$ with $\Gamma^{ij}_k(x)$ defining the Christoffel symbols of a *contravariant connection* ∇ for $T^*_\pi M$ on T^*M without torsion (see [33]). In these coordinates, the ODE defined by V for $\alpha = (x, p)$ imply that $p(t)$ is ∇-contravariantly constant along $x(t) \in M$. One can think of V as a contravariant analogue of a Riemannian spray in $\mathfrak{X}(TM)$ whose trajectories are the underlying Riemannian geodesics.

Given a Poisson spray V for (M, π) there is an associated local symplectic groupoid $(\mathcal{G}_V \rightrightarrows M, \omega_V)$ called *spray groupoid* integrating (M, π), [18, 19]. The symplectic structure was introduced in [31] and is given by an average along the flow of V of the canonical $\omega_c \in \Omega^2(T^*M)$,

$$\omega_V = \int_0^1 (\phi^V_t)^* \omega_c \, dt.$$

It is well defined and symplectic on a neighborhood $U_\omega \subset T^*M$ of 0_M. We now describe the structure maps of $\mathcal{G}_V \rightrightarrows M$ recalling that, for local groupoids, each structure map has a domain of definition including the corresponding identity arrows. The identity for $x \in M$ is given by the zero covector $0_x \in T^*M$ and, hence, the structure of $\mathcal{G}_V$ will be supported in appropriate opens in T^*M. The source map $s = q : T^*M \to M$ is the bundle projection while the target map is $t = q \circ \phi^V_{t=1} : U_{tar} \subset T^*M \to M$ with ϕ^V_t being the flow of the spray V on T^*M. Notice that, since V vanishes on the zero section $0_M \subset T^*M$, there exists a neighborhood $U_1 \subset T^*M$ of 0_M on which the flow is defined up to $t = 1$ and which we can shrink as needed for the domain of other structure maps. The inverse of the local groupoid is given by $inv(\alpha) = \phi^V_{t=-1}(-\alpha)$ for α close enough to zero. Finally, the multiplication map is defined by $m(\alpha_1, \alpha_2) = \beta(t = 1)$, where $\beta(t) \in T^*M$ is the solution to the ODE with initial value:

$$\omega_V(\dot\beta(t), \cdot) = (q \circ \phi^V_{t=1})^* \phi^V_t(\alpha_1),\ \beta(0) = \alpha_2.$$

The map m is defined for α_1, α_2 satisfying the composability condition $s(\alpha_1) = t(\alpha_2)$ and close enough to the zero section, for which the above ODE has solution defined up to $t = 1$. The fact that these maps satisfy the axioms of a local groupoid is [18, Thm. 3.8] and the fact that it integrates (M, π) can be found in [31]. More details in the present Poisson case can be found in [19].

When M is a coordinate domain, one can provide even more explicit formulas for its integration.

Remark 4.— (Coordinate M and a generating function) Let $M \subset \mathbb{R}^n$ be an open subset with coordinates x^j. We denote $M^* = (\mathbb{R}^n)^*$ the corresponding dual vector space with dual coordinates p_j. Let us consider the flat contravariant connection ∇ so that $\Gamma^{ij}_k = 0$ in the formula for the spray V. One can show (see [11, 13]) that $(\mathcal{G}_V \rightrightarrows M, \omega_V)$ is isomorphic to a local symplectic groupoid $(\mathcal{G}_\pi \rightrightarrows M, \omega_c)$ with the same units $1_x = 0_x = (x, 0) \in T^*M = M \times M^*$ but with canonical symplectic ω_c. Moreover, all the other structure maps are determined by a *generating function* (see [11, 21])

$$S : U_S \subset M^* \times M^* \times M \to \mathbb{R}, (p_1, p_2, x) \mapsto S(p_1, p_2, x)$$

where U_S is a neighborhood of $\{(p_1 = 0, p_2 = 0, x) : x \in M\}$. The source and target maps are

$$s(x, p) = \partial_{p_2} S(p, 0, x),\ t(x, p) = \partial_{p_1} S(0, p, x)$$

the inverse is $inv(x, p) = (x, -p)$, and the multiplication is defined by the relation

$$m((x_1, p_1), (x_2, p_2)) = \\ = (x_3, p_3 = \partial_x S(p_1, p_2, x_3)), \\ \text{with } x_j = \partial_{p_j} S(p_1, p_2, x_3),\ j = 1, 2.$$

This relation can be understood as the fact that the Lagrangian $graph(m) \hookrightarrow T^*M \times T^*M \times T^*M \simeq T^*M^3$ is suitably given as the graph of the exact 1-form dS with respect to the projection $T^*M^3 \to M^* \times M^* \times M$. Moreover, one can give formulas for $S(p_1, p_2, x)$ in terms of Hamiltonian flows in T^*M, see [11, Sec. 3.4]. We observe that for the linear Poisson manifold $M = \mathfrak{g}^*$ of Example 1 then S recovers the Baker-Campbell-Hausdorff formula for any Lie algebra $\mathfrak{g}$; we then think of S as a non-linear analogue of this classical formula. For the source map, one can also provide the following implicit definition due to Karasev (see [38, 39])

$$\int_0^1 \phi^V_t(s(x, p), p)\, dt = x$$

for all $p \sim 0$ small enough and with V being the flat spray above. From this, one can deduce (see [13]) a universal Butcher-type Taylor expansion in terms of rooted trees which defines a universal *symplectic realization* $s : (U_s \subset T^*M, \omega_c) \to (M, \pi)$ for any such coordinate Poisson manifold (M, π).

3 Applications and ramifications

In this last section, we outline some relatively recent developments in research lines which grow out of the general Poisson geometry realm of the previous sections.

3.1 Lie theory for multiplicative structures

We have seen that the Lie theoretic integration of Poisson brackets on M consist of a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a compatible symplectic structure $\omega \in \Omega^2(\mathcal{G})$ in its space of arrows. Remarkably, this situation generalizes to other types of geometric structures on M which are integrated by a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a compatible and non-degenerate version of the same geometry. In general, this integration $\mathcal{G} \rightrightarrows M$ serves as a powerful tool to study the original geometry on M. Here, we shall review some results of the general formalism behind this phenomenon via the description of *Lie theory for multiplicative forms on Lie groupoids* following [7].

As already mentioned, a *multiplicative k-form* $\omega \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a form which satisfies the cocycle condition in eq. 2. It will be interesting shortly to observe that eq. 2 is equivalent to the fact that the induced map

$$\overline{\omega} : \underbrace{T\mathcal{G} \oplus_{\mathcal{G}} T\mathcal{G} \oplus_{\mathcal{G}} \cdots \oplus_{\mathcal{G}} T\mathcal{G}}_{k} \to \mathbb{R},$$

$$(X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k)$$

defines a Lie groupoid morphism with respect to the direct sum of the *tangent lifted Lie groupoid structure* $T\mathcal{G} \rightrightarrows TM$ (see [41]) and to $\mathbb{R} \rightrightarrows *$ seen as a group. We denote $\Omega^k_M(\mathcal{G})$ the space of multiplicative k-forms.

Let $A \to M$ be the Lie algebroid of $\mathcal{G} \rightrightarrows M$. The infinitesimal analogues of multiplicative forms are called *infinitesimally multiplicative (IM) k-forms* and admit the following two descriptions. First, an IM k-form on A can be described as a k-form $\omega_A \in \Omega^k(A)$ on the total space of the vector bundle A such that the induced map

$$\overline{\omega}_A : \underbrace{TA \oplus_A TA \oplus_A \cdots \oplus_A TA}_{k} \to \mathbb{R},$$

$$(V_1, \dots, V_k) \mapsto \omega_A(V_1, \dots, V_k),$$

defines a Lie algebroid morphism with respect to the natural direct sum of the *tangent lifted Lie algebroid structure* on $TA \to TM$ (see [41]) and to $\mathbb{R} \to *$ seen as a Lie algebra. We denote $\Omega^k_{IM}(A)$ the space of such IM k-forms. One can then show ([7, Sec. 4.1]) that, given a multiplicative $\omega \in \Omega^k_M(\mathcal{G})$, the Lie-functor applied to the morphism $\overline{\omega}$ will yield an IM-form $\omega_A \in \Omega^k_{IM}(A)$ via $Lie(\overline{\omega}) = \overline{\omega}_A$. One can also provide a second, more minimalistic description of IM forms, as follows. There is a $1-1$ correspondence (see [7, Secs. 2.2 and 3.3]) between $\omega_A \in \Omega^k_{IM}(A)$ and pairs (μ, ν) of vector bundle maps $\mu : A \to \Lambda^{k-1}T^*M$ and $\nu : A \to \Lambda^k T^*M$ over id_M satisfying the following *IM equations*: for all $u, v \in \Gamma(A)$,

$$i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u),$$

$$\mu([u,v]) = L_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u) - i_{\rho(v)}\nu(u),$$

$$\nu([u,v]) = L_{\rho(u)}\nu(v) - i_{\rho(v)}d\nu(u).$$

The main theorem [7, Thm. 2] which provides a complete Lie-theoretic description for such forms says that the correspondance

$$\Omega^k_M(\mathcal{G}) \to \Omega^k_{IM}(A), \ \omega \mapsto \omega_A \equiv (\mu, \nu)$$

is $1 : 1$ when $\mathcal{G} \rightrightarrows M$ has 1-connected source fibers. Moreover, this correspondence commutes with de Rham differential.

Let us now describe some applications to Lie-theoretic integration of geometric structures. In the case of a Poisson structure (M, π) outlined in the previous sections, we have an obvious map $\mu = id : A = T^*_\pi M \to T^*M$ defining an IM 2-form with $\nu = 0$. It corresponds to the canonical symplectic 2-form $\omega_A = \omega_c \in \Omega^2(T^*M)$ on the total space of $A = T^*M$. When $\mathcal{G} \rightrightarrows M$ is a Lie groupoid integrating $T^*_\pi M$ which is source 1-connected, then this IM 2-form integrates to a closed 2-form $\omega \in \Omega^2(\mathcal{G})$. Moreover, since ω_c is non-degenerate, one can show that ω is non-degenerate as well, thus defining a multiplicative symplectic structure on $\mathcal{G} \rightrightarrows M$. This provides a Lie theoretic explanation of the additional "symplectic" nature of the integration when M is Poisson. Another example is given by *twisted Dirac structures* (L, H) on M, see [9]. These are strucutres motivated by both constraints in mechanical systems and field-theoretic applications. It consists of a subbundle $L \subset TM \oplus T^*M$ which is maximally isotropic with respect to the natural symmetric pairing on $TM \oplus T^*M$ and also its sections ΓL are closed under the Courant bracket $[,]_H$ twisted by a background closed 3-form $H \in \Omega^3(M)$. In such a structure, $A = L \to M$ defines a Lie algebroid and the map $\mu : L \to T^*M$ given by the projection onto the covector component can be shown to define an IM 2-form together with $\nu(v \oplus \alpha) = i_v H \in \Lambda^2 TM$, $v \oplus \alpha \in TM \oplus T^*M$. This corresponds to the pullback $\omega_A = \rho^*\omega_c \in \Omega^2(L)$ along the anchor $\rho(v \oplus \alpha) = v \in TM$. Applying the Lie theoretic integration to a source 1-connected $\mathcal{G} \rightrightarrows M$ integrating $A = L$, we obtain a multiplicative $\omega \in \Omega^2_M(\mathcal{G})$ which is not closed by satisfies $d\omega = s^*H - t^*H$ and is not non-degenerate but satisfies the weaker condition $Ker(Ds) \cap Ker(Dt) \cap Ker(\omega) = 0$ at identity arrows. Such a structure $(\mathcal{G} \rightrightarrows M, \omega)$ is called *twisted presymplectic* groupoid as identified in [9] to be the integration of Dirac structures.

The Lie theory described here for k-forms can be extended to *multiplicative multivectors* $\mathfrak{X}^k_M(\mathcal{G})$ ([7, Sec. 6]) and, also, to more general *multiplicative tensors* on $\mathcal{G}$ ([10]). The former includes the important examples of *Poisson-Lie groups and groupoids* while the latter include multiplicative complex structures among other types of multiplicative geometries. Other related ramifications include: higher Dirac structures, reduced Poisson and Dirac structures ([20]); vector bundles and representations up to homotopy ([8]); among others.

3.2 Quantization

As mentioned earlier, the idea of *quantization* of Poisson brackets $\{,\}$ to make contact with the formalism for quantum mechanics has been a powerful driving force in the field, largely motivating the study of Lie theory for $\{,\}$. Here we outilne some of the main notions and some recent results connecting quantization back to the Lie theoretic aspects of Poisson brackets. Some general references for the interested reader are [5, 39, 6, 14, 22, 36, 42, 45].

The quantization of a Poisson $\{,\} \equiv \pi$ on M is given by a *star product*, see [6]. This is a binary operation $\star_\hbar$ motivated by important examples of symbol calculus (see [35, 47]) in which there is a "quantization map" $f \mapsto Q_\hbar(f)$ promoting a classical observable $f \in C^\infty(M)$ to a quantum observable given by a suitable operator on a Hilbert space of states and where one encounters the relation $Q_\hbar(f_1) \circ Q_\hbar(f_2) = Q_\hbar(f_1 \star_\hbar f_2)$. Here $\hbar > 0$ represents Planck's constant seen as a scale parameter, taking different values at different scales, with $\hbar \to 0$ representing the *classical limit*. The intrinsic description of a star product is as a 1-parameter family $\hbar \mapsto \star_\hbar$ of binary operations on $C^\infty(M)$ satisfying three key axioms:

Q1) Deformation:
$$f_1 \star_\hbar f_2 = f_1 f_2 + O(\hbar);$$

Q2) Correspondence:
$$\frac{1}{i\hbar}(f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1) = \{f_1, f_2\} + O(\hbar);$$

Q3) Associativity:
$$(f_1 \star_\hbar f_2) \star_\hbar f_3 = f_1 \star_\hbar (f_2 \star_\hbar f_3) + O(\hbar^\infty).$$

The correspondence axiom (Q2) fixes the underlying Poisson brackets and, in this case, we say that $\star_\hbar$ *quantizes* $(M, \pi \equiv \{,\})$. Often, one focuses on asymptotic expansions and interprets $\hbar$ as a formal parameter so that $f_1 \star_\hbar f_2 \in C^\infty(M)[[\hbar]]$ and $O(\hbar^\infty)$ terms are quotiented to zero. In such case, $\star_\hbar$ defines a formal 1-parameter family of (non-commutative if $\{,\} \neq 0$) associative algebras deforming $C^\infty(M)$ with the pointwise multiplication.

The problem of finding a quantization $\star_\hbar$ of an arbitrary Poisson (M, π) is notably difficult and was finally solved by Kontsevich [40] for formal $\hbar$. Before this solution, a Lie theoretic path going through symplectic groupoids was a well-known program (see [22] and references therein) to try to solve this quantization problem. In the rest of this subsection, we outline some recent connections between Kontsevich's solution and the symplectic groupoid formalism following [11].

Let us consider, for simplicity, $(M \subset \mathbb{R}^n, \pi)$ a coordinate Poisson manifold as in Remark 4. Denoting $p_1, p_2 : M \to \mathbb{R}$ linear functions and $x \in M$, Kontsevich's star product $\star_\hbar$ satisfies the following structural relation (see [21])
$$e^{\frac{i}{\hbar}p_1} \star_\hbar e^{\frac{i}{\hbar}p_2}|_x = \bar{a}_\hbar(p_1, p_2, x) e^{\frac{i}{\hbar}\bar{S}(p_1, p_2, x)}, \qquad (3)$$
where $\bar{a}_\hbar = \bar{a}_0 + \hbar\bar{a}_1 + \hbar^2\bar{a}_2 + \dots$ and with $\bar{S}, \bar{a}_n \in C^\infty(M)[[p_1, p_2]]$ formal power series on $p_1, p_2 \in M^*$. These formal power series admit a formula in terms of Kontsevich graphs and complicated Kontsevich weights, see [40]. On the other hand, one of the main results in [11] says that the leading exponent $\bar{S}(p_1, p_2, x)$ coincides with the formal Taylor expansion at $t = 0$ along $t \mapsto t\pi$ of the (non-formal) generating function $S_{t\pi}(p_1, p_2, x)$ for the underlying local symplectic groupoid $(\mathcal{G}_{t\pi} \rightrightarrows M, \omega_c)$, as described in Remark 4. Moreover, the principal symbol of the right multiplication operator $\cdot \star_\hbar x^j$ by a coordinate function $x^j : M \to \mathbb{R}$ is given by the j-th coordinate of source map $s^j(x, p)$ described in the same remark (see also [37]). In [13], it was moreover shown that the Bucher series expansion of $s^j(x, p)$ mentioned in Remark 4 coincides with the expansion in terms of Kontsevich tree graphs and weights coming from $\star_\hbar$ (see also [21]). Altogether, these results establish an explicit correspondence between concrete Lie theoretic constructions for Poisson geometry and Kontsevich's formulas for $\star_\hbar$ in the quantization context. These results are also extended to the field-theoretic context of the *Poisson sigma model* [23] in [11, Sec. 5].

To finish this subsection, we mention some recent related work. The relation 3 above can be understood in the context of semi-classical Fourier integral operators (FIOs, see [35]). The coefficient $\bar{a}_0(p_1, p_2, x)$ in that expansion corresponds to a half-density along the graph of the multiplication map $graph(m) \simeq \mathcal{G}^{(2)}$ of the underlying local symplectic groupoid $(\mathcal{G}_\pi \rightrightarrows M, \omega_c)$, which is subject to an associativity condition. In this way, we observe that, behind a star product, there is an underlying *enhancement* of the symplectic groupoid structure (see [17]). Non-formal versions of star products $\star_\hbar$, $\hbar > 0$ can be defined in the context of FIOs (see [14]) and recent work in progress ([15]) indicates that non-integrability of the underlying (M, π) must manifest itself in a failure of associativity for large strings $f_1 \star_\hbar \cdots \star_\hbar f_N$, $N >> 1$.

3.3 Numerical integrators

In this final subsection, we briefly mention a more recent branch of applications to numerical methods adapted to the underlying geometry [25, 26, 12, 16, 3]. The general idea is that, as explained in Section 2, a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \omega)$ integrating (M, π) is meant to provide a finite dimensional description of Hamiltonian flows and it thus turns out to be very useful also in their approximation.

Let (M, π) be a Poisson manifold and $H \in C^\infty(M)$ a given Hamiltonian function. The idea is to provide an approximation scheme for the flow φ_t^H of the corresponding Hamiltonian vector field $X_H = \pi(dH, \cdot) \in \mathfrak{X}(M)$ which retains its key geometric features. Moreover, we want this method of approximation to be implementable numerically and computationally. Thinking of an atlas for M, most of the discussion can be restricted to the coordinate case $(M \subset \mathbb{R}^n, \pi)$ with an arbitrary Poisson structure $\pi = (1/2)\pi^{ij}(x)\partial_{x^i} \wedge \partial_{x^j}$, although the constructions are geometric in nature.

The first approach to this problem was introduced in [25]. It is based on the underlying symplectic realization $s : (\mathcal{G}, \omega) \to (M, \pi)$ together with the Lagrangian section $1 : M \hookrightarrow (\mathcal{G}, \omega)$. Let $L \hookrightarrow (\mathcal{G}, \omega)$ be another deformed Lagrangian which is a *bisection* in the sense that both $s|_L$ and $t|_L$ are (local) diffeomorphisms $L \to M$. Then, there is an induced (locally defined) map $\varphi_L : M \to M$ given by

$$\varphi_L(x) := s(t|_L^{-1}(x)).$$

The key fact (see [27]) is that, as a consequence of L being Lagrangian in $(\mathcal{G}, \omega)$, then φ_L is a Poisson automorphism of (M, π). In particular, it preserves the geometry underlying π, just as Hamiltonian flows φ_t^H do. Moreover, given H and for small enough time $t \sim 0$, there always exists a family of Lagrangians L_t^H such that $\varphi_{L_t^H} = \varphi_t^H$ reproduces the desired Hamiltonian flow.

Since finding the exact L_t^H can be as hard as solving for φ_t^H, the idea introduced in [25] is to consider an *approximate Lagrangian bisection* $\hat{L}_t^H \hookrightarrow (\mathcal{G}, \omega)$ which is close to the exact one L_t^H in an appropriate sense and, thus, defines an approximation $\varphi_{\hat{L}_t^H}$ for φ_t^H. By construction, this approximation $\varphi_{\hat{L}_t^H}$ preserves the underlying Poisson geometry since $\hat{L}_t^H$ is Lagrangian. On appropiate Darboux charts around $1(M) \subset \mathcal{G}$, one can provide a concrete, computationally amenable ansatz for this approximating Lagrangian $\hat{L}_t^H$ in terms of a truncated solution of the Hamilton-Jacobi equation for an underlying generating function, see [25, 26].

In this way, the Lie theoretic integration $(\mathcal{G} \rightrightarrows M, \omega)$ can be used to provide geometric numerical approximations to Hamiltonian ODEs on the underlying Poisson manifold (M, π). We finish by mentioning recent related work. In [12], the authors explore the idea of also approximating the realization $s : \mathcal{G} \to M$ itself, thus enlarging the domain of aplicability to all cases in which the realization geometry is not known exactly. Truncation of the explicit formulas mentioned in Remark 4 for the realization map s can be used in this context. In [16], the authors explore the role of groupoid multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ in these methods which, so far, has not been used. In particular, preliminary results show that the universal generating function S for the groupoid multiplication of Remark 4 can be used as a tool for refining the numerical methods. Analogous methods for other types of geometries on M are explored in [3].

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